Final practice
Directions

Answer each question in the space provided. Please write clearly and legibly. Show all of your work—your work must both justify and clearly identify your final answer. No books, notes or calculators are allowed. You must simplify results of function evaluations when it is possible to do so.

For instructor use only

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Original Solutions by Mike Willis. Typos corrected by me.
1. [10 pts] Integrate the function \( f(x, y, z) = x \) over the solid region in the first octant \((x \geq 0, y \geq 0, z \geq 0)\) bounded above by \( z = 8 - 2x^2 - y^2 \) and below by \( z = y^2 \).

\[
\begin{align*}
\int_W f(x, y, z) \, dV &= \int_D \int_{z=8-2x^2-y^2}^{z=y^2} x \, dz \, dA \\
&= \int_D x \left. z \right|_{z=y^2}^{z=8-2x^2-y^2} \, dA \\
&= \int_D \left( 8x - 2x^3 - 2xy^2 \right) \, dA \\
\end{align*}
\]

Switch to polar for \( D \): (and take out common factor of \( x \))

\[
\begin{align*}
&= \int_0^{\frac{\pi}{2}} \int_0^2 \left( 8r^2 \cos \theta - 2r^3 \sin^2 \theta \right) (r \cos \theta) \, r \, dr \, d\theta \\
&= \int_0^{\frac{\pi}{2}} \int_0^2 \left( 8r^3 \cos^2 \theta - 2r^4 \cos \theta \right) \, dr \, d\theta \\
&= \int_0^{\frac{\pi}{2}} \left( \frac{8}{3} r^3 - \frac{2}{5} r^5 \right) \cos \theta \bigg|_0^2 \, d\theta \\
&= \int_0^{\frac{\pi}{2}} \left( \frac{32}{3} - \frac{256}{125} \right) \cos \theta \, d\theta \\
&= \left( \frac{128}{15} \right) \left( \frac{8}{5} \right) - \left( \frac{192}{125} \right) \left( \frac{8}{5} \right) \\
&= \frac{128}{15}
\end{align*}
\]
2. [10 pts] Use the transformation \( x = 2u + v, \ y = u + 2v \) to evaluate \( \int \int_R (x - 3y) \, dA \) where \( R \) is the triangular region with vertices \((0,0), (2,1), (1,2)\) in the \( xy \)-plane. Be sure to draw both the original region \( R \) AND the resulting region in the \( uv \)-plane.

\[
\begin{align*}
G(0,0) &= (0,0) \\
G(2,1) &= (2,1) \\
G(1,2) &= (1,2)
\end{align*}
\]

\[
|dG| = 3
\]

So \( \int \int_R (x-3y) \, dA = \int_{uv} (2uv - 3(u+2v)) \, |dG| \, dA \)

\[
= -3 \int_0^1 \int_0^v (u + 5v) \, dv \, du
\]

\[
= -3 \int_0^1 \left( uv + \frac{5}{2} v^2 \right) \bigg|_{v=0}^{v=u} \, du
\]

\[
= -3 \int_0^1 u^2 + \frac{5}{2} u^2 \, du
\]

\[
= -\frac{21}{2} \int_{u=0}^1 u^2 \, du
\]

\[
= -\frac{7}{2} - 3
\]
3. [8 pts] Find the volume enclosed by the cylinder $x^2 + y^2 = 4$, bounded above by the plane $x + 2y + z = 5$, and bounded below by the plane $x + y + z = -1$.

\[
\text{Volume} = \int_{\theta=0}^{2\pi} \int_{r=0}^{2} \int_{z=1-r^2}^{1-r^2} r \, dz \, dr \, d\theta
\]

\[
= \int_{\theta=0}^{2\pi} \int_{r=0}^{2} (r (1-r^2)) \, dr \, d\theta
\]

\[
= \int_{\theta=0}^{2\pi} (8 - \frac{8}{3} \sin \theta) \, d\theta
\]

\[
= \frac{16\pi}{3}
\]

4. [8 pts] Prove that \( \text{curl} \, \vec{F}(x, y, z) = 0 \) for any vector field \( \vec{F} \).

