Double integral in polar coordinates.

When the area lies in some angular sector $0_1 \leq \theta \leq 0_2$, want to find the largest and smallest radius for a given angle $\theta$. Then region is $0_1 \leq \theta \leq 0_2$, $r_1(\theta) \leq r \leq r_2(\theta)$.

Double integral over region given by:

$$\iint_R f(x,y)\,dA = \int_{0_1}^{0_2} \int_{r_1(\theta)}^{r_2(\theta)} f(r\cos\theta, r\sin\theta) \,r\,dr\,d\theta.$$  

Example:

Evaluate $\iint_R y\,dA$ where $R$ is the region bounded by $x^2 + y^2 \leq 1$ and $(x-1)^2 + y^2 \leq 1$.

This region is in the intersection of two circles.
This shape isn't that easy to work as a radially simple region. It lies within the sector $-\pi/3 \leq \theta \leq \pi/3$. However at different values of $\theta$, the largest value of $r$ is given by the different circles. In particular, the pts of intersection are $(\pm \tfrac{1}{2}, \pm \sqrt{3}/2)$ and, which in polar coordinates is $(1, \pm \pi/3)$.

So for $-\pi/3 \leq \theta \leq \pi/3$ we have $0 \leq r \leq 1$.

For $-\pi/2 \leq \theta \leq -\pi/3$, $\pi/3 \leq \theta \leq \pi/2$, we have $0 \leq r \leq 2 \cos \theta$.

We can join these together into one description by setting

$$g(\theta) = \begin{cases} 
1 & \text{if } \theta \in [-\pi/3, \pi/3], \\
2 \cos \theta & \text{if } \theta \in [-\pi/2, -\pi/3) \cup (\pi/3, \pi/2].
\end{cases}$$

Then the region is given by

$$-\pi/2 \leq \theta \leq \pi/2, \quad 0 \leq r \leq g(\theta).$$

Now,

$$\iint_R y \, dA = \int_{-\pi/2}^{\pi/2} \int_0^{g(\theta)} r \sin \theta \, r \, dr \, d\theta$$

$$= \int_{-\pi/2}^{\pi/2} \sin \theta \int_0^{g(\theta)} r^2 \, dr \, d\theta$$
\[ \int_{-\pi/2}^{\pi/2} \sin \theta r^3 \frac{1}{3} |g(\theta)| \, d\theta \]

\[ = \int_{-\pi/2}^{\pi/2} \sin \theta \frac{g(\theta)^3}{3} \, d\theta \]

\[ = \frac{8}{3} \int_{-\pi/3}^{\pi/3} \sin \theta \cos^3 \theta \, d\theta + \frac{8}{3} \int_{\pi/3}^{\pi/2} \sin \theta \cos \theta \, d\theta + \frac{1}{3} \int_{\pi/3}^{\pi/2} \sin \theta \, d\theta \]

\[ = \frac{1}{3} \int_{-\pi/3}^{\pi/3} \sin \theta \, d\theta \text{ by symmetry.} \]

\[ = 0 \text{ since } \sin \theta \text{ odd.} \]

**Note:** We could have guessed this from the start as the region is symmetric around the x-axis; and y is an odd function.

**Question:** Using polar coordinates, evaluate

\[ \int_{-2}^{2} \int_{0}^{\sqrt{4-x^2}} (x^2+y^2) \, dy \, dx. \]

**Answer:**

The region we are integrating over is where

\[ 0 \leq y \leq \sqrt{4-x^2} < \text{ upper semicircle.} \]
this lies in sector $0 \leq \theta \leq \pi$. Note above, the pt of intersection is $(2, 2\pi/3)$. So it follows the region is given by

for $0 \leq \theta \leq 2\pi/3$, we have $0 \leq r \leq 2$
and for $2\pi/3 \leq \theta \leq \pi$, we have $0 \leq r \leq \sec(\theta - \pi) = -\sec\theta$

(Remember general eq of line: $r = d \sec(\theta - \alpha)$

so

\[ \int_{-1}^{1} \int_{0}^{\sqrt{4-x^2}} x^2 + y^2 \, dy \, dx = \int_{0}^{2\pi/3} \int_{0}^{2} r^3 \, dr \, d\theta + \int_{2\pi/3}^{\pi} \int_{0}^{-2} r^3 \, dr \, d\theta \]

\[ = \int_{0}^{\pi} 4 \, d\theta + \int_{2\pi/3}^{\pi} \frac{\sec^4(\theta)}{4} \, d\theta \]

