Q1
(a) \( \frac{d}{dx} (x^2 e^{2x}) = 2x e^{2x} + 2x^2 e^{2x} \)
\[ = 2x (x+1) e^{2x} \]

(b) \( \frac{d}{dx} (e^{\sin x}) = \cos(x) e^{\sin x} \)

(c) \( \frac{d}{dx} (\tan(e^{5-6x})) = \sec^2(e^{5-6x}) \cdot \frac{d}{dx} (e^{5-6x}) \)
\[ = -6 e^{5-6x} \sec^2(e^{5-6x}) \]

(d) \( \frac{d}{dx} (e^{\frac{1}{x}}) = e^{\frac{1}{x}} \frac{d}{dx} \left( \frac{1}{x} \right) \)
\[ = -e^{\frac{1}{x}} \frac{1}{x^2} \]

(e) \( \frac{d}{dx} (4^{-2x}) = \frac{d}{dx} \left( e^{-2\ln(4)x} \right) \)
\[ = -2 \ln(4) e^{-2\ln(4)x} \]
\[ = -2 \ln(4) 4^{-2x} \]
Q2

(a) \[ \frac{d}{dx} (x \ln(x) - x) = \ln(x) + \frac{x}{x} - 1 \]
\[ = \ln(x) \]

(b) \[ \frac{d}{dx} (\ln((\ln x)^3)) = \frac{d}{dx} (3 \ln(\ln x)) \]
\[ = \frac{1}{\ln x} \cdot \frac{d}{dx} (3 \ln x) \]
\[ = \frac{3}{x \ln x} \]

Alternatively,
\[ \frac{d}{dx} (\ln((\ln x)^3)) = \frac{1}{(\ln x)^3} \cdot 3(\ln x)^2 \cdot \frac{1}{x} \]
\[ = \frac{3}{x \ln x} \]

(c) \[ \frac{d}{dx} \ln \left( \frac{x+1}{x^3 + 1} \right) = \frac{d}{dx} \ln(x+1) - \frac{d}{dx} \ln(x^3 + 1) \]
\[ = \frac{1}{x+1} - \frac{3x^2}{x^3 + 1} \]
Rule of thumb: + and - are easier to deal with than \( \times \) and \( \div \), so use log to change them when possible.

i.e., if we didn't use the property \( \ln(a/b) = \ln(a) - \ln(b) \) in last question, we get that

\[
\frac{d}{dx} \ln\left( \frac{x+1}{x^3+1} \right) = \frac{1}{x+1/x^3+1} \cdot \frac{(x+1)'(x^3+1) - (x^3+1)'(x+1)}{(x^3+1)^2}
\]

\[
= \frac{x^3+1 - (x+1) \cdot 3x^2}{(x+1)(x^3+1)}
\]

\[
= \frac{1 - 3x^3}{(x+1)(x^3+1)}
\]

which requires more work.

(d) The complicated powers suggest logarithmic differentiation over a straight application of the quotient rule.
(d) continued:

\[
y = \frac{(x+12)^{3/2}}{(x-6)^{1/5}}
\]

\[
\ln y = \frac{5}{2} \ln(x+12) - \frac{1}{5} \ln(x-6)
\]

\[
\frac{1}{y} \cdot \frac{dy}{dx} = \frac{5}{2(x+12)} - \frac{1}{5 \ln(x-6)}
\]

\[
\Rightarrow \frac{dy}{dx} = \left( \frac{5}{2(x+12)} - \frac{1}{5 \ln(x-6)} \right) \frac{(x+12)^{3/2}}{(x-6)^{1/5}}
\]
Q3

1) We have that \( f'(x) = \ln b \cdot b^x \).

Since \( 1 < b \), \( \ln b > 0 \) and as \( b^x > 0 \)
we conclude that \( f'(x) > 0 \) and so
\( f \) is strictly increasing.

2) Since \( \ln(x), x > 0 \) is continuous we have
\[
\lim_{x \to \infty} \ln \left( 1 + \frac{1}{x} \right) = \ln \left( \lim_{x \to \infty} 1 + \frac{1}{x} \right)
\]
\[
= \ln(1)
\]
\[
= 0
\]

3) We have by the change of base formula
\[
f(x) = \log_b(x) = \frac{\ln(x)}{\ln(b)}
\]
and so
\[
f'(x) = \frac{1}{x \ln(b)}
\]
As \( x > 0 \), \( 0 < b < 1 \) \( \Rightarrow \ln(b) < 0 \) we see
that \( f'(x) < 0 \) and so \( f \) is decreasing.
(4) Let $u = e^t + 1$, then $du = e^t dt$ and we have

$$
\int e^t \sqrt{e^t + 1} \, dt = \int \sqrt{u} \, du
$$

$$
= \frac{2}{3} u^{3/2} + C
$$

$$
= \frac{2}{3} (e^t + 1)^{3/2} + C
$$

(5) Let $u = \ln(t)$, then $du = \frac{1}{t} dt$.