Let \( \vec{F} = (F_1, F_2, F_3) \)

Then \( \text{curl} \, (\vec{F}) = \left( \frac{\partial F_3}{\partial y} - \frac{\partial F_2}{\partial z}, \frac{\partial F_1}{\partial z} - \frac{\partial F_3}{\partial x}, \frac{\partial F_2}{\partial x} - \frac{\partial F_1}{\partial y} \right) \)

So \( \text{div} \, \text{curl} \, \vec{F} = \frac{\partial^2 F_2}{\partial x \partial y} - \frac{\partial^2 F_3}{\partial x \partial z} + \frac{\partial^2 F_1}{\partial y \partial z} - \frac{\partial^2 F_3}{\partial y \partial x} + \frac{\partial^2 F_2}{\partial z \partial x} - \frac{\partial^2 F_1}{\partial z \partial y} \)

\[
= 0
\]
5. Suppose $f(x, y, z)$ is a scalar function, and $\mathbf{F}(x, y, z)$ is a vector field (these two may be totally unrelated). Suppose we have a closed curve $\mathcal{C}$ in $\mathbb{R}^3$ and we know that $\int_{\mathcal{C}} f \, ds = 5$ and $\int_{\mathcal{C}} \mathbf{F} \cdot d\mathbf{r} = 6$.

(a) [3 pts] What is the value of $\int_{\mathcal{C}} f \, ds$? No justification needed.

\[ S \]

(b) [3 pts] What is the value of $\int_{\mathcal{C}} \mathbf{F} \cdot d\mathbf{r}$? No justification needed.

\[ -6 \]

(c) [4 pts] Is it possible that $\mathbf{F} = \nabla f$? Explain.

\[ \text{No.} \quad \text{If } \mathbf{F} = \nabla f, \text{ then we must have } \int_{\mathcal{C}} \mathbf{F} \cdot d\mathbf{r} = 0 \text{ since } \mathcal{C} \text{ is closed.} \]

\[ \text{But we know } \int_{\mathcal{C}} \mathbf{F} \cdot d\mathbf{r} = 6. \]
6. [6 pts] Let $C$ be the unit circle in $\mathbb{R}^2$, oriented counter-clockwise. Find $\int_C (2x - y, 2y + x) \cdot d\vec{r}$.

1F $\vec{F} = (2x - y, 2y + x)$, then $\text{curl}_{\mathbb{R}^2}(\vec{F}) = 1 - (-1) = 2$

Thus $\oint_C \vec{F} \cdot d\vec{r} = \iint_D \text{curl}_{\mathbb{R}^2}(\vec{F}) dA = 2\pi$.

7. Let $f(x, y) = x^2 + 3y$, and let $L$ be the line segment from $(3, 0)$ to $(0, 4)$.

(a) [5 pts] Find $\int_L f(x, y) ds$.

Parametrize $L$ via $\vec{r}(t) = (1-t)(3, 0) + t(0, 4) = (3-3t, 4t)$

$\vec{r}'(t) = (-3, 4)$, $|\vec{r}'(t)| = 5$

So $\int_L f ds = \int_{t=0}^{t=1} [(3-3t)^2 + (4t)^2] \cdot 5 dt = 5 \int_{t=0}^{t=1} (9 - 18t + 9t^2 + 16t^2) dt$

$= 5 [9t - 9t^2 + 3t^3]_{t=0}^{t=1} = 5(9 - 9 + 3) = 30$.

(b) [5 pts] Find $\int_L \nabla f(x, y) \cdot d\vec{r}$

$\int_L \nabla f \cdot d\vec{r} = f(3, 0) - f(3, 0)$

$= (0 + 12) - (9 + 0) = 3$.
8. Let $S$ be the cone given by equation $\phi = \frac{\pi}{4}$ in spherical coordinates.

