Derivation of vector valued functions

Given \( \mathbf{r}(t) = \langle x(t), y(t), z(t) \rangle \), then \( \mathbf{r}'(t) = \langle x'(t), y'(t), z'(t) \rangle \).
Geometrically, we think of \( \mathbf{r}'(t) \) as the tangent vector to the curve \( \mathbf{r}(t) \) at \( t \).

The derivative behaves similarly to the one-dimensional case. Given scalar function \( f: \mathbb{R} \rightarrow \mathbb{R} \) we have:

- **Product rule:** \( (f(t)\mathbf{r}(t))' = f'(t)\mathbf{r}(t) + f(t)\mathbf{r}'(t) \)
- **Chain rule:** \( (\mathbf{r}(f(t)))' = \mathbf{r}'(f(t))f'(t) \)

The dot and cross product also obey the product rule:

\[
\begin{align*}
(\mathbf{r}_1(t) \cdot \mathbf{r}_2(t))' &= \mathbf{r}'_1(t) \cdot \mathbf{r}_2(t) + \mathbf{r}_1(t) \cdot \mathbf{r}'_2(t) \\
(\mathbf{r}_1(t) \times \mathbf{r}_2(t))' &= \mathbf{r}'_1(t) \times \mathbf{r}_2(t) + \mathbf{r}_1(t) \times \mathbf{r}'_2(t)
\end{align*}
\]

**Group Questions:**

1) Given \( \mathbf{r}_1(t) = \langle t^2, 1, 2t \rangle \) and \( \mathbf{r}_2(t) = \langle 1, 2, e^t \rangle \), calculate \( (\mathbf{r}_1(t) \cdot \mathbf{r}_2(t))' \) in two ways:
   a) Take dot product first and then differentiate
   b) Differentiate using product rule

2) If \( \|\mathbf{r}(t)\| \) is constant, show that \( \mathbf{r}(t) \) and \( \mathbf{r}'(t) \) are orthogonal.
   That: \( \|\mathbf{r}(t)\|^2 = \mathbf{r}(t) \cdot \mathbf{r}(t) \)

3) Show that \( (\mathbf{a} \times \mathbf{r}(t))' = \mathbf{a} \times \mathbf{r}'(t) \) for any constant vector \( \mathbf{a} \).
Answers:

1) a) \( r_1(t) \cdot r_2(t) = t^2 + 2 + 2te^t \)
   
   So \( (r_1(t) \cdot r_2(t))' = 2t + 2e^t + 2te^t \)

   b) \( (v_1(t) \cdot r_2(t))' = v_1'(t) \cdot r_2(t) + r_1(t) \cdot r_2'(t) \)
      
      = \langle 2t, 0, 2 \rangle \cdot \langle 1, 2, e^t \rangle + \langle f', 1, 2t \rangle \cdot \langle 0, 0, e^t \rangle \)
      
      = 2t + 2e^t + 2te^t.

2) \( K = ||r(t)||^4 \) some constant.

   Hence, \( K = r(t) \cdot r(t) \) and differentiating:

   \[ 0 = r'(t) \cdot r(t) + r(t) \cdot r'(t) \]
   
   \[ \Rightarrow r'(t) \cdot r(t) = 0 \]
   
   \[ \Rightarrow r'(t) \text{ and } r(t) \text{ orthogonal.} \]

3) Note, we can think of \( a \) as a constant vector function

   \( g(t) = a \), i.e. \( g'(t) = 0 \). Hence by product rule,

   \[ (a \times r(t))' = (a \times r(t))' \]
   
   \[ = g'(t) \times r(t) + g(t) \times r'(t) \]
   
   \[ = a \times r'(t) \text{ since } g'(t) = 0. \]
Midterm 2 (Practice Test)
Calculus of Several Variables
(Math 32A)

Name: _______________________________  U ID: _______________________

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1. 5 points Find the equation of the plane which contains the point \((-1, 0, 2)\) and is parallel to the plane \(2x - y - z = 3\).

Parallel to plane means same normal vector. 
\[ \hat{n} = \langle 2, -1, -1 \rangle. \]

Hence the plane is of the form \(2x - y - z = d\) for some \(d\). Since it contains \((-1, 0, 2)\) we get

\[ 2(-1) - 0 - 2 = d \]
\[ \therefore d = -4. \]

Therefore, the plane is: \(2x - y - z = -4\).
2. [5 points] Let \( \vec{u} \) and \( \vec{v} \) be two unit vectors such that \( \|\vec{u} + \vec{v}\| = \frac{3}{2} \). Then compute \( \|\vec{u} - \vec{v}\| \).

Use that \( a \)

\[
\|\vec{u} + \vec{v}\|^2 = \|\vec{u}\|^2 + 2\vec{u} \cdot \vec{v} + \|\vec{v}\|^2
\]

\[
\|\vec{u} - \vec{v}\|^2 = \|\vec{u}\|^2 - 2\vec{u} \cdot \vec{v} + \|\vec{v}\|^2
\]

Hence, \( \|\vec{u} + \vec{v}\|^2 + \|\vec{u} - \vec{v}\|^2 = 2\|\vec{u}\|^2 + 2\|\vec{v}\|^2 = 2 \cdot 1. \)

Now, we are given \( \|\vec{u} + \vec{v}\|^2 = \frac{9}{4} \) and \( \|\vec{u}\|^2 = \|\vec{v}\|^2 = 1 \).

Therefore,

\[
\frac{9}{4} + \|\vec{u} - \vec{v}\|^2 = 4
\]

\[
\|\vec{u} - \vec{v}\|^2 = \frac{7}{4}
\]

\[
\therefore \|\vec{u} - \vec{v}\| = \frac{\sqrt{7}}{2}
\]
3. 5 points Consider the following parametric equation:

\[ x = a \cos \theta + a \sin \theta, \quad y = -b \sin \theta + b \cos \theta, \]  

where \( \theta \) is a parameter.

Find a relation between \( x \) and \( y \) by eliminate the parameter \( \theta \).

We have that

\[
\left( \frac{x}{a} \right)^2 = \cos^2 \theta + 2 \cos \theta \sin \theta + \sin^2 \theta = 1 + 2 \cos \theta \sin \theta
\]

\[
\left( \frac{y}{b} \right)^2 = \sin^2 \theta - 2 \cos \theta \sin \theta + \cos^2 \theta = 1 - 2 \cos \theta \sin \theta
\]

Hence

\[
\frac{x^2}{a^2} + \frac{y^2}{b^2} = 2
\]
4. 5 points Find the solution of the differential equation with respect to the given initial conditions:

\[ \vec{r}''(t) = \langle e^t, \sin t, \cos t \rangle, \quad \vec{r}'(0) = \langle 1, 0, 1 \rangle \text{ and } \vec{r}'(0) = \langle 0, 2, 2 \rangle. \]

\[ \vec{r}'(t) = \int \vec{r}''(t) \, dt = \langle e^t, -\cos t, \sin t \rangle + \vec{c} \]

when \( t=0 \), \( \vec{r}'(0) = \langle 0, 2, 2 \rangle = \langle 1, -1, 0 \rangle + \vec{c} \)
\[ \therefore \vec{c} = \langle -1, 3, 2 \rangle. \]

so \( \vec{r}'(t) = \langle e^t - 1, 3 - \cos t, 2 + \sin t \rangle \)

Repeating,
\[ \vec{r}(t) = \int \vec{r}'(t) \, dt = \langle e^t - t, 3t - \sin t, 2t - \cos t \rangle + \vec{c} \]
\[ \vec{r}(0) = \langle 1, 0, 1 \rangle = \langle 1, 0, -1 \rangle + \vec{c} \]

Hence \( \vec{c} = \langle 0, 0, 2 \rangle \)
\[ \therefore \vec{r}(t) = \langle e^t - t, 3t - \sin t, 2 + 2t - \cos t \rangle \]
5. **5 points** If $\vec{r}(t)$ is a vector of constant length for all $t$, then prove that $\vec{r}'(t)$ is orthogonal to $\vec{r}'(t)$.

*As above.*