From textbook:

**THEOREM 2 Derivative of the Inverse** Assume that $f$ is differentiable and one-to-one with inverse $g(x) = f^{-1}(x)$. If $b$ belongs to the domain of $g$ and $f'(g(b)) \neq 0$, then $g'(b)$ exists and

$$g'(b) = \frac{1}{f'(g(b))}$$


Example: Let $f(x) = \sqrt{3-x}$, $x \leq 3$. Let $g$ be the inverse function of $f$. Find $g(2)$ and $g'(2)$.

**Answer 1 (Textbook answer)**

Let $b = g(2)$, this is the number such that $f(b) = 2$ by definition, i.e.

$$\sqrt{3-b} = 2$$

$$3-b = 4$$

$$\therefore b = -1.$$ 

Hence we have $g(2) = -1$.

Now, $f'(x) = \frac{-1}{2 \sqrt{3-x}}$ and so we have that

$$f'(g(2)) = f'(-1) = \frac{-1}{2 \sqrt{4}} = -\frac{1}{4}.$$
Hence by above formula \( g'(2) = -4 \).

**Answer 2.** (A bit more old school I suppose)

Some background: Notice that the above formula can be rewritten in terms of Liebnitz notation as \( g'(b) = \frac{1}{\frac{dy}{dx}}|_{y=g(b)} \) where \( y = f(x) \). In other words you can think of \( g'(b) \) as \( "\frac{dx}{dy}\b (g(b)) \). This agrees with thinking of the inverse function as flipping the role of \( x \) and \( y \). So we can do this problem another way: we set \( y = f(x) = \sqrt{3-x} \)

\[ y^2 = 3 - x \]

differentiating gives

\[ 2y \frac{dy}{dx} = -1 \]

\[ \Rightarrow \frac{dy}{dx} = \frac{-1}{2y} \]
\[ \Rightarrow \frac{dx}{dy} = -2y \] (This is somewhat informal, I really mean \( \frac{1}{\frac{dy}{dx}} \))

Now, we want the derivative of the inverse when \( y = 2 \), so \( g'(2) = \frac{dx}{dy} \bigg|_{y=2} = -4 \)

**Example:** Find \( \int \frac{(\ln x)^2}{x} \, dx \)

We use **u-substitution**, let \( u = \ln x \), then \( du = \frac{1}{x} \, dx \) and so \( \int \frac{(\ln x)^2}{x} \, dx = \int u^2 \, du = \frac{u^3}{3} + C \)

Substituting back gives

\[ \int \frac{(\ln x)^2}{x} \, dx = \frac{(\ln x)^3}{3} + C \]

**Example:** Differentiate \( f(x) = \frac{x(x+1)^3}{(3x-1)^2} \)

We will use a trick called logarithmic differentiation.

Note: Whenever you see something that has a lot of multiplication, you should immediately think logarithmic differentiation. 
think of using logarithms since they turn multiplication (hard) into addition (easy)

we have \( \ln f(x) = \ln(x) + 3\ln(x+1) - 2\ln(3x-1) \)

by log rules. Differentiating gives

\[
\frac{f'(x)}{f(x)} = \frac{1}{x} + \frac{3}{x+1} - \frac{6}{3x-1}
\]

\[
\Rightarrow f'(x) = f(x) \left( \frac{1}{x} + \frac{3}{x+1} - \frac{6}{3x-1} \right)
\]

\[
= \frac{x(x+1)^3}{(3x-1)^2} \left( \frac{1}{x} + \frac{3}{x+1} - \frac{6}{3x-1} \right)
\]