Week 8 Notes

**Question 1.** The involute of a circle has parameterisation given by

\[ r(\theta) = (R(\cos(\theta) + \theta \sin(\theta)), R(\sin(\theta) - \theta \cos(\theta))) \]

Find the arclength parameterisation.

\[ r'(\theta) = \left(-R\sin(\theta) + R\cos(\theta) + R\theta \cos(\theta), R\cos(\theta) - R\cos(\theta) + R\sin(\theta)\right) \]

\[ = R\theta \left(\cos(\theta), \sin(\theta)\right) \]

Hence

\[ s(\theta) = \int_0^\theta \| r'(u) \| \, du \]

\[ = \int_0^\theta R \, du \]

\[ = R \left[ u \right]_0^\theta \]

\[ = \frac{R \theta^2}{2} \]

Hence, \( \theta(s) = \sqrt{\frac{2s}{R}} \) and the arclength parameterisation is given by

\[ r_1(s) = r\left(\sqrt{\frac{2s}{R}}\right) \].

**Question 2.** Show that the curvature at an inflection point of a plane curve \( y = f(x) \) is zero.

An inflection point is when \( f''(x) = 0 \). Hence using the formula for curvature of a plane curve:

\[ \kappa(x) = \frac{|f''(x)|}{\left(f'(x)\right)^2} \]

\[ k(x) = \frac{|f''(x)|}{(1 + (f'(x))^2)^{3/2}} \]

we get \( k(x) = 0 \).

**Question 3.** Given a frenet frame \( (T, N, B) \) with arclength parameterisation.

(a) Show \( \frac{d\vec{B}}{ds} = \vec{T} \times \frac{d\vec{N}}{ds} \) and conclude that \( \frac{d\vec{B}}{ds} \) is orthogonal to \( \vec{T} \).

(b) Show that \( \frac{d\vec{B}}{ds} \) is orthogonal to \( \vec{B} \). Hint: Differentiate \( \vec{B} \cdot \vec{B} = 1 \).

(c) Show that \( \frac{d\vec{B}}{ds} \) is a multiple of \( \vec{N} \).

a) By definition, \( \vec{B} = \vec{T} \times \vec{N} \), and so using product rule for cross product:

\[
\frac{d\vec{B}}{ds} = \frac{d\vec{T}}{ds} \times \vec{N} + \vec{T} \times \frac{d\vec{N}}{ds}
\]

Since \( \frac{d\vec{T}}{ds} \parallel \vec{N} \), \( \frac{d\vec{T}}{ds} \times \vec{N} = 0 \).

Hence \( \frac{d\vec{B}}{ds} = \vec{T} \times \frac{d\vec{N}}{ds} \).

Since cross product output orthogonal vector, we get \( \frac{d\vec{B}}{ds} \perp \vec{T} \).

b) \( \vec{B} \cdot \vec{B} = 1 \), differentiate via product rule gives

\[
\frac{d\vec{B}}{ds} \cdot \vec{B} + \vec{B} \cdot \frac{d\vec{B}}{ds} = 0 \implies 2 \vec{B} \cdot \frac{d\vec{B}}{ds} = 0.
\]

Hence \( \frac{d\vec{B}}{ds} \perp \vec{B} \).

c) Since \( \frac{d\vec{B}}{ds} \perp \vec{B} \) and \( \frac{d\vec{B}}{ds} \perp \vec{T} \), \( \vec{T}, \vec{N}, \vec{B} \) form an othornormal system, \( \frac{d\vec{B}}{ds} \) must be parallel to \( \vec{N} \). is a multiple.
Question 4. A particle has orbit given by

\[ \mathbf{r}(t) = (\ln(t), t, t^2/2) \] for \( t \geq 0. \)

Find the equation for the osculating plane to this particle at \( t = 1 \)

we need to find a normal vector to the plane spanned by \( \mathbf{T}, \mathbf{N} \).
We can find this via \( \mathbf{r}' \times \mathbf{r}'' \) since \( \mathbf{r}'' \) is in the osculating plane.

\[ \mathbf{r}'(t) = \left( \frac{1}{t}, 1, t \right) \]
\[ \mathbf{r}''(t) = \left( -\frac{1}{t^2}, 0, 1 \right) \]
\[ \mathbf{r}'(1) \times \mathbf{r}''(1) = \left( 1, 1, 1 \right) \times \left( -1, 0, 1 \right) = \left( 1, -2, 1 \right) \]

Hence the osculating plane is given by \( (\mathbf{r}'(1) = \left( 1, 1, 1 \right)) \)

\[ 1(x-1) - 2(y-1) + 1(z-1) = 0 \]
\[ x - 2y + z = 0 \]

Question 5. Show that for a vector function \( \mathbf{r}(t) \), both \( \mathbf{r}'(t) \) and \( \mathbf{r}'' \) lie in the osculating plane. Hint: differentiate \( \mathbf{r}'(t) = \mathbf{v}(t) \mathbf{T}(t) \).

\[ \mathbf{r}'(t) = \mathbf{v}(t) \mathbf{T}(t), \text{ hence } \mathbf{r}'(1) \text{ in osculating plane.} \]

\[ \mathbf{r}''(t) = \mathbf{v}'(t) \mathbf{T}(t) + \mathbf{v}(t) \mathbf{T}'(t) \]
\[ = \mathbf{v}'(t) \mathbf{T}(t) + \mathbf{v}(t) \mathbf{K}(t) \mathbf{N}(t) \]

Hence \( \mathbf{r}''(1) \) in osculating plane.
Question 6. Find the domain for the following functions

(a) \( f(x, y) = \frac{1}{\sqrt{x^2 + y^2} - 1} \)

(b) \( f(x, y) = \frac{y \sin(x)}{1 + y} \)

(c) \( f(x, y) = -\frac{1}{\sin(xy)} \)

(a) As long as the denominator is nonzero.

\[ \sqrt{x^2 + y^2} = 1 \] not included.

\( x^2 + y^2 = 1. \)

\[ D = \{ (x, y) \in \mathbb{R}^2 \mid x^2 + y^2 \neq 1 \} \]

(b) Similarly, \( y \neq -1. \)

(c) Similarly, \( \sin(xy) \neq 0 \)

\( xy \neq n\pi, \forall n \in \mathbb{Z}. \)

\[ D = \{ (x, y) \in \mathbb{R}^2 \mid xy \neq n\pi, \forall n \in \mathbb{Z} \}. \]