When $t = e$, $u = 1$

When $t = e^2$, $u = 2$

Hence, after $u$-substitution we get

$$
\int_{e}^{e^2} \frac{1}{t \ln(t)} \, dt = \int_{1}^{2} \frac{du}{u} = \left. \ln(u) \right|_{1}^{2} = \ln(2)
$$
Q4

(1) we check that \( g(f(x)) = x \) on domain of \( f \) and \( f(g(x)) = x \) on domain of \( g \).

\[
g(f(x)) = \frac{1 - \frac{1}{1+x}}{1/1+x} = \frac{1+x - 1}{1} = x
\]

\[
f(g(x)) = \frac{1}{1+\frac{1}{1+x}} = \frac{x}{x+1-x} = x
\]

Hence \( f \) and \( g \) are inverses.

(2) \( g'(b) = \frac{1}{f'(g(b))} \)

(3) by inspection, \( f(1) = 0 \) and so \( f^{-1}(0) = 1 \)

Now, by the above formula, we have

\[
f^{-1}(0) = \frac{1}{f'(f^{-1}(0))} = \frac{e^x}{(e^0 - e)^2 + 1}
\]

\[
= e
\]
\[ f(x) = x \ln(2) + 2x \ln(3) + x^2 \ln(e) \]

Differentiating,

\[ \frac{f'(x)}{f(x)} = \ln(2) + 2 \ln(3) + 2x \]

\[ f''(x) = (\ln(2) + 2 \ln(3) + 2x) 2^x 3^{2x} e^{x^2} \]

\[ = (\ln(18) + 2x) 2^x 3^{2x} e^{x^2} \]

Note, if the question didn't specify logarithmic differentiation, we could do the following:

\[ f(x) = 2^x 3^{2x} e^{x^2} \]

\[ = e^{\ln(2)x} e^{2x \ln(3)} e^{x^2} \]

\[ = e^{\ln(2)x + 2x \ln(3) + x^2} \]

\[ f'(x) = (\ln(2) + 2 \ln(3) + 2x) 2^x 3^{2x} e^{x^2} \]
Q 5

(i) \( \lim_{x \to 0} \left( \frac{1}{\sin(x)} - \frac{1}{x} \right) = \lim_{x \to 0} \left( \frac{x - \sin(x)}{x \sin(x)} \right) \)

\[ = \frac{0}{0} \text{ which is indeterminate.} \]

Hence, L'hôpital's gives

\[ \text{LHS} = \lim_{x \to 0} \frac{1 - \cos(x)}{\sin(x) + x \cos(x)} = \frac{0}{0} \]

L'hôpital's again gives

\[ \text{LHS} = \lim_{x \to 0} \frac{\sin(x)}{2 \cos(x) - x \sin(x)} = \frac{0}{0} \]

Hence, \( \lim_{x \to 0} \left( \frac{1}{\sin(x)} - \frac{1}{x} \right) = 0. \)

Alternate method without L'hôpital's:

We already know the limit \( \lim_{x \to 0} \frac{x}{\sin(x)} = 1. \)

Consider,

\[ \lim_{x \to 0} \frac{x - \sin(x)}{x \sin(x)} = \lim_{x \to 0} \frac{x - \sin(x)}{x^2} \]

\[ = \frac{0}{0} \text{ which is indeterminate.} \]
\[= \lim_{x \to 0} \frac{1}{x} - \lim_{x \to 0} \frac{\sin(x)}{x} \cdot \frac{1}{x} \quad \text{by limit law}\]

\[= 0 - 1 \cdot 0 = 0\]

Now,

\[\lim_{x \to 0} \frac{x - \sin(x)}{x \sin(x)} \cdot \frac{\sin(x)}{x} \cdot \frac{x}{\sin(x)} = \lim_{x \to 0} \frac{x - \sin(x)}{x \sin(x)} \cdot \frac{\sin(x)}{x} \cdot \frac{x}{\sin(x)} \quad \text{by limit law}\]

\[= 0 \cdot 1 = 0\]

\((2)\) \[\lim_{x \to 1} \frac{x(\ln x - 1) + 1}{(x-1)\ln x} = \frac{0}{0} \quad \text{which is indeterminate}\]

by l'Hopitals

\[\lim_{x \to 1} \frac{x(\ln x - 1) + 1}{(x-1)\ln x} = \lim_{x \to 1} \frac{\ln x - 1 + \frac{x}{x}}{x - 1 + \ln x} \quad \text{which is again indeterminate}\]
10 by L'Hôpital's,

\[
\text{LHS} = \lim_{x \to 1} \frac{1/x}{x^2 + 1/x} = \lim_{x \to 1} \frac{x}{x + 1} = \frac{1}{2}
\]

Hence \[\lim_{x \to 1} \frac{x(\ln x - 1)}{(x-1)\ln x} = \frac{1}{2}\]

(3) \[\lim_{x \to 0} \frac{\cos(x+\pi/2)}{\sin(x)} = \frac{0}{0}\] which is indeterminate.

by L'Hôpital's,

\[\text{LHS} = \lim_{x \to 0} \frac{-\sin(x+\pi/2)}{\cos(x)} = -\frac{1}{1} = -1\]