(a) [4 pts] Find a parametrization $G(u,v)$ for $S$.

\[
G(u,v) = \left( \frac{1}{2} u \cos v, \frac{1}{2} u \sin v, \frac{\sqrt{3}}{2} u \right)
\]

(b) [4 pts] Find the angle between the tangent vectors $\mathbf{T}_u$ and $\mathbf{T}_v$ at the point $(1,0,\sqrt{3})$ (given in $(x,y,z)$ coordinates).

\[
\mathbf{T}_u = \left( \frac{1}{2} \cos v, \frac{1}{2} \sin v, \frac{\sqrt{3}}{2} \right)
\]
\[
\mathbf{T}_v = \left( -\frac{1}{2} u \sin v, \frac{1}{2} u \cos v, 0 \right)
\]

At point $(1,0,\sqrt{3})$, $u=2$, and thus $v=0$.

(c) [6 pts] Find an equation for the tangent plane to the surface at that same point.

\[
\mathbf{N} = \mathbf{T}_u \times \mathbf{T}_v = \left( -\frac{\sqrt{3}}{2}, 0, \frac{1}{2} \right)
\]

So we use equation $(\mathbf{P} - (x,y,z)) \cdot \mathbf{N} = 0$

\[
(1,0,\sqrt{3}) - (x,y,z) \cdot \mathbf{N} = 0
\]
\[
-\frac{\sqrt{3}}{2} (1-x) + 0 + \frac{1}{2} (\sqrt{3}-z) = 0
\]
\[
\frac{\sqrt{3}}{2} x - \frac{1}{2} z = 0
\]

(Or any multiple of this equation.)
The plane \( z + 2y + 2z = 3 \) has a finite portion \( S \) lying in the first octant \((x, y, z \geq 0)\). Suppose that this plane carries a charge density of \( \delta_Q(x, y, z) = e^z \). Find the total charge \( Q \) on \( S \). Be sure to have a good sketch of your parametrization domain \( D \) if necessary.

Write as \( x = 3 - 2y - 2z \), so \( u = \frac{x}{z} \Rightarrow G(u,v) = \left( 3 - 2u - 2v, u, v \right) \)

New domain needs

\[
\begin{align*}
x &\geq 0 \quad \Rightarrow \quad 3 - 2u - 2v \geq 0 \quad \Rightarrow \quad v \leq \frac{3}{2} - u \\
y &\geq 0 \quad \Rightarrow \quad u \geq 0 \\
z &\geq 0 \quad \Rightarrow \quad v \geq 0
\end{align*}
\]

\[
\vec{T}_u = (-2, 1, 0) \\
\vec{T}_v = (-2, 0, 1) \\
\vec{N} = \pm \langle 1, 1, 2, 2 \rangle \\
|\vec{N}| = 3
\]

So

Total Charge

\[
Q = \iint_S \delta_Q \, dS = \iint_D \delta_Q(G(u,v)) \, |\vec{N}| \, dA
\]

\[
= \iint_D e^v \cdot 3 \, dA \\
= 3 \int_{u=0}^{\frac{3}{2}} \int_{v=0}^{\frac{3}{2}-u} e^v \, dv \, du \\
= 3 \int_{u=0}^{\frac{3}{2}} \left( e^{\frac{3}{2} - u} - 1 \right) \, du \\
= 3 \left[ -e^{\frac{3}{2} - u} \right]_{u=0}^{u=\frac{3}{2}} \\
= 3 \left[ -e^0 + e^{\frac{3}{2}} - 0 \right] \\
= 3 \left[ e^{\frac{3}{2}} - \frac{15}{2} \right] = 3e^{\frac{3}{2}} - \frac{15}{2}
\]
10. [10 pts] Compute the surface integral (flux) \( \iint_S \mathbf{F} \cdot d\mathbf{S} \) of \( \mathbf{F} = (y, x, e^{x^2}) \) over the surface \( S \) defined by the equations \( x^2 + y^2 = 9, \ x \geq 0, \ y \geq 0, \ -3 \leq z \leq 3 \), with outward-pointing normal. Be sure to have a good sketch of your parametrization domain \( D \) if necessary.

\[ x^2 + y^2 = 9 \text{ is cylinder of radius } 3, \text{ and we are limited in 1st quadrant with } z \text{ going from } -3 \text{ to } 3 \]

Use cylindrical coordinates to parameterize.

