**Parameterisations:**

- Curves can be parameterised in more than one way. (Actually infinitely many ways)
  For instance, the circle $S^1$ can be parameterised by $r_1(t) = (\cos t, \sin t)$ or $r_2(t) = (\cos^3 t, \sin^3 t)$.
  Both go around the circle but do so at different speeds at different times.

- We say a parameterisation $r(t)$ is an **arc length parameterisation** if $\|r'(t)\|=1$ for all $t$.

- Given a parameterisation $r(t)$, the arclength $\gamma$ is given by the function $\gamma(t) = \int_0^t \|r'(u)\|\,du$.

  We can then construct an arclength parameterisation by $r_1(s) = r(\gamma^{-1}(s))$.

**Example:** $r(t) = (\cos 4t, \sin 4t, 3t)$ we find an arclength param.

\[
\gamma(t) = \int_0^t \|r'(u)\|\,du
\]

\[
= \int_0^t \sqrt{16\sin^2 4u + 16\cos^2 4u + 9}\,du
\]

\[
= \int_0^t 5\,du = 5t \quad \Rightarrow \quad \gamma^{-1}(s) = \frac{s}{5}
\]

Hence arclength param $r_1(s) = r(\gamma^{-1}(s)) = (\cos \frac{4s}{5}, \sin \frac{4s}{5}, \frac{3}{5}s)$.

**Question:**
25. Let \( \mathbf{r}(t) = (3t + 1, 4t - 5, 2t) \).

(a) Evaluate the arc length integral \( s(t) = \int_0^t \| \mathbf{r}'(u) \| \, du \).

(b) Find the inverse \( g(s) \) of \( s(t) \).

(c) Verify that \( \mathbf{r}_1(s) = \mathbf{r}(g(s)) \) is an arc length parametrization.

\[
(a) \quad s(t) = \int_0^t \| (3, 4, 2) \| \, du = \int_0^t \sqrt{9 + 16 + 4} \, du = \sqrt{29} t \quad \Rightarrow \quad t = \frac{s}{\sqrt{29}}.
\]

(b) Hence \( g(s) = \frac{s}{\sqrt{29}} \). 

(c) \( \mathbf{r}_1(s) = \mathbf{r}\left(\frac{s}{\sqrt{29}}\right) = \left< \frac{3}{\sqrt{29}} s + 1, \frac{4}{\sqrt{29}} s - 5, \frac{2}{\sqrt{29}} s \right> \)

and \( \mathbf{r}'_1(s) = \frac{1}{\sqrt{29}} \left< 3, 4, 2 \right> \)

\( \| \mathbf{r}'_1(s) \| = 1 \). Hence arc length param.

**Curvature:**

The curvature \( K(t) \) is a measurement of how much a curve bends at a point. There are a few ways to calculate it, but in practice, the easiest way is given by formula:

\[
K(t) = \frac{\| \mathbf{r}'(t) \times \mathbf{r}''(t) \|}{\| \mathbf{r}'(t) \|^{3/2}} \quad \text{when } \mathbf{r}(t) \text{ is some param.}
\]

\( \text{(or } \mathbf{a} \text{)} \)

**Example:** \( \mathbf{r}(t) = (r \cos t, r \sin t, 0) \) circle of radius \( r \).

Then \( \mathbf{r}'(t) = (-r \sin t, r \cos t, 0) \).
\[ r''(t) = \langle -r \cos \theta, -r \sin \theta, 0 \rangle \]

\[ r'(t) \times r''(t) = \begin{vmatrix} i & j & k \\ -r \sin \theta \cos \theta & r \cos \theta & 0 \\ -r \cos \theta \sin \theta & -r \sin \theta & 0 \end{vmatrix} = \langle 0, 0, r^2 \rangle \]

so \[ K(t) = \frac{\|v(t) \times r''(t)\|}{\|v'(t)\|^3} = \frac{r^2}{r^3} = \frac{1}{r} \]

**Question:** Given a graph \( y = f(x) \). This is a curve in the xy-plane which we can consider as the xy-plane in 3-space. What is the curvature at a graph?

**Answer:** We can parameterize a graph by \( x = t, y = f(t), z = 0 \).

i.e. \( r(t) = \langle t, f(t), 0 \rangle \).

