Triple Integrals

Reminders:

• A simple region consists of points \((x, y, z)\) between two surfaces \(z = z_1(x, y)\), \(z = z_2(x, y)\), where \(z_1(x, y) \leq z \leq z_2(x, y)\), lying over domain \(D\) in \(xy\)-plane. So \(W\) is defined by \((x, y) \in D\) \(z_1(x, y) \leq z \leq z_2(x, y)\).

In other words, all lines parallel to \(z\)-axis that intersect \(W\) do so unbroken.

• Triple integrals over \(W\) equal to iterated integral

\[
\iiint_W f(x, y, z) \, dV = \iint_D \int_{z_1}^{z_2} f(x, y, z) \, dz \, dA.
\]

• Nothing special about \(z\) above, works for \(x, y\) too.

Example: evaluate \(\iiint_W e^z \, dV\) where \(W: x + y + z = 1\), \(x \geq 0, y \geq 0, z \geq 0\).

Now \(x + y + z = 1\) is the plane through \((1, 0, 0), (0, 1, 0)\), and \((0, 0, 1)\). Hence it follows that \(W\) is the wedge.

Now, I want \(z\) to be the inner integral. So I think for each fixed \((x, y)\), what is the possible range of \(z\)-values in this region?

Well, it's below the plane \(z = 1\) given by
$D$ range of $z$-values in this region? Well, it's below the plane (given by $z \leq 1-x-y$) and above $xy$-plane ($z \geq 0$).

Hence we get, $0 \leq z \leq 1-x-y$.

So $\iiint_W e^z \, dV = \int_0^1 \int_0^{1-x} \int_0^{1-x-y} e^z \, dz \, dy \, dx$ where $D$ is the "shadow" of $W$ onto the $xy$-plane, which we figure out to be given by $0 \leq x \leq 1$, $0 \leq y \leq 1-x$.

Hence,

$$\iiint_W e^z \, dV = \int_0^1 \int_0^{1-x} \int_0^{1-x-y} e^z \, dz \, dy \, dx$$

$$= \int_0^1 \int_0^{1-x} (e^{1-x-y} - 1) \, dy \, dx$$

$$= \int_0^1 \left[ -e^{1-x-y} - y \right]_0^{1-x} \, dx$$

$$= \int_0^1 -1 + e^{1-x} - 1 + x \, dx$$

$$= -e^{1-x} + \frac{x^2}{2} - 2x \bigg|_0^1$$

$$= -1 + e + \frac{1}{2} - 2 = e - \frac{5}{2}.$$ 

**Questions**

Find $\iiint_W z \, dV$ where $W$ is the shape:

![Diagram of a region in 3D space](attachment:image.png)
Answer: The top surface is given by \( z = \frac{1}{4} y \) (think of corresponding line in \( yz \)-plane). Hence if \( 0 \leq z \leq \frac{1}{4} y \). The shadow of \( W \) on the \( xy \)-plane is the box \([0, 3] \times [0, 4]\). Hence

\[
\iiint_W \, dV = \int_0^3 \int_0^4 \int_0^{\frac{1}{4} y} z \, dz \, dy \, dx
\]

\[
= \int_0^3 \int_0^4 \left[ \frac{1}{2} z^2 \right]_0^{\frac{1}{4} y} dy \, dx
\]

\[
= \int_0^3 \int_0^4 \frac{1}{32} y^2 dy \, dx
\]

\[
= \int_0^3 \left[ \frac{1}{32} \frac{y^3}{3} \right]_0^4 dx
\]

\[
= \int_0^3 \frac{2}{3} \, dx = 2.
\]

**Question**

Find volume of the solid in octant \( x > 0, y > 0, z > 0 \) bounded by \( x + y + z = 1 \) and \( x + y + 2z = 1 \)
Answer

Draw the region: this is the slice between the two planes $x+y+z=1$, $x+y+2z=1$.

Let $W$ be this solid. For each fixed $(x,y)$, the $z$-values are below the point $z = x+y+2z = 1$ which gives $z = 1-x-y$, i.e. $z \leq 1-x-y$.

And they are above points $(x,y)$ such that $z = \frac{1}{2} (1-x-y)$ i.e. $z \geq \frac{1}{2} (1-x-y)$, hence it follows

$$V = \iiint_W \, dV = \int_0^1 \int_0^{1-x} \int_{\frac{1}{2} (1-x-y)}^{1-x-y} \, dz \, dy \, dx$$

$$= \int_0^1 \int_0^{1-x} \frac{1}{2} (1-x-y) \, dy \, dx$$

$$= \frac{1}{2} \int_0^1 \int_0^{1-x} \left( y - xy - \frac{y^2}{2} \right) \, dy \, dx$$
\[ \frac{1}{2} \int_0^1 \left( -x - x(1-x) - \frac{(1-x)^2}{2} \right) \, dx \]

\[ = \frac{1}{2} \int_0^1 \left( -2x + x^2 - \frac{1}{2} - 2x + x^2 \right) \, dx \]

\[ = \frac{1}{2} \int_0^1 \left( -x + x^2 \right) \, dx = \frac{1}{4} \int_0^1 (1-x)^2 \, dx \]

\[ = -\frac{1}{12} (1-x)^3 \bigg|_0^1 = \frac{1}{12} \]

**Question:** Let \( W \) be the region bounded by \( z = 1 - y^2 \), \( y = x^2 \) and plane \( z = 0 \). Write volume of \( W \) as triple integral in the order \( dz \, dy \, dx \), \( dx \, dz \, dy \) and \( dy \, dz \, dx \).