Cylindrical \( r = 3 \) fixed, while \( \theta : 0 \to \frac{\pi}{2} \)

\[ z : -3 \to 3 \]

To make \( \mathbf{N} \) unitary, choose

\[ \mathbf{N} = \left< 3\cos u, 3\sin u, 0 \right> \]

So \( \iint_S \mathbf{F} \cdot d\mathbf{S} = \int_0^{\frac{\pi}{2}} \int_{-3}^{3} \left< 3\sin u, 3\cos u, e^{3u^2} \right> \cdot \left< 3\cos u, 3\sin u, 0 \right> \, dA \)

\[ = \int_0^{\frac{\pi}{2}} \int_{-3}^{3} 18\sin u \cos u \, dv \, du \]

\[ = 108 \int_0^{\frac{\pi}{2}} \sin u \cos u \, du \]

\[ = 54 \sin^2 u \bigg|_{u=0}^{\frac{\pi}{2}} = 54 \]
11. [6 pts] Prove that, if \( \vec{F} \) is a conservative vector field, then the flux of \( \text{Curl}(\vec{F}) \) through any surface \( S \) (closed or not closed) must equal zero.

\[
\text{IF } \vec{F} \text{ is conservative, THEN } \vec{F} = \nabla f \text{ AND } \text{Curl}(\vec{F}) = \vec{0}.
\]

\[
\text{Thus } \int_{S} \text{Curl}(\vec{F}) \cdot d\vec{S} = 0 \text{ regardless of } S.
\]

12. [6 pts] Find the volume of a region \( \mathcal{V} \) if we know that

\[
\int_{\mathcal{W}} \left( x + xy + z, x + 3y - \frac{1}{2}y^2, 4z \right) \cdot d\vec{S} = 16
\]

By Divergence Theorem, this integral matches

\[
\iiint_{\mathcal{W}} \left( 1 + y + 3 - y + y \right) \, dV
\]

\[
\iiint_{\mathcal{W}} & \, dV
\]

\[
8 \text{ Volume(} \mathcal{W} \text{)} \quad \text{Thus } \text{Volume(} \mathcal{W} \text{)} = 2
\]
13. [6 pts] Evaluate \( \mathbf{\int}_C \mathbf{F} \cdot d\mathbf{r} \) where \( C \) is the closed curve illustrated below (HINT: Stokes' Theorem).

\[
\begin{aligned}
\text{curl } \mathbf{F} &= \begin{pmatrix} 1 \\ -1 \\ 1 \end{pmatrix} \\
\text{and we can form surface } S \text{ in } yz-plane, \\
\text{with unit normal } \mathbf{n} &= \begin{pmatrix} -1 \\ 0 \\ 0 \end{pmatrix} \text{ to match orientation of } C
\end{aligned}
\]

So \( \mathbf{\int}_C \mathbf{r} \cdot d\mathbf{r} = \mathbf{\iint}_S (1, -1, 1) \cdot \mathbf{n} \, dS \)

\[
= -\pi \chi_{\alpha} = -\frac{\pi}{4}
\]

14. TRUE/FALSE (circle your answer, no justification needed)

(a) [3 pts] Suppose that \( \text{Curl} \mathbf{F}(x, y, z) = \mathbf{0} \) throughout some simply connected domain. Then there is one and only one potential function \( f(x, y, z) \) such that \( \mathbf{F} = \nabla f \).

\[\text{TRUE} \quad \text{FALSE}\]

(b) [3 pts] If \( \mathbf{F}(x, y, z) \) is conservative on some domain (simply connected or not), then we must have \( \text{Curl} \mathbf{F} = \mathbf{0} \).

\[\text{TRUE} \quad \text{FALSE}\]