1. The equation of the osculating circle to a curve parametrized by $\mathbf{r}(t) = (f(t), g(t))$ at the point $(2, 1)$ is given by:

$$x^2 + y^2 - 4x + 2y + 1 = 0.$$

(Note that $(2, 1)$ is not the center of the osculating circle, it’s the point where the osculating circle touches the curve $\mathbf{r}(t)$.)

(a) **5 points** Find the curvature of the curve $\mathbf{r}'(t)$ at the point $(2, 1)$.

(b) **3 points** Find the unit normal $\hat{N}$ to the curve $\mathbf{r}'(t)$ at the point $(2, 1)$.

(c) **2 points** Find a vector parametrization of the tangent line to $\mathbf{r}'(t)$ at the point $(2, 1)$. (Note that this is not the same thing as finding the unit tangent vector $\mathbf{T}$ at $(2, 1)$, you need to find a vector parametrization of the whole tangent line).

[Hint: First rewrite the equation as $(x-a)^2 + (y-b)^2 = r^2$. Then look at the Figure 1 on Page 1 and use its geometry (i.e., the relation between the curve $\mathbf{r}'(t)$ and its osculating circle) to solve this problem.]

\[a) \quad x^2 + y^2 - 4x + 2y + 1 = 0 \]

\[\Rightarrow \quad (x-2)^2 - 4 + (y+1)^2 - 1 + 1 = 0 \]

\[\quad (x-2)^2 + (y+1)^2 = 4.\]

Hence osc circle has radius $R = 2$ and curvature $k = 1/R = 1/2$.

\[b) \quad \text{The circle has centre } (2, -1) \text{ and so by geometry, } \]

$$\hat{N} = \frac{(2, -1) - (2, 1)}{\| (2, -1) - (2, 1) \|} = \frac{(0, -2)}{\| (0, -2) \|} = (0, -1)$$

\[c) \quad \text{we need to find a vector perpendicular to } \hat{N}. \text{ This will be the direction vector of the line. } \]

$$\mathbf{v} = (9, b). \quad \mathbf{v} \cdot \hat{N} = 0 \Rightarrow -b = 0 \quad \text{so } b = 0.$$
\( \mathbf{v} = (9, 5) \).  \( \mathbf{v} \cdot \mathbf{n} = 0 \Rightarrow -b = 0 \), so \( b = 0 \) and we can take \( a = 1 \).

Hence \( \mathbf{\hat{v}} = (1, 0) \) is the direction vector to the line.

\( r(t) = (2, 1) + t(1, 0) \) is a parameterization of tangent line.

2. \[10 \text{ points} \] Decompose the acceleration vector \( \mathbf{a}(t) \) of \( \mathbf{r}(t) = (t^3, 2t, \ln t) \) into its tangential and normal components at \( t = \frac{1}{2} \), i.e., find \( a_{T(\frac{1}{2})}, \mathbf{T}(\frac{1}{2}), a_{N(\frac{1}{2})} \) and \( \mathbf{N}(\frac{1}{2}) \), and express \( \mathbf{a}(\frac{1}{2}) \) as

\[
\mathbf{a}(\frac{1}{2}) = a_{T(\frac{1}{2})} \mathbf{T}(\frac{1}{2}) + a_{N(\frac{1}{2})} \mathbf{N}(\frac{1}{2}).
\]

From textbook:

**Theorem 1: Tangential and Normal Components of Acceleration**

In the decomposition \( \mathbf{a} = a_T \mathbf{T} + a_N \mathbf{N} \), we have

\[
a_T = \frac{\mathbf{a} \cdot \mathbf{T}}{\|\mathbf{T}\|}, \quad a_N = \mathbf{a} \cdot \mathbf{N} = \sqrt{\|\mathbf{a}\|^2 - \|a_T\|^2}
\]

and

\[
a_T \mathbf{T} = \left( \frac{\mathbf{a} \cdot \mathbf{T}}{\|\mathbf{T}\|} \right) \mathbf{T}, \quad a_N \mathbf{N} = \mathbf{a} - a_T \mathbf{T} = \mathbf{a} - \left( \frac{\mathbf{a} \cdot \mathbf{T}}{\|\mathbf{T}\|} \right) \mathbf{T}
\]

So we need to find \( \dot{\mathbf{v}} \) and \( \ddot{\mathbf{a}} \).

