Learning outcomes

(1) Be able to calculate tangent planes

(2) Understand linear approximation as replacing functions with a tangent plane to get estimates

(3) Be able to use linear approximation to get estimates for simple functions.

(4) Understand differentiable at a point as meaning error of function and linear approximation goes to zero faster than a linear function.

(5) Understand tangent plane as the plane of all tangent vectors of curves that go through a specific point.

Tangent planes

If $f(x, y)$ is differentiable at $(9,6)$, then the tangent plane at $(9,6)$ is given by:
\[ z = f(a,b) + f_x(a,b)(x-a) + f_y(a,b)(y-b) \]

Remember:
- \( f_x(a,b) \) is slope of graph in positive \( x \)-direction
- \( f_y(a,b) \) is slope of graph in \( y \)-direction

Problem 1. Find tangent plane at \( g(x,y) = e^{xy} \) at the point \((2,1)\).

Problem 2.
(a) When is the tangent plane horizontal?
(b) What is a normal vector for the tangent plane?
(c) Can the tangent plane be vertical?

Linear approximation

The basic idea of linear approximation is to replace a function with its tangent plane (called the "linearization") and use this to estimate function values.

Linear functions are super nice since changing one of the inputs induces a proportional change.
In the outputs,

**Example** \( f(x, y) = 6 + 3x + 4y \).

Then at \((1, 1)\) we have: \( f(1, 1) = 13 \).

- For every unit change in \( x \), this causes a 3 unit change in output.
  - \( f(1, 1) = 13 \)
  - \( f(2, 1) = 16 \)
  - \( f(3, 1) = 19 \)
  - \( f(0, 1) = 10 \)
  - etc...

- For every unit change in \( y \), this causes a 4 unit change in output.
  - \( f(1, 1) = 13 \)
  - \( f(1, 2) = 17 \)
  - \( f(1, 3) = 21 \)

- Moreover, this change is additive, i.e., I can do the change in each variable and add the changes. i.e., increase \( x \) and \( y \) by 1 then output increases by \( 3 + 4 = 7 \).
Note, change in a variable is written as $\Delta x$ or $\Delta y$. So the relationship can be written as:

$$\Delta f = f_x(a,b) \Delta x + f_y(a,b) \Delta y$$

- $\Delta x$: change in output
- $\Delta y$: change in $y$

**Problem 3**

Suppose $f$ is a linear function with $f(1,2) = 9$ and $f_x(1,2) = 3$, $f_y(1,2) = -2$. Find:

(a) $f(2,2)$
(b) $f(2,4)$
(c) $f(4,1) - f(2,1)$

The linearization of a function at a point is just the tangent line.

i.e., linearization at $(a,b)$ is:

$$L(x,y) = f(a,b) + f_x(a,b)(x-a) + f_y(a,b)(y-b).$$

**Example:** Look at $e^{-xy}$ in desmos. Tangent plane at $(2,2)$. Error for different pts.
Problem 9: Use linear approximation of \( f(x,y) = \sqrt{\frac{x}{y}} \) at \((9,4)\) to estimate \( \sqrt{9.1/3.9} \).

That is

- Find linearization (tangent plane) at \((9,4)\)
- Treat \( f \) as if it was this linearization to estimate \( \sqrt{9.1/3.9} \).

Differentiability

- Linearization can also be calculated if the function isn’t differentiable (sometimes). In fact, differentiability is defined by linearization.

A function \( f(x,y) \) is differentiable at a point \((a,b)\) if

\[
\lim_{(x,y) \to (a,b)} \frac{f(x,y) - L(x,y)}{\|\langle x-a, y-b \rangle\|} = 0
\]

where \( L(x,y) \) is the linearization of \( f \).

Let’s break this down:

- \( f(x,y) - L(x,y) \) is the error of the linear
approximation at \((x,y)\)

If \(f(x,y)\) is continuous then \(\lim_{(x,y)\to(a,b)} f(x,y) - l(x,y) = 0\)

- \(\|\langle x-a, y-b \rangle\|\) is the distance of \((x,y)\) to \((a,b)\), we have \(\lim_{(x,y)\to(a,b)} \|\langle x-a, y-b \rangle\| = 0\).

- We can think of \(\lim_{(x,y)\to(a,b)} \frac{f(x,y) - l(x,y)}{\|\langle x-a, y-b \rangle\|} = 0\)

as saying the linear error \(f(x,y) - l(x,y)\) goes to zero faster than the distance from \((a,b)\) to \((x,y)\) goes to zero.

It \(l(x,y)\) is a "very good" approximation.

**Problem 5** True or false

1) If the partials \(\frac{\partial f}{\partial x}, \frac{\partial f}{\partial y}\) exist, then the function is differentiable at a point.

2) If a function is differentiable at a point,
Then it is continuous at that point.

Alternate description of tangent plane

Given any curve parametrized \( \mathbf{r}(t) = \langle g_1(t), g_2(t) \rangle \) such that \( \mathbf{r}(0) = \langle g_1(0), g_2(0) \rangle = \langle a, b \rangle \). Then if \( f(x,y) \) is differentiable at \((a,b)\), then consider the curve in 3-space given by

\[
\mathbf{r}(t) = \langle g_1(t), g_2(t), f(g_1(t), g_2(t)) \rangle
\]

we then have a tangent \( \mathbf{T}(0) \) and this is contained inside the tangent plane at \((a,b)\). Indeed, for any vector contained inside the tangent plane, we can find a curve \( \mathbf{q}(t) \) such that the tangent \( \mathbf{T}(\theta) \) to the corresponding \( \mathbf{r}(t) \) is that vector.
• You can use this to show something isn't differentiable at a point. i.e.
  • have a candidate tangent plane from the partial derivatives
  • show that for some curve \( \gamma(t) = (\gamma_1(t), \gamma_2(t)) \) that the tangent vector \( \vec{F}(t) \) at the curve \( \vec{F}(t) = (\gamma_1(t), \gamma_2(t), f(\gamma_1(t), \gamma_2(t))) \)
  isn't in the tangent plane.