(1) The gravitational potential \( V \) at \( P \) due to a point mass \( m \) sitting at the point \( Q \) is given by

\[
V(P) = -\frac{Gm}{r},
\]

where \( r = \|P - Q\| \) is the distance from \( P \) to \( Q \) and \( G \) is the gravitational constant.

Suppose that instead of a point mass, we have a thin surface \( S \) with mass density \( \delta(x, y, z) \); then the gravitational potential is given by

\[
V(P) = -G \int_S \frac{\delta(x, y, z) \, dS}{\sqrt{(x-a)^2 + (y-b)^2 + (z-c)^2}},
\]

where \( P = (a, b, c) \).

(a) Suppose the surface \( S \) is a hollow sphere of radius \( R \) centered at the origin with mass density \( \delta = m/(4\pi R^2) \). By symmetry, we only need to compute \( V(P) \) at a single point \( P = (0, 0, r) \) with \( r \neq R \). Use spherical coordinates to show that

\[
V(0, 0, r) = -\frac{Gm}{4\pi} \int_0^\pi \int_0^{2\pi} \frac{\sin \phi \, d\theta \, d\phi}{\sqrt{R^2 + r^2 - 2Rr \cos \phi}}.
\]

I'm lazy to derive this, so from textbook:

- Sphere of radius \( R \), centered at the origin:

\[
G(\theta, \phi) = (R \cos \theta \sin \phi, R \sin \theta \sin \phi, R \cos \phi)
\]

Unit radial vector: \( e_r = (\cos \theta \sin \phi, \sin \theta \sin \phi, \cos \phi) \)

Outward normal: \( N = T_{\phi} \times T_{\theta} = (R^2 \sin \phi) \, e_r \)

\[
dS = \|N\| \, d\phi \, d\theta = R^2 \sin \phi \, d\phi \, d\theta
\]

Hence \( V(0, 0, r) = -\frac{Gm}{4\pi R^2} \int_0^\pi \int_0^{2\pi} \frac{R^2 \sin \phi \, d\phi \, d\theta}{\sqrt{R^2 + r^2 - 2Rr \cos \phi}} \)

\[
= -\frac{Gm}{(4\pi R^2)} \int_0^\pi \int_0^{2\pi} \frac{R^2 \sin \phi \, d\phi \, d\theta}{(R^2 \cos^2 \phi + R^2 \sin^2 \phi + (R \cos \phi - r)^2)^{1/2}}
\]

\[
= -\frac{Gm}{(4\pi R^2)} \int_0^\pi \int_0^{2\pi} \frac{R^2 \sin \phi \, d\phi \, d\theta}{(R^2 \sin^2 \phi + R^2 \cos^2 \phi + 2Rr \cos \phi + r^2)^{1/2}}
\]

\[
= -\frac{Gm}{(4\pi R^2)} \int_0^\pi \int_0^{2\pi} \frac{\sin \phi \, d\phi \, d\theta}{r}
\]
\[
\frac{-Gm}{(4\pi)^2} \int_0^{2\pi} \int_0^\pi \frac{\sin \phi \, d\phi \, d\theta}{\sqrt{R^2 + r^2 - 2Rr \cos \phi}}
\]

(b) To evaluate this integral, it’s helpful to use the substitution \( u = R^2 + r^2 - 2Rr \cos \phi \). Show that when \( \phi = 0 \), \( u = (R - r)^2 \), and when \( \phi = \pi \), \( u = (R + r)^2 \). Use this to show that

\[
V(0, 0, r) = \frac{-Gm}{2Rr} \left( |R + r| - |R - r| \right).
\]

when \( \phi = 0 \), \( u = R^2 + r^2 - 2Rr = (R - r)^2 \)

\( \phi = \pi \), \( u = R^2 + r^2 + 2Rr = (R + r)^2 \)

\( du = 2Rr \sin \phi \, d\phi \)

Hence,

\[
V(0, 0, r) = \frac{-Gm}{8\pi Rr^2} \int_0^{2\pi} \int_{(R - r)^2}^{(R + r)^2} \frac{du}{\sqrt{u}} \, d\theta
\]

\[
= \frac{-Gm}{4Rr} \int_{(R - r)^2}^{(R + r)^2} \frac{du}{\sqrt{u}}
\]

\[
= \frac{-Gm}{2Rr} \left( |R + r| - |R - r| \right)
\]

(c) Suppose first that \( P \) is outside the sphere (so \( r > R \)). Show that \( V(P) = \frac{-Gm}{r} \).

If \( r > R \), then \( |R + r| = R + r \) and \( |R - r| = r - R \).

Hence,

\[
V(P) = \frac{-Gm}{2Rr} \left( R + r - (r - R) \right)
\]

\[
= \frac{-Gm}{r}
\]
(d) If $P$ is inside the sphere (so $r < R$) show that $V(P) = -\frac{GM}{R}$.

If $r < R$, then $|r + r| = r + r$, $|r - r| = r - r$.

Hence $V(P) = -\frac{GM}{2R} (r + r - (r - r))$

$= -\frac{GM}{R}$

(e) Let’s interpret this result (keep in mind that $R$ is a constant and $r$ is a variable). If the object is inside the sphere, then the sphere exerts no gravitational force on the object (remember that $F = -\nabla V$). On the other hand, if the object is outside the sphere, then the sphere behaves like a point mass as far as gravity is concerned (look at equation (1)). Newton was the first to prove this somewhat surprising fact.

(2) Let $S$ be the ellipsoid $\left(\frac{x}{a}\right)^2 + \left(\frac{y}{b}\right)^2 + \left(\frac{z}{c}\right)^2 = 1$. Calculate the flux of $\mathbf{F} = z\mathbf{i}$ over the portion of $S$ where $x, y, z \geq 0$ with upward-pointing normal.

Hint: parametrize $S$ using a modified form of spherical coordinates.

Parameterization: $\mathbf{g}(s,t) = (4\cos s \sin t, 3\sin s \sin t, 2\cos t)$

$\mathbf{g}_s(s,t) = (-4\sin s \sin t, 3\cos s \sin t, 0)$

$\mathbf{g}_t(s,t) = (4\cos s \cos t, 3\sin s \cos t, -2\sin t)$

$\mathbf{g}_s(s,t) \times \mathbf{g}_t(s,t) = \begin{vmatrix}
\mathbf{i} & \mathbf{j} & \mathbf{k} \\
-4\sin s \sin t & 3\cos s \sin t & 0 \\
4\cos s \cos t & 3\sin s \cos t & -2\sin t
\end{vmatrix}$

$\mathbf{N}(s,t) = \langle -6\cos s \sin^2 t, -8\sin s \sin^2 t, -12\sin t \cos t \rangle$

Test a point to see if outward facing: $t = 0, s = \frac{\pi}{2}$

$\mathbf{N}(0,\frac{\pi}{2}) = \langle 0, 0, -12 \rangle$ so inward. Hence swap signs.
\( \mathbf{N}(s,t) = \langle 6 \cos s \sin^2 t, 8 \sin s \sin^2 t, 12 \sin t \cos t \rangle \).

\( \mathbf{F}(\mathbf{G}(s,t)) = \langle 2 \cos t, 0, 0 \rangle \)

\( \mathbf{F}(\mathbf{G}(s,t)) \cdot \mathbf{N}(s,t) = 12 \cos s \sin^2 t \cos t \)

So integral is:

\[
\iint_S \mathbf{F} \cdot d\mathbf{s} = \int_0^{\pi/2} \int_0^{\pi/2} 12 \cos s \sin^2 t \cos t \, ds \, dt
\]

\[
= 12 \left( \int_0^{\pi/2} \cos s \, ds \right) \left( \int_0^{\pi/2} \sin^2 t \cos t \, dt \right)
\]

Now, \( \int_0^{\pi/2} \cos s \, ds = \sin s \bigg|_0^{\pi/2} = 1 \)

\( \int_0^{\pi/2} \sin^2 t \cos t \, dt = \frac{\sin^3 t}{3} \bigg|_0^{\pi/2} = \frac{1}{3} \)

Hence, \( \iint_S \mathbf{F} \cdot d\mathbf{s} = 4 \).