Note that $\sec^4 x = \sec^2 x \tan^2 x + \sec^2 x$, so

\[ = \frac{8\pi}{3} + \frac{1}{4} \int_{2\pi/3}^{\pi} \frac{\sec^2 \theta \tan^2 \theta + \sec^2 \theta}{4} \, d\theta \]

\[ = \frac{8\pi}{3} + \frac{1}{4} \left[ \frac{\tan^3 \theta}{3} + \tan \theta \right]_{2\pi/3}^{\pi} \]
\[\frac{8\pi}{3} + \frac{1}{4} \left( \sqrt{3} + \sqrt{3} \right) = \frac{8\pi}{3} + \frac{\sqrt{3}}{2}\]

**Correction:** In discussion I said the line \(x = -1\) was \(r = \sec \theta\) since "this cuts out the second line for \(\frac{\pi}{2} < \theta < \frac{3\pi}{2}\)"

This is wrong as for these values of \(\theta\), \(\sec \theta < 0\) and thus would imply \(r\) is negative, which by definition isn't true. So actually, \(r = \sec \theta\) has no solutions for \(\frac{\pi}{2} < \theta < 3\pi\). Instead, we should have used that \(x = -1\) is \(x = 1\) rotated by \(\pi\) degrees, i.e., \(r = \sec (\theta - \pi) = -\sec \theta\).

**Cylinder Coordinates:**

This is just a triple integral where we write the shaded region in polar coordinates.

**Question:**

Evaluate \(\iiint_R z \, dV\) where \(R\) is the region given \(x^2 + y^2 \leq z \leq 9\).

**Answer:**

The region is above the paraboloid \(x^2 + y^2 = z\) and below \(z = 9\).
writing this as a \( z \)-simple region, we get

\[
\iiint_{R} z \, dV = \int_{D} \int_{x^2+y^2}^{z} z \, r \, dr \, d\theta
\]

where \( D \) is the shadow in the \( xy \)-plane, which is the circle centred at the origin with radius 3.

In polar coordinates, this is given by

\[
0 \leq r \leq 3
\]

\[-\pi \leq \theta \leq \pi.\]

Hence

\[
\iiint_{R} z \, dV = \int_{0}^{3} \int_{-\pi}^{\pi} \int_{0}^{\sqrt{r^2 - x^2}} z \, r \, dr \, d\theta \, dx
\]

\[
= \int_{0}^{3} \int_{-\pi}^{\pi} \left. \frac{x^2}{2} \right|_{0}^{r} \, r \, d\theta \, dr
\]

\[
= \int_{0}^{3} \int_{-\pi}^{\pi} \frac{8}{2} r - \frac{r^5}{2} \, d\theta \, dr
\]

\[
= \pi \int_{0}^{3} \frac{8}{2} r \, dr - \frac{r^5}{2}
\]

\[
= \pi \left. \left[ \frac{8}{2} r^2 - \frac{r^6}{6} \right] \right|_{0}^{3}
\]
\[
\pi \left(1 - \frac{\sqrt{1 - 1/9}}{2}\right) 10 = 243\pi.
\]

Integration in spherical coordinates.

Remember \( x = \rho \sin \phi \cos \theta \) (compare to polar coordinates! \( \theta \) is usual angle in xy-plane)

\( y = \rho \sin \phi \sin \theta \)

\( z = \rho \cos \phi \)

Now, we have \( dV = \rho^2 \sin \phi \, d\rho \, d\phi \, d\theta \) after transformation.

Example:

Evaluate \( \iiint_W y \, dV \) where \( W \) is the region given by

\( x^2 + y^2 + z^2 \leq 1 \), \( x, y, z \leq 0 \).