Hence \[\lim_{x \to 0} \frac{\cos(x+\pi/2)}{\sin(x)} = -1\]
Q6

(1) \[ f(x) = e^{\arccos(x)} \]

\[ f'(x) = \frac{d}{dx} \left( \arccos(x) \right) e^{\arccos(x)} \]

\[ = -\frac{e^{\arccos(x)}}{\sqrt{1 - x^2}} \]

(2) \[ \int_0^3 \frac{dx}{x^2 + 3} = \frac{1}{3} \int_0^3 \frac{dx}{\frac{x^2}{3} + 1} \]

Let \( u = \frac{x}{\sqrt{3}} \), then \( du = \frac{1}{\sqrt{3}} \) \( dx \)

When \( x = 0 \), \( u = 0 \)

When \( x = 3 \), \( u = \sqrt{3} \)

Hence \( \text{LHS} = \frac{1}{\sqrt{3}} \int_0^{\sqrt{3}} \frac{du}{u^2 + 1} \)

\[ = \frac{1}{\sqrt{3}} \arctan(u) \bigg|_0^{\sqrt{3}} \]

\[ = \frac{1}{\sqrt{3}} \arctan(\sqrt{3}) \]

\[ = \frac{\pi}{3 \sqrt{3}} \]
3) we have \( \sinh(x) \to \infty \) as \( x \to \infty \)

and \( \tanh(x) \to 1 \) as \( x \to \infty \).

Putting these together gives:

\[
\lim_{x \to \infty} \tanh(\sinh(x)) = \tanh\left( \lim_{x \to \infty} \sinh(x) \right) = \infty.
\]

Note, it is useful to know the graphs of the hyperbolic functions.

(4) \( \cosh^{-1}(x) \) is the inverse of \( \cosh(x) \) for \( x > 1 \).

From the previous question, we know that

\[
\frac{d}{dx} \cosh^{-1}(x) = \frac{1}{\cosh'(\cosh^{-1}(x))} \quad \text{for when } x \text{ in domain of } \cosh^{-1}(x) \ (x > 1)
\]

Now, the derivative of \( \cosh \) is \( \sinh \), hence

\[
\frac{d}{dx} \cosh^{-1}(x) = \frac{1}{\sinh(\cosh^{-1}(x))}
\]
\[
\sqrt{\cosh(\cosh^{-1}(x))^2 - 1} = 1
\]

\[
= \frac{1}{\sqrt{x^2 - 1}}
\]

Alternatively,

\[y = \cosh^{-1}(x) \Rightarrow \cosh(y) = x\] and implicitly differentiating gives

\[
\sinh(y), \frac{dy}{dx} = 1
\]

\[
\frac{dy}{dx} = \frac{1}{\sinh(y)} = \frac{1}{\sqrt{\cosh^2(y) - 1}} = \frac{1}{\sqrt{x^2 - 1}}
\]

\[x = \cosh(y).\]
\( \int \arcsin^{-1}(x) \, dx = x \arcsin^{-1}(x) - \int \frac{x}{\sqrt{1-x^2}} \, dx \)

Let \( u = 1 - x^2 \), \( du = -2x \, dx \)

Then LHS = \( x \arcsin^{-1}(x) + \frac{1}{2} \int \frac{1}{\sqrt{u}} \, du \)

\( = x \arcsin^{-1}(x) + u^{1/2} + c \)

\( = x \arcsin^{-1}(x) + (1-x^2)^{1/2} + c \)

\( \int_0^1 xe^{-x} \, dx = -xe^{-x} \bigg|_0^1 + \int_0^1 e^{-x} \, dx \)

\( = e^{-1} - e^{-x} \bigg|_0^1 \)

\( = e^{-1} - e^{-1} + 1 \)

\( = 1 \)
Q8

(1) \[
\frac{3x^2 + 5x - 4}{(x-2)(x+1)^2} = \frac{A}{x-2} + \frac{B}{x+1} + \frac{C}{(x+1)^2}
\]

and so

\[
3x^2 + 5x - 4 = A(x+1)^2 + B(x-2)(x+1) + C(x-2)
\]

The general method is to equate the coefficients and then solve the system of 3 equations in 3 unknowns. However, we can use a few tricks here.

When \(x=2\), we have (\#) is:

\[
3(2)^2 + 5(2) - 4 = 3^2 A
\]

\[
12 + 10 - 4 = 9A \implies A = 2
\]

Equating the coefficient of \(x^2\) in (\#) gives

\[
3 = A + B \implies B = 1
\]

When \(x=-1\) in (\#), we get

\[
3(-1)^2 - 5 - 4 = -3c
\]

\[
-6 = -3c \implies c = 2.
\]

Hence, \(A = 2, B = 1, C = 2\).
(2) we have

\[ \int \frac{2x^2 - 2x + 4}{(x-1)(x^2+1)} \, dx = \int \frac{2}{x-1} - \frac{2}{x^2+1} \, dx = 2 \ln |x-1| - 2 \arctan(x) + C. \]