Question 1. Use L’Hopital’s rule to evaluate the following limits or state that it does not apply:

(a) \( \lim_{x \to 9} \frac{x^{1/2} - x + 6}{x^{3/2} - 27} \)

(b) \( \lim_{x \to 0} \frac{\cos(2x) - 1}{\sin(5x)} \)

(c) \( \lim_{x \to 0} \frac{\sin(4x)}{x^2 + 3x + 1} \)

(d) \( \lim_{x \to 1} \frac{e^x - e}{2x - 2} \)

Solution to Question 1.

(a) We have \( \lim_{x \to 9} \frac{x^{1/2} - x + 6}{x^{3/2} - 27} = 0 \) and so L-Hopital’s applies. Differentiating we get that

\[
\lim_{x \to 9} \frac{1}{2} x^{-1/2} - 1 = -\frac{5}{27}.
\]

Hence \( \lim_{x \to 9} \frac{x^{1/2} - x + 6}{x^{3/2} - 27} = -\frac{5}{27} \).

(b) We have \( \lim_{x \to 0} \frac{\cos(2x) - 1}{\sin(5x)} = 0 \) and so L-Hopital’s applies. Differentiating we get that

\[
\lim_{x \to 0} \frac{-2 \sin(2x)}{5 \cos(5x)} = 0.
\]

Hence \( \lim_{x \to 0} \frac{\cos(2x) - 1}{\sin(5x)} = 0 \).

(c) We have \( \lim_{x \to 0} \frac{\sin(4x)}{x^2 + 3x + 1} = 0 \) and so L-Hopital’s does not apply.

(d) We have \( \lim_{x \to 1} \frac{e^x - e}{2x - 2} = 0 \) and so L-Hopital’s applies. Differentiating we get that

\[
\lim_{x \to 1} \frac{e^x}{2} = \frac{e}{2}.
\]

Hence \( \lim_{x \to 1} \frac{e^x - e}{2x - 2} = \frac{e}{2} \).

Question 2. Compute without calculator:

(a) \( \arcsin(\sin \frac{\pi}{4}) \)

(b) \( \arcsin(\sin \frac{4\pi}{3}) \)

(c) \( \arctan(\tan \frac{3\pi}{4}) \)

(d) \( \cos(\arctan(x)) \)
Solution to Question 2.

(a) Since the range of arcsin is \([\pi/2, \pi/2]\), we have that \(\arcsin(\sin \frac{\pi}{3}) = \frac{\pi}{3}\).

(b) Since \(\frac{4\pi}{3}\) is not in the range of arcsin, we cannot use the previous solution. We want to find a value \(y \in [\pi/2, \pi/2]\) such that \(\sin(\frac{4\pi}{3}) = \sin(y)\), since this will then give us

\[
\arcsin(\sin \frac{4\pi}{3}) = \arcsin(\sin y) = y.
\]

In order to do this, we will use the identity \(\sin(x) = \sin(\pi - x)\) (note, the general version of this identity is \(\sin(x) = \sin((-1)^n(x - n\pi))\), \(n \in \mathbb{Z}\) which can be used in the general case).

Now, we have that \(\sin(\frac{4\pi}{3}) = \sin(\pi - \frac{4\pi}{3}) = \sin(-\frac{\pi}{3})\) and as \(-\frac{\pi}{3}\) is in the range of arcsin we have that

\[
\arcsin(\sin \frac{4\pi}{3}) = \arcsin(\sin \left(-\frac{\pi}{3}\right)) = -\frac{\pi}{3}.
\]

(c) Similarly to the previous question. \(\frac{3\pi}{4}\) is not in the range of arctan. However, we do have that \(\tan(x)\) is \(\pi\) periodic \((\tan(x) = \tan(x + n\pi)\) for all \(n \in \mathbb{Z}\)) and so

\[
\arctan(\tan \frac{3\pi}{4}) = \arctan(\tan(\frac{3\pi}{4} - \pi)) = \arctan(\tan \left(-\frac{\pi}{4}\right)) = -\frac{\pi}{4}.
\]

(d) We use a triangle method. We have the triangle

\[
\begin{array}{c}
\sqrt{x^2 + 1} \\
\text{arctan}(x) \\
1
\end{array}
\]

Hence, we have that \(\cos(\arctan(x)) = \frac{1}{\sqrt{x^2 + 1}}\).

Question 3. Show that \(e = \lim_{x \to 0} (1 + x)^{1/x}\).

Solution to Question 3.
\(\lim_{x \to 0} (1 + x)^{1/x} = 1^\infty\) which is an indeterminate form. We consider the logarithm,

\[
\ln((1 + x)^{1/x}) = \frac{\ln(1 + x)}{x} \to \frac{0}{0} \quad \text{as} \quad x \to 0.
\]

Hence we try L'Hopitals and get that

\[
\frac{(\ln(1 + x))'}{(x)'} = \frac{(1 + x)^{-1}}{1} \to 1 \quad \text{as} \quad x \to 0.
\]

Therefore, \(\lim_{x \to 0} (1 + x)^{1/x} = 1\) and as exponential is continuous we have that

\[
\lim_{x \to 0} (1 + x)^{1/x} = \lim_{x \to 0} \exp(\ln((1 + x)^{1/x})) = \exp\left(\lim_{x \to 0} \ln((1 + x)^{1/x})\right) = \exp(1) = e.
\]
**Question 4.** Find the derivatives of the following functions:

(a) \( \arcsin(e^x) \)  
(b) \( \arccos(\ln(x)) \)  
(c) \( \sec^{-1}(t+1) \)  
(d) \( \tan^{-1}\left(\frac{1+t}{1-t}\right) \).

**Solution to Question 4.**

(a) \( e^x \sqrt{1-e^{2x}} \)  
(b) \( \frac{-1}{x\sqrt{1-\ln^2(x)}} \)  
(c) \( \frac{1}{|t+1|\sqrt{(t+1)^2-1}} \)  
(d) \( \frac{1}{1+t^2} \).

**Question 5.** Evaluate the following integrals:

(a) \( \int \frac{dt}{\sqrt{1-16t^2}} \)  
(b) \( \int \frac{dx}{x\sqrt{x^4-1}} \)  
(c) \( \int \frac{\ln(\cos^{-1}(x))dx}{(\cos^{-1}(x))\sqrt{1-x^2}} \).

**Solution to Question 5.**

(a) \( \frac{1}{4} \arcsin(4t) + C \). Use substitution \( u = 4t \).  
(b) \( \frac{1}{2} \sec^{-1}(x^2) + C \). Use substitution \( u = x^2 \).

(c) \( -(\ln(\arccos(x)))^2 + C \). Use substitution \( u = \arccos(x) \).

**Homework Questions**

Section 7.7  
6, 10, 12, 16, 18, 26, 38, 48, 50, 53, 54, 60, 62.  
Section 7.8  
30, 32, 34, 38, 48, 54, 56, 58, 62, 72, 112.

**Extra Questions**

**Question 6.** Show that \( 0^\infty \) is not an indeterminate form by showing that for any positive functions \( f \) and \( g \) such that \( \lim_{x \to 0} f(x) = 0 \) and \( \lim_{x \to 0} g(x) = \infty \), then

\[
\lim_{x \to 0} f(x)^{g(x)} = 0.
\]

In contrast, show that \( 1^\infty \) is an indeterminate form by finding an example of positive functions \( f, g \) such that \( \lim_{x \to 0} f(x) = 1, \lim_{x \to 0} g(x) = \infty \) and \( \lim_{x \to 0} f(x)^{g(x)} = 1 \). And then find another pair of functions \( f, g \) with corresponding limits as \( x \to \infty \) but \( \lim_{x \to 0} f(x)^{g(x)} \neq 1 \).

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**Solution to Question 6.**
Consider the logarithm \( \ln(f(x)^{g(x)}) = g(x)(x) \) which always exists since \( f \) is positive. By the limit laws we have
\[
\lim_{x \to 0} g(x) \ln f(x) = \lim_{x \to 0} g(x) \cdot \lim_{x \to 0} \ln f(x) = \infty \times -\infty = -\infty.
\]
Hence
\[
\lim_{x \to 0} f(x)^{g(x)} = \lim_{x \to 0} \exp(\ln f(x)^{g(x)}) = \exp(\lim_{x \to 0} f(x)^{g(x)}) = 0.
\]
Now consider the functions \( f(x) = 1 \) and \( g(x) = 1/x \), then \( \lim_{x \to 0} f(x) = 1 \), \( \lim_{x \to 0} g(x) = \infty \) and we have that
\[
\lim_{x \to 0} f(x)^{g(x)} = 1.
\]
If instead we take \( f(x) = (1 + x) \), then we still have that \( \lim_{x \to 0} f(x) = 1 \). However, from Question 3 we have that
\[
\lim_{x \to 0} f(x)^{g(x)} = e.
\]

**Question 7.** Evaluate the following integrals:

(a) \( \int 2^x e^{4x} \, dx \)

(b) \( \int \frac{e^x \, dx}{\sqrt{1 - 16e^{2x}}} \)

(c) \( \int \cos(x) 5^{-2 \sin(x)} \, dx \)

(d) \( \int \frac{dx}{x\sqrt{25x^2 - 1}} \).

**Solution to Question 7.**

(a) \( \frac{2^x e^{4x}}{4 + \ln 2} + C \). Note \( 2^x e^{4x} = e^{\ln(2)x+4x} \).

(b) \( \frac{1}{4} \arcsin(4e^x) + C \). Use \( u = 4e^x \).

(c) \( \frac{-5^{-2 \sin(x)}}{2 \ln(5)} + C \). Use \( u = \sin(x) \).

(d) \( \sec^{-1}(5x) + C \). Use \( u = 5x \).