Discussion Questions

**Question 1.** Find the interval of convergence for the following

(a) \[ \sum_{n=2}^{\infty} \frac{x^n}{\ln(n)} \]

(b) \[ \sum_{n=1}^{\infty} n(x - 3)^n \]

**Solution to Question 1.**

(a) Using the ratio test we find that

\[ \left| \frac{a_{n+1}}{a_n} \right| = \left| \frac{x^{n+1}}{\ln(n+1)} \right| \left| \frac{\ln(n)}{x^n} \right| \to |x|. \]

Hence the power series converges absolutely for \(|x| < 1\). Now we check the end points.

When \(x = 1\), we compare with the harmonic series to see that it diverges. When \(x = -1\) we can apply the alternating series test and we see that it converges. Hence the series converges on the interval \([-1, 1)\).

(b) Using the ratio test we see that

\[ \left| \frac{a_{n+1}}{a_n} \right| = \frac{n+1}{n} |x - 3| \to |x - 3| \text{ as } n \to \infty. \]

Hence the series converges absolutely for \(|x - 3| < 1\). We now check the end points. When \(x = \pm 4\), this diverges by the n-th term divergence test. Hence we see that the interval of convergence is \((2, 4)\).

**Question 2.** We have that

\[ \frac{1}{1-x} = \sum_{n=0}^{\infty} x^n \text{ for } |x| < 1. \]

Use this and the equality \( \frac{1}{1-x} = \frac{-1}{1+(x-2)} \) to show that

\[ \frac{1}{1-x} = \sum_{n=0}^{\infty} (-1)^{n+1}(x-2)^n \text{ for } |x-2| < 1. \]

**Solution to Question 2.**

We have that

\[ \frac{1}{1-x} = \frac{-1}{1+(x-2)} \]

\[ = - \sum_{n=0}^{\infty} (-x-2)^n \text{ for } |x-2| < 1 \]

\[ = - \sum_{n=0}^{\infty} (-1)^n(x-2)^n \]

\[ = \sum_{n=0}^{\infty} (-1)^{n+1}(x-2)^n \]
**Question 3.** Find the following Maclaurin series and the interval the expansion is valid by using previously known series.

(a) \( f(x) = \frac{1 - \cos(x)}{x} \)

(b) \( f(x) = (x^2 + 1) \sin(x) \)

_Solution to Question 3._

(a) We know that \( \cos(x) = \sum_{n=0}^{\infty} (-1)^n \frac{x^{2n}}{(2n)!} \) for all \( x \). Hence it follows for all \( x \) that

\[
\frac{1 - \cos(x)}{x} = \frac{1 - \sum_{n=0}^{\infty} (-1)^n \frac{x^{2n}}{(2n)!}}{x} = \sum_{n=1}^{\infty} (-1)^n \frac{x^{2n-1}}{(2n)!} x
\]

(b) Similarly, we know that \( \sin(x) = \sum_{n=0}^{\infty} (-1)^n \frac{x^{2n+1}}{(2n+1)!} \) for all \( x \). Hence we have that

\[
(x^2 + 1) \sin(x) = (x^2 + 1) \sum_{n=0}^{\infty} (-1)^n \frac{x^{2n+1}}{(2n+1)!} = x^2 \sum_{n=0}^{\infty} (-1)^n \frac{x^{2n+3}}{(2n+1)!} + \sum_{n=0}^{\infty} (-1)^n \frac{x^{2n+1}}{(2n+1)!} \]

\[
= x + \sum_{n=0}^{\infty} (-1)^n \frac{x^{2n+3}}{(2n+1)!} + \sum_{n=0}^{\infty} (-1)^n \frac{x^{2n+1}}{(2n+1)!} \]

\[
= x + \sum_{n=0}^{\infty} (-1)^n \frac{x^{2n+3}}{(2n+1)!} + \sum_{n=0}^{\infty} (-1)^{n+1} \frac{x^{2n+3}}{(2n+3)!} \]

\[
= x + \sum_{n=0}^{\infty} (-1)^n \left( 1 - \frac{1}{(2n+3)(2n+2)} \right) (-1)^n \frac{x^{2n+3}}{(2n+1)!} \]

\[
= x + \sum_{n=0}^{\infty} (-1)^n \left( \frac{4n^2 + 10n + 5}{(2n+3)!} \right) x^{2n+3} \]

**Question 4.** Show that

\[
\pi - \frac{\pi^3}{3!} + \frac{\pi^5}{5!} - \frac{\pi^7}{7!} + \cdots
\]

converges to zero. How many terms must be computed to get within 0.01 of zero?

_Solution to Question 4._

We have that \( \sin(x) = \sum_{n=0}^{\infty} (-1)^n \frac{x^{2n+1}}{(2n+1)!} \) for all \( x \) and so we see this series converges to \( \sin(\pi) = 0 \). Power
series coincide with their taylor expansion and so we can use the error estimate for the taylor polynomial of sin(x) around x = 0 to understand how many terms we need to compute the series to get it within 0.01 of zero. i.e, the first N terms of the series are exactly T_{2N-1}(\pi).

We find that

$$|\sin(\pi) - T_{2N-1}(\pi)| \leq \max_{x \in [0,\pi]} \frac{|\sin^{(2N)}(x)| \pi^{2N}}{(2N)!} \leq \frac{\pi^{2N}}{(2N)!}$$

We want this less than 10^2 and so we see that N = 10 is enough after putting this into a calculator.