**Answer:** We first try and picture this.

\[ dz \, dy \, dx \] order:

First look at possible \( z \)-values for given \((x, y)\)

We see by above \( 0 \leq z \leq 1 - y^2 \)
First look at possible \( z \)-values for given \((x,y)\)
we see by above \( 0 \leq z \leq 1-y^2 \)
Then possible \( y \)-value for given \( x \): \( x^2 \leq y \leq 1 \)
and finally possible \( x \)-values: \( -1 \leq x \leq 1 \)

\[ V = \int_{-1}^{1} \int_{x^2}^{1} \int_{0}^{\sqrt{1-z}} dz \, dy \, dx \]

\( dx \, dz \, dy \) order:
possible \( x \)-values (given \((y,z)\)): \( -\sqrt{y} \leq x \leq \sqrt{y} \)
possible \( z \)-values: \( 0 \leq z \leq 1-y^2 \)
possible \( y \)-values: \( 0 \leq y \leq 1 \)

\[ \therefore V = \int_{0}^{1} \int_{\sqrt{y}}^{\sqrt{1-y}} \int_{-\sqrt{y}}^{\sqrt{y}} dx \, dz \, dy. \]

\( dy \, dz \, dx \) order:
possible \( y \)-values (given \((x,z)\)): \( x^2 \leq y \leq \sqrt{1-z} \)
shadow on \( xz \)-plane:

so we get \( -1 \leq x \leq 1, \quad 0 \leq z \leq 1-x^2 \)
Hence \[ V = \int_{-1}^{1} \int_{x^2}^{1} \int_{0}^{\sqrt{1-x^2}} dz \, dy \, dx. \]
Checking these give same volume; (not part of Q).
\[ \int_{-1}^{1} \int_{x^2}^{1-x^2} dz dy dx = \int_{-1}^{1} \int_{x^2}^{1-x^2} 1 - y^2 dy dx \]
\[ = \int_{-1}^{1} y - \frac{y^3}{3} \bigg|_{x^2}^{1} dx \]
\[ = \int_{-1}^{1} \frac{2}{3} - x^2 + \frac{x^6}{3} dx \]
\[ = 2 \left( \frac{2}{3} x - \frac{x^3}{3} + \frac{x^7}{7} \right) \bigg|_{-1}^{1} \]
\[ = 2 \left( \frac{1}{3} + \frac{1}{21} \right) = 16/21 \]

\[ \int_{0}^{1} \int_{0}^{\sqrt{y}} \int_{0}^{\sqrt{y}} \sqrt{y} dz dy dx \]
\[ = \int_{0}^{1} \int_{0}^{\sqrt{y}} \left( 2 \sqrt{y} - 2 \right) dy dx \]
\[ = 2 \left( \frac{2}{3} y^{3/2} - \frac{2 y^{5/2}}{5} \right) \bigg|_{0}^{1} \]
\[ = 2 \left( \frac{2}{3} - \frac{2}{5} \right) = 2 \frac{19}{15} \]
\[ = 16/21 \]

\[ \int_{-1}^{1} \int_{x^2}^{\sqrt{1-x^2}} dy dz dx = \int_{-1}^{1} \int_{x^2}^{\sqrt{1-x^2}} -x^2 dz dx \]
\[ \int_{-1}^{1} \int_{-1}^{1} x^2 = \int_{-1}^{1} \left[ -\frac{2}{3} (1-x^3) - \frac{x^2}{2} \right]_0^1 dx = \int_{-1}^{1} \left[ -\frac{2}{3} x^3 - \frac{x^2}{2} + \frac{2}{3} \right] dx = 2 \int_{0}^{1} \left[ \frac{1}{3} x^3 - \frac{x^2}{2} + \frac{2}{3} \right] dx = 2 \left( \frac{1}{21} - \frac{1}{3} + \frac{2}{3} \right) = 2 \left( \frac{1}{21} + \frac{1}{3} \right) = \frac{16}{21} \]  

So everything checks out.

**Polar Coordinates:**

- The transformation
  \[ (x, y) \rightarrow (\sqrt{x^2+y^2}, \tan^{-1}(\frac{y}{x})) \] 
  is the mapping from rectangular to polar coordinates.
- The inverse transformation is
\((r, \theta) = (r \cos \theta, r \sin \theta)\)

**Quick Question**: describe what the following eq. look like.

a) \(r = 2\)  
   b) \(\theta = 2\)  
   c) \(r = 2 \sec \theta\)  
   d) \(r = 2 \csc \theta\)

**Answer**:

a) circle radius 2  
   b) line from origin  
   c) vertical line \(x = 2\)  
   d) horizontal line \(y = 2\)

**Question**: convert the equation \(r = 2 \sin \theta\) to an equation in rectangular coordinate.

**Answer**:

\[ r = 2 \sin \theta \iff r^2 = 2 r \sin \theta \text{ as } r \neq 0. \]

and \[ r^2 = x^2 + y^2 \]

\[ r \sin \theta = y \]

Therefore equation becomes \(x^2 + y^2 = 2y\)

and completing the square \(x^2 + (y-1)^2 = 1\)

so circle radius 1 centred at \((0,1)\).