Discussion Questions

Question 1. Let \( f(x) = x + \cos(x) \) and \( g(x) \) it’s inverse. Calculate \( g(1) \) and \( g'(1) \).

Solution to Question 1.
Let \( b = g(1) \), this is a number such that \( f(b) = 1 \) since it’s an inverse. i.e,

\[
b + \cos(b) = 1.
\]
By inspection, we find that \( b = 0 \) and so \( g(1) = 0 \). Now, we will use the formula for the derivative of the inverse to find \( g'(1) \). We first differentiate \( f, f'(x) = 1 - \sin(x) \). Hence we have

\[
g'(1) = \frac{1}{f'(g(1))} = \frac{1}{f'(0)} = 1.
\]

Question 2. Calculate \( \int \frac{\cos(x)}{2\sin(x) + 3} \, dx \). Hint: \( u \)-substitution.

Solution to Question 2.

\[
\int \frac{\cos(x)}{2\sin(x) + 3} \, dx = \int \frac{1}{2u + 3} \, du \text{ after } u = \sin(x) \\
= \frac{1}{2} \ln |2\sin(x) + 3| + C \\
= \frac{1}{2} \ln |2\sin(x) + 3| + C
\]

Question 3. Find the derivative of \( f(x) = \frac{x(x^2 + 1)}{\sqrt{x + 1}} \).

Solution to Question 3.
We have that

\[
\ln(f(x)) = \ln(x) + \ln(x^2 + 1) - \frac{1}{2} \ln(x + 1).
\]
Differentiating gives us

\[
\frac{f'(x)}{f(x)} = \frac{1}{x} + \frac{2x}{x^2 + 1} - \frac{1}{2(x + 1)}.
\]
Hence

\[
f'(x) = \frac{x(x^2 + 1)}{\sqrt{x + 1}} \left( \frac{1}{x} + \frac{2x}{x^2 + 1} - \frac{1}{2(x + 1)} \right)
\]
Extra Questions

**Question 4.** Let \( f(x) = x^3 + 2x + 4 \) and \( g \) it’s inverse. Without finding a formula for \( g(x) \) (no seriously, don’t even try) calculate \( g(7) \) and then \( g'(7) \).

**Solution to Question 4.**

\( g(7) = b \) where \( b \) is the number such that \( f(b) = 7 \). i.e, \( b^3 + 2b + 4 = 7 \) and so \( b = 1 \) by inspection. Now, by the derivative of the inverse formula we have that

\[
g'(7) = \frac{1}{f'(g(7))} = \frac{1}{f'(1)} = \frac{1}{5}.
\]

**Question 5.** Calculate the following derivatives

(a) \( y = \ln(x^2 6^x) \)     (c) \( y = 8^{\cos(x)} \)
(b) \( y = \ln \left( \frac{x+1}{x^3+1} \right) \)     (d) \( y = x^e \)

**Solution to Question 5.**

(a) \[
y = \ln(x^2 6^x) \\
y = 2 \ln(x) + x \ln(6) \\
\frac{dy}{dx} = \frac{2}{x} + \ln(6).
\]

(b) \[
y = \ln \left( \frac{x+1}{x^3+1} \right) \\
y = \ln(x+1) - \ln(x^3 + 1) \\
\frac{dy}{dx} = \frac{1}{x+1} - \frac{3x^2}{x^3 + 1}.
\]

(c) \[
y = 8^{\cos(x)} \\
y = e^{\ln(8) \cos(x)} \\
\frac{dy}{dx} = - \ln(8) \sin(x) e^{\ln(8) \cos(x)} \\
\quad = - \ln(8) \sin(x) 8^{\cos(x)}.
\]

(d) \[
y = x^e \\
y = e^{\ln(x) e^x} \\
\frac{dy}{dx} = \frac{d}{dx} (\ln(x) e^x) \cdot e^{\ln(x) e^x} \\
\quad = \left( \frac{e^x}{x} + \ln(x) e^x \right) \cdot x e^x.
\]
**Question 6.** Prove that \( \frac{d}{dx} (\arcsin(x)) = \frac{1}{\sqrt{1-x^2}} \). Remember that \( \arcsin(x) \) is the inverse of \( \sin \) after restricting the domain to \([-\pi/2, \pi/2]\).

**Solution to Question 6.**

Let \( y = \arcsin(x) \). Then \( \sin(y) = x \) and differentiating gives us that

\[
\cos(y) \frac{dy}{dx} = 1.
\]

Note that by Pythagoras we have that \( \cos(y) = \sