Vector-Valued Functions.

A function of the form \( \mathbf{r}(t) = \langle x(t), y(t), z(t) \rangle = x(t)\mathbf{i} + y(t)\mathbf{j} + z(t)\mathbf{k} \)

Think of \( t \) as time, \( \mathbf{r}(t) \) as a moving position vector that traces out some path in \( \mathbb{R}^3 \), i.e., vector-valued functions are really parameterisations of some space curve.

Example: \( \mathbf{r}(t) = \langle \cos t, \sin t, t \rangle \) - \( \infty < t < \infty \).

The first two components are just the parameterisation of the circle and \( z \) is fixed at 1. Hence the curve traced by \( \mathbf{r}(t) \) is a circle at height 1. The path followed by \( \mathbf{r}(t) \) is that of \( t \to \infty \) if start at the point \((1,0,1)\) and \( t \to -\infty \) if you wind counterclockwise.

Group Questions:

Given two paths \( \mathbf{r}_1(t), \mathbf{r}_2(t) \). We say they intersect if there is a point \( P \) lying on both curves. We say that \( \mathbf{r}_1(t), \mathbf{r}_2(t) \) collide if \( \mathbf{r}_1(t_0) = \mathbf{r}_2(t_0) \) for some \( t_0 \).

Is it true that:
1) if \( \mathbf{r}_1(t), \mathbf{r}_2(t) \) intersect, then they collide?
2) if \( \mathbf{r}_1(t), \mathbf{r}_2(t) \) collide, then they intersect?
3) intersection depends only on the curves traced by \( \mathbf{r}_1, \mathbf{r}_2 \). While collision depends only on underlying parameterisations?

Given \( \mathbf{r}_1(t) = \langle t^2 + 3, t + 1, 6t^{-1} \rangle \)

\( \mathbf{r}_2(t) = \langle 4t, 2t^2 - 2, t^2 - 7 \rangle \)

Determine if these collide or intersect?

Answer:
1) false 2) true 3) true.

They collide if \( r_1(t) = r_2(t) \) has a solution.

Components $\Rightarrow$
\[
\begin{align*}
t^2 + 3 &= 9t \quad \cdot (1) \\
t + 1 &= 2t - 2 \quad \cdot (2) \\
6t^2 - 1 &= t^2 - 2 \quad \cdot (3)
\end{align*}
\]

So (2): $t + 1 = 2t - 2 \Rightarrow t = 3$, plug into (1): $12 = 12$.

Plug into (3): $2 = 2$.

Hence $t = 3$ is a solution and the two curves collide. Since they collide, they also intersect.

Parameterising intersection of surfaces.

Example (From textbook)

How do we parameterise the intersection of the two surfaces:
\[
x^2 - y^2 = z - 1 \quad \text{and} \quad x^2 + y^2 = 4.
\]

First way: try and write two of the variables in terms of the last one and make this the parameterising variable.

In this example, we write \( y \) and \( z \) in terms of just \( x \).

\[
x^2 + y^2 = 4 \Rightarrow y = \pm \sqrt{4 - x^2}
\]

So \( x^2 - y^2 = z - 1 \Rightarrow z = 1 + x^2 - y^2 \)
\[
= 1 + x^2 - (4 - x^2) \\
= 2x^2 - 3.
\]

Hence, let \( x = t \). We then parameterise the intersection by the two paths:
\[
r_1(t) = (t, \sqrt{4 - t^2}, 2t^2 - 3), \quad r_2(t) = (t, -\sqrt{4 - t^2}, 2t^2 - 3).
\]

Second method: use a known parameterisation.
Second method: use a known parameterisation

The equation \( x^2 + y^2 = 4 \) can be parameterised by \( x = 2 \cos t, \ y = 2 \sin t \).

Hence, subbing these into the second equation gives
\[
2 = 4 \sin^2 t - 4 \cos^2 t.
\]

Hence we can parameterise the intersection as
\[
r(t) = (2 \cos t, 2 \sin t, 1 + 4 \sin^2 t - 4 \cos^2 t) \]

Questions

Parameterise the intersection of the two cylinders: \( x^2 + y^2 = 1 \)
and \( x^2 + z^2 = 1 \).

Answer:
\( x^2 + y^2 = 1 \) can be parameterised by \( x = \cos t, \ y = \sin t \) and then by
the second equation \( z^2 = 1 - x^2 \Rightarrow z = \sqrt{1 - x^2} \). Hence we get
the two paths
\[
z = \pm \sqrt{1 - \cos^2 t} = \pm \sqrt{\sin^2 (t)} = \pm \sin (t).
\]
Hence we get the two paths:
\[
r_1(t) = (\cos t, \sin t, \pm \sin (t)) \quad r_2(t) = (\cos t, \sin (t), - \pm \sin (t)) \]

Question:
The intersection of the surfaces:
\[
z = x^2 - y^2 \quad z = x^2 + xy - 1.
\]

Answer: we have on the intersection \( x^2 - y^2 = x^2 + xy - 1 \)
\[
1 = y^2 + xy \quad \Rightarrow \quad \frac{1}{y} = x + \frac{1}{y} \Rightarrow x = \frac{1}{y} - y.
\]
Then \( z = (\frac{1}{y} - y)^2 - y \)
\[
= \frac{1}{y^2} - 2 + y^2 - y^2 = \frac{1}{y^2} - 2 \quad \Rightarrow \quad z = \frac{1}{y^2} - 2.
\]
Hence let $y = \xi$ and then the parameterisation is

$$r(t) = \left( \frac{1}{\xi} - \xi t, \xi t, \frac{1}{\xi^2} - 2 \right)$$