Limits and continuity

Informally:

A limit of a function at some input $x = a$ is some value $L$ (the limit) such that no matter what you input into the function, if the inputs get closer and closer to $a$, (but not equal) then the output get closer and closer to $L$.

- This idea is the same for one variable functions and multivariable functions.

- The difficulty comes from the fact in 1-variable, you can approach an input either from the left and right only. In multiple variables, there are infinitely many ways to approach something.

\[\begin{array}{cc}
\text{one-variable.} & \text{2-variables.}
\end{array}\]
You usually prove multivariable limits don't exist by showing that the limit along two different paths to the value are different.

**Example**  \[
\lim_{(x,y) \to (0,0)} \frac{x^2 - y^2}{x^2 + y^2}
\]
doesn't exist.

Why? Along the x-axis, y = 0. Hence,

\[
\lim_{(x,0) \to (0,0)} \frac{x^2 - 0}{x^2 + 0} = \lim_{(x,0) \to (0,0)} 1 = 1
\]

Along y-axis, x = 0: \[
\lim_{(0,y) \to (0,0)} \frac{0 - y^2}{0 + y^2} = -1.
\]

These are different, so the limit doesn't exist.

**Problem 1:** Does \[
\lim_{(x,y) \to (0,0)} \frac{x^2y}{x^4 + 2y^2}
\]
exist?

**Hint:** A line isn't good enough.

Showing a limit exists is either easy or very hard.
Simple case: the function is continuous. Then you can substitute the value into the function.

(Continuity itself means that the limit at a point is equal to the value of the function at that point. That is \( \lim_{(x,y) \to (a,b)} f(x,y) = f(a,b) \).)

**Warning**: To show a limit exists, it is not enough to show the limit along all lines or some paths all converge to the same thing.

To show a limit exists without continuity usually means using the squeeze theorem in some way.

Setup for squeeze: \( \lim_{(x,y) \to (a,b)} f(x,y)g(x,y) \)

where one function, say \( g(x,y) \) is wild but bounded, and \( f(x,y) \) is nice with known limit. Then we take:

\[
m \leq g(x,y) \leq M
\]

\[
mf(x,y) \leq f(x,y)g(x,y) \leq Mf(x,y) \quad (\text{if } f \geq 0)
\]
take limit on both sides, if both are equal then the middle limit is also this.

**Example** \( \lim_{(x,y) \to (0,0)} (x^2+y^2) \cos \left(\frac{1}{x^2+y^2}\right) \)

- wild but bounded function: \( \cos \left(\frac{1}{x^2+y^2}\right) \)
- nice function: \( x^2+y^2 \)

\[
-1 \leq \cos \left(\frac{1}{x^2+y^2}\right) \leq 1
\]

\[
-(x^2+y^2) \leq (x^2+y^2) \cos \left(\frac{1}{x^2+y^2}\right) \leq x^2+y^2
\]

The left and right functions are continuous, so

\[
\lim_{(x,y) \to (0,0)} -(x^2+y^2) = 0 = \lim_{(x,y) \to (0,0)} (x^2+y^2).
\]

Hence \( \lim_{(x,y) \to (0,0)} (x^2+y^2) \cos \left(\frac{1}{x^2+y^2}\right) = 0. \)

**Problem 2** Prove \( \lim_{(x,y) \to (0,0)} \tan^2x \sin \left(\frac{1}{|x|+|y|}\right) = 0. \)
Partial derivatives.

Informally: The partial derivatives of a function \( f(x,y) \) at a point \( P=(a,b) \) can be viewed as:

- \( \frac{\partial f}{\partial x} \bigg|_{P} = \text{the slope of } f \text{ in the direction of positive } x \)-axis.

- \( \frac{\partial f}{\partial y} \bigg|_{P} = \text{--- positive } y \)-axis.

Example:

Demonstration with Desmos app with \( f(x,y) = \cos(x+y) \).

- Calculating partial derivatives is as simple as considering the other variables as constant.

Problem 3: Find the partial derivatives of \( f(x,y) = \frac{x}{\sqrt{x^2+y^2}} \).
"We can differentiate again to get higher order partials.

Example: Consider \( f(x,y) = 6x^2 + 2xy + 3y^2 \)

\[ \frac{\partial f}{\partial x} = 12x + 2y, \quad \text{and} \quad \frac{\partial f}{\partial y} = 2x + 6y \]

\[ \frac{\partial}{\partial x} \left( \frac{\partial f}{\partial x} \right) = 12 \quad \text{i.e.} \quad \frac{\partial^2 f}{\partial x^2} = 12 \]

\[ \frac{\partial}{\partial y} \left( \frac{\partial f}{\partial x} \right) = 2 \quad \text{and} \quad \frac{\partial^2 f}{\partial y \partial x} = 2 \quad \text{etc...} \]

Clairaut's Theorem: under nice conditions, \( f_{xy} = f_{yx} \), i.e. it doesn't matter what order you differentiate.

Extra things to talk about:
* absolute value in squeeze
* last problem in HW.