Question 1. Determine whether the following improper integrals converge or diverge. If they converge, what do they converge to?

(a) \[ \int_1^\infty \frac{1}{x^3} \, dx \]

(b) \[ \int_0^2 x^2 \, dx \]

(c) \[ \int_1^2 \frac{1}{x \ln(x)} \, dx \]

Solution to Question 1.

(a)

\[ \int_1^\infty x^3 \, dx = \lim_{R \to \infty} \int_1^R x^3 \, dx \]
\[ = \lim_{R \to \infty} \frac{x^4}{4} \bigg|_1^R \]
\[ = \lim_{R \to \infty} \frac{R^4}{4} - \frac{1}{4} \]
\[ = \infty. \]

Hence this diverges

(b)

\[ \int_0^2 x^2 \, dx = \lim_{r \to 0} \int_r^2 x^2 \, dx \]
\[ = \lim_{r \to 0} \frac{x^3}{3} \bigg|_r^2 \]
\[ = \lim_{r \to 0} \frac{8}{3} - \frac{r^3}{3} \]
\[ = \frac{8}{3}. \]

Hence converges to \( \frac{8}{3} \).

(c)

\[ \int_1^2 \frac{1}{x \ln(x)} \, dx = \lim_{r \to 1} \int_r^2 \frac{1}{x \ln(x)} \, dx \]
\[ = \lim_{r \to 1} \ln(\ln(x)) \bigg|_r^2 \]
\[ = \lim_{r \to 0} \ln(\ln(2)) - \ln(\ln(r)) \]
\[ = \infty. \]

Hence this diverges.
Question 2. Use the comparison test to determine whether the following integrals converge or diverge.

(a) \( \int_1^\infty \frac{dx}{\sqrt{x^4 + 3}} \)

(b) \( \int_0^1 \frac{dx}{x^4 + \sqrt{x}} \)

Solution to Question 2.

(a) We have the inequality \( \sqrt{x^4 + 3} > \sqrt{x^4} = x^2 \) for \( x > 1 \). Hence \( 0 \leq \frac{1}{\sqrt{x^4 + 3}} \leq \frac{1}{x^2} \). Since \( \int_1^\infty \frac{1}{x^2} \, dx \) converges (either by above method or as a \( p \)-integral), the comparison test implies that \( \int_1^\infty \frac{dx}{\sqrt{x^4 + 3}} \) converges.

(b) We have the inequality \( x^4 + \sqrt{x} \geq \sqrt{x} \) for \( x \in [0,1] \). Hence \( 0 \leq \frac{1}{x^4 + \sqrt{x}} \leq \frac{1}{\sqrt{x}} \). Since \( \int_0^1 \frac{1}{\sqrt{x}} \, dx \) converges (either by above method or as a \( p \)-integral), the comparison test implies that \( \int_0^1 \frac{dx}{x^4 + \sqrt{x}} \) converges.

Question 3. Compute the arclength of \( y = \left( \frac{x}{2} \right)^4 + \frac{1}{2x^2} \) over the interval \([1,4]\).

Solution to Question 3.

We have that \( y' = \frac{x^3}{4} - \frac{1}{x^3} \) and so

\[
1 + (y')^2 = 1 + \left( \frac{x^3}{4} - \frac{1}{x^3} \right)^2 \\
= 1 + \frac{x^6}{16} - \frac{1}{2} + \frac{1}{x^6} \\
= \frac{x^6}{16} + \frac{1}{2} + \frac{1}{x^6} \\
= \left( \frac{x^3}{4} + \frac{1}{x^3} \right)^2.
\]

Hence arclength is given by

\[
\text{Arclength} = \int_1^4 \sqrt{1 + (y')^2} \, dx \\
= \int_1^4 \frac{x^3}{4} + \frac{1}{x^3} \, dx \\
= \frac{509}{32}
\]

Question 4. (More Challenging) Compute the arc length of \( y = \ln \left( \frac{e^x + 1}{e^x - 1} \right) \) over the interval \([1,3]\).
Solution to Question 4.
We have the following:

\[
\frac{dy}{dx} = \frac{-2e^x}{e^{2x} - 1}
\]

\[
\left(\frac{dy}{dx}\right)^2 = \frac{4e^{2x}}{(e^{2x} - 1)^2}
\]

\[
1 + \left(\frac{dy}{dx}\right)^2 = \frac{(e^{2x} + 1)^2}{(e^{2x} - 1)^2}
\]

\[
\sqrt{1 + \left(\frac{dy}{dx}\right)^2} = \frac{e^{2x} + 1}{e^{2x} - 1}
\]

= coth(x) for \( x \geq 0 \).

Hence we have that

\[
\text{Arclength} = \int_1^3 \sqrt{1 + \left(\frac{dy}{dx}\right)^2} \, dx = \int_1^3 \coth(x) \, dx = \ln(\sinh(3)) - \ln(\sinh(1)).
\]