Hence, \( r'(t) = \langle 1, f'(t), 0 \rangle \)

\( r''(t) = \langle 0, f''(t), 0 \rangle \)

\[ r'(t) \times r''(t) = \begin{vmatrix} i & j & k \\ 1 & f'(t) & 0 \\ 0 & f''(t) & 0 \end{vmatrix} = f''(t) \]

Hence, \( K(t) = \frac{\|r'(t) \times r''(t)\|}{\|r'(t)\|^3} = \frac{|f''(t)|}{(1 + f'(t))^3/2} \)

**Question:** Find the curvature of the Cornu Spiral

\( x(t) = \int_0^t \sin \frac{u^2}{2} \, du \)
\( y(t) = \int_0^t \cos \left( \frac{u^2}{2} \right) \, du \)

**Answer:** \( r(t) = \langle \int_0^t \sin \frac{u^2}{2} \, du, \int_0^t \cos \frac{u^2}{2} \, du, 0 \rangle \)
\[ r'(t) = \left< \sin \frac{t^2}{2}, \cos \frac{t^2}{2}, 0 \right> \]
\[ r''(t) = \left< \frac{t^2}{2}, -t \sin \frac{t^2}{2}, 0 \right> \]

\[ r'(t) \times r''(t) = \begin{vmatrix} i & j & k \\ \sin \frac{t^2}{2} & \cos \frac{t^2}{2} & 0 \\ t \cos \frac{t^2}{2} & -t \sin \frac{t^2}{2} & 0 \end{vmatrix} = -t \]

Hence, \[ k(t) = \frac{|r'(t) \times r''(t)|}{|r'(t)|^3} = |t| \]

Given a parametric curve \( r(t) \), we have:

- unit tangent vector \( T(t) = \frac{r'(t)}{|r'(t)|} \)
- unit normal vector \( N(t) = \frac{T'(t)}{|T'(t)|} \)
- the binormal vector \( B(t) = T(t) \times N(t) \)

These form a right-handed system \( \{T, N, B\} \).

**Question:** Find \( T, N, B \) for \( r(t) = \langle t^2, et \rangle \) at \( (0,0,0) \)

**Answer:** \[ r'(t) = \langle 1, 1, et \rangle \]
\[ |r'(t)| = \sqrt{2 + e^{2t}} \]
Answer: \( r'(t) = \langle 1, 1, e^t \rangle \). \( \|r'(t)\| = \sqrt{2 + e^{2t}} \)

Hence, \( T(t) = \frac{1}{\sqrt{2 + e^{2t}}} \langle 1, 1, e^t \rangle \).

\( T'(t) = \frac{1}{\sqrt{2 + e^{2t}}} \langle 1, 1, e^t \rangle + \frac{1}{\sqrt{2 + e^{2t}}} \langle 0, 0, e^t \rangle \)

\( T'(0) = \frac{1}{3 \sqrt{3}} \langle 1, 1, 1 \rangle + \frac{1}{\sqrt{3}} \langle 0, 0, 1 \rangle \)

\( = \frac{1}{3 \sqrt{3}} \left( -\langle 1, 1, 1 \rangle + \langle 0, 0, 3 \rangle \right) \)

\( = \frac{1}{3 \sqrt{3}} \langle -1, -1, 2 \rangle \).

\( \|T'(0)\| = \frac{\sqrt{9 + 2}}{3 \sqrt{3}} = \frac{\sqrt{11}}{3} \)

Hence, \( N(0) = \frac{T'(0)}{\|T'(0)\|} = \frac{1}{\sqrt{6}} \langle -1, -1, 2 \rangle \).

\( \text{and} \quad B(0) = T(0) \times N(0) = \left( \frac{1}{\sqrt{3}} \langle 1, 1, 1 \rangle \right) \times \left( \frac{1}{\sqrt{6}} \langle -1, -1, 2 \rangle \right) \)

\( = \frac{1}{3 \sqrt{2}} \langle 1, 1, 1 \rangle \times \langle -1, -1, 2 \rangle. \)

\( \langle 1, 1, 1 \rangle \times \langle -1, -1, 2 \rangle = \left| \begin{array}{ccc}
1 & 1 & k \\
1 & 1 & 1 \\
1 & 1 & 2
\end{array} \right| = \langle 3, -3, 0 \rangle. \)

Hence, \( B(0) = \frac{1}{3 \sqrt{2}} \langle 3, -3, 0 \rangle \).

Hence, \( \{\bar{T}, N, B\} \text{ at } t = 0 \) is \( \left\{ \frac{1}{\sqrt{3}} \langle 1, 1, 1 \rangle, \frac{1}{\sqrt{6}} \langle -1, -1, 2 \rangle, \frac{1}{\sqrt{2}} \langle 3, -3, 0 \rangle \right\} \).
Question:

56. (a) What does it mean for a space curve to have a constant unit tangent vector $T$?
(b) What does it mean for a space curve to have a constant normal vector $N$?
(c) What does it mean for a space curve to have a constant binormal vector $B$?

(a) If $T$ is constant, it must be moving in a line.

(b) Since $||T|| = 1 \Rightarrow T'$ is perpendicular to $T$. Moreover, $N$ is always pointing towards the centre of the oscillating circle so it follows that $T$ must also be parallel, since if this changes, it must do so in the plane perpendicular to $N$ but then the oscillating circle would be in this plane.

(c) The curve must stay in the plane perpendicular to $B$ and the bending of the curve must be on the same side as the director always.