This region is the part of the sphere in the \((-,-,-\)) octant. Hence, in spherical coordinates it is given by \( \frac{\pi}{2} \leq \phi \leq \pi \), \( \pi \leq \theta \leq \frac{3\pi}{2} \), and \( 0 \leq \rho \leq 1 \).
Hence,
\[
\iiint_W y \, dV = \int_{\pi/2}^{\pi} \int_{0}^{\pi} \int_{0}^{3\pi/2} \rho \sin \theta \sin \phi \, d\rho \, d\theta \, d\phi
\]
\[
= \int_{\pi/2}^{\pi} \sin^2 \phi \, d\phi \int_{0}^{3\pi/2} \sin \theta \, d\theta \int_{0}^{1} \rho^3 \, d\rho
\]
\[
= \left(\int_{\pi/2}^{\pi} \frac{1}{2} (1 - \cos 2\phi) \, d\phi\right) \left[-\cos \theta \bigg|_{\pi/4}^{3\pi/2}\right] \cdot \frac{1}{4}
\]
\[
= \left(\frac{1}{2} \phi - \frac{1}{4} \sin 2\phi\right|_{\pi/2}^{\pi} \right) (-1) \cdot \frac{1}{4}
\]
\[
= \left(-\frac{\pi}{2} + \frac{\pi}{4}\right) (-\frac{1}{4})
\]
\[
= -\frac{\pi}{16}
\]

Question:
Evaluate \( \iiint_W \sqrt{x^2 + y^2 + z^2} \, dV \) where \( W \) is the region given by \( x^2 + y^2 + z^2 \leq 4 \), \( z \leq 1 \), \( x \geq 0 \).

Answer: This is the part of the sphere of radius 2 below \( z = 1 \) and right from \( x = 0 \) shaped like a.
the intersection of sphere and
plane \( z = 1 \) happens when \( x^2 + y^2 + 1 = 4 \)
\( x^2 + y^2 = 3 \).

Hence, in terms of spherical coordinate, the intersection
happens when \( \phi = \tan^{-1}\left(\sqrt{3}\right) = \frac{\pi}{3} \).
So we get when \( \frac{\pi}{3} \leq \phi \leq \pi \), \( 0 \leq \rho \leq 2 \).

When \( 0 \leq \phi \leq \frac{\pi}{3} \), the length of a ray is
bounded above by \( z = 1 \) i.e. \( \rho \cos \phi = 1 \iff \rho = \sec \phi \).
Hence \( 0 \leq \rho \leq \sec \phi \) in this case.

Hence we get altogether that
\[
\iiint_n \sqrt{x^2 + y^2 + 1} \, dV = \int_{-\pi/2}^{\pi/2} \left( \int_0^{\pi/3} \sec \phi \, \rho \, d\rho \right) \, d\phi \, d\theta
\]
\[
= \left( \int_{-\pi/2}^{\pi/2} \left( \int_0^{\pi/3} \rho \, d\rho \right) \right) \left( \int_0^{\pi/3} \sec \phi \, d\phi \right)
\]
\[
= \left( \int_{-\pi/2}^{\pi/2} \left( \frac{\rho^2}{2} \right) \right) \left( \int_0^{\pi/3} \sec \phi \, d\phi \right)
\]
\[
= \left( \frac{\pi}{4} \left( \frac{\pi}{3} \right) \right) \left( \ln \sec \frac{\pi}{3} + \tan \frac{\pi}{3} \right)
\]
\[
\begin{align*}
&= \int_{-\pi/3}^{\pi/3} \left( \int_0^{\pi/3} \frac{r^2}{\rho} \left. \sec \phi \right|_0^{\pi/3} + \int_{\pi/3}^{\pi} \frac{r^2}{\rho} \left. \sec \phi \right|_0^{\pi/3} \right) d\phi d\theta \\
&= \frac{1}{4} \int_{-\pi/2}^{\pi/2} \left( \int_0^{\pi/3} \sec \phi d\phi + \int_{\pi/3}^{\pi} 4 d\phi \right) d\theta \\
&= \frac{\pi}{4} \left\{ \int_0^{\pi/3} \sec^2 \phi + \tan^2 \phi + \sec^2 \phi d\phi + \frac{8\pi}{3} \right\} \\
&\quad \text{since } \sec^2 x = \tan^2 x + 1 \\
&= \frac{\pi}{4} \left\{ \frac{\tan^3 \phi}{3} + \tan \phi \right|_0^{\pi/3} + \frac{8\pi}{3} \right\} \\
&= \frac{\pi}{4} \left\{ 2\sqrt{3} + \frac{8\pi}{3} \right\}
\end{align*}
\]