\( \mathbf{r}(t) = (t^3, 2t, \ln t) \)

\( \mathbf{r}'(t) = (3t^2, 2, \frac{1}{t}) \) \( \dot{\mathbf{v}} = \mathbf{r}'(\frac{1}{2}) = (1, 2, 2) \), \( \|\dot{\mathbf{v}}\| = 3 \)

\( \mathbf{r}''(t) = (6t, 0, -\frac{1}{t^2}) \) \( \ddot{\mathbf{a}} = \mathbf{r}''(\frac{1}{2}) = (3, 0, -4) \), \( \|\ddot{\mathbf{a}}\| = \sqrt{20} \)

\( a_T = \frac{\mathbf{a} \cdot \mathbf{T}}{\|\dot{\mathbf{v}}\|} = \frac{(2, 2, -4) \cdot (1, 2, 2)}{\|1, 2, 2\|} = \frac{2 - 8}{\sqrt{1 + 4 + 4}} = -6 \quad \frac{3}{2} = -2 \)

\( a_N = \sqrt{\|\mathbf{a}\|^2 - |a_T|^2} = \sqrt{20 - 9} = 4 \)
\[ a_T T = \frac{a \cdot v}{v \cdot v} v = \frac{(2, 0, -4) \cdot (1, 2, 2)}{(1, 2, 2)} (1, 2, 2) = -\frac{2}{3} (1, 2, 2) \]

\[ a_N N = a - a_T T = (2, 0, -4) + \frac{2}{3} (1, 2, 2) = \left( \frac{8}{3}, \frac{4}{3}, -\frac{8}{3} \right) \]

3. Let \( \vec{T}(t) \) be a parametrization of a curve, and \( \vec{N}(t) \) and \( \vec{B}(t) \) are respectively the unit tangent, unit normal and the binormal vector.

(a) 6 points Prove that \( \vec{T}'(t) \times \vec{T}''(t) \) is a multiple of \( \vec{B}(t) \).

(b) 4 points Using Part (a) conclude that \( \vec{B}(t) = \frac{\vec{T}'(t) \times \vec{T}''(t)}{||\vec{T}'(t) \times \vec{T}''(t)||} \).

[Hint: You may use the fact that \( \vec{T}'(t) = v(t) \vec{T}(t) \).]

\[ \vec{T}'(t) = v(t) \vec{T}(t) \]

\[ \vec{T}''(t) = \vec{T}'(t) \vec{T}(t) + v(t) \vec{T}'(t) \vec{T}(t) \]

\[ = v(t) \vec{T}'(t) \vec{T}(t) + v(t) \vec{T}'(t) ||\vec{T}'(t)|| \vec{N}(t) \quad \text{since} \quad \vec{N}(t) = \frac{\vec{T}'(t)}{||\vec{T}'(t)||} \]

by definition.

\[ \vec{T}'(t) \times \vec{T}''(t) = \left( \vec{T}'(t) \times \vec{T}(t) \right) \times \left( \vec{T}'(t) \vec{T}(t) + v(t) \vec{T}'(t) ||\vec{T}'(t)|| \vec{N}(t) \right) \]

\[ = v v' \vec{T} \times \vec{T} \vec{T} + v^2 ||\vec{T}'(t)|| \vec{T} \times \vec{N} \]

\[ = v^2 ||\vec{T}'(t)|| \vec{B} \quad \text{since} \quad \vec{T} \times \vec{T} = 0 \quad \text{and} \quad \vec{B} = \vec{T} \times \vec{N} \quad \text{by definition.} \]

Hence \( \vec{B} \) is a multiple of \( v' \vec{x} v'' \). 

b. \( \vec{B} \) has unit length, so if we divide \( v' \vec{x} v'' \) by \( ||v' \vec{x} v''|| \) we have

\[ \frac{v' \vec{x} v''}{||v' \vec{x} v''||} = \pm \vec{B}. \quad \text{However, by above,} \quad v^2 ||\vec{T}'(t)|| > 0 \quad \text{and so} \quad \vec{B} \text{ is a positive multiple (in some direction).} \]

Hence \( \vec{B} = \frac{v' \vec{x} v''}{||v' \vec{x} v''||} \).
Partial derivatives (possibly skipping ahead)

Given a function \( f(x, y) \). The partial derivatives are the derivatives of each variable separately, i.e.

\[
\frac{\partial f}{\partial x} = f_x = \lim_{h \to 0} \frac{f(x+h, y) - f(x, y)}{h}
\]

\[
\frac{\partial f}{\partial y} = f_y = \lim_{h \to 0} \frac{f(x, y+h) - f(x, y)}{h}
\]

In practice, you essentially treat the other variable as constant and differentiate.

**Example:** \( f(x, y) = x^2 y^3 \).

\[
\frac{\partial (x^2 y^3)}{\partial x} = y^3 \frac{\partial x^2}{\partial x} = y^3 (2x) = 2x y^3 \quad \text{(treat y constant)}
\]

\[
\frac{\partial (x^2 y^3)}{\partial y} = x^2 \frac{\partial (y^3)}{\partial y} = x^2 (3y^2) = 3x^2 y^2 \quad \text{(treat x constant)}
\]

**Question:** Find the partials at the following:

a) \( f(x, y) = \sin(x^2 y^3) \)

b) \( f(x, y) = x^3 + y e^x \)

**Answers:**

a) \( f_x = \cos(x^2 y^3) \cdot (2xy^3) = 2xy^3 \cos(x^2 y^3) \)
\[ f_y = 3x^2y^2 \cos(x^2y^3) \]

b) \[ f_x = 3x^2 + ye^x \]
\[ f_y = e^x \]

Geometrically, you can think of \( f_x \) as the rate of change in the \( x \) direction.

Similarly for \( f_y \).

Textbook Q's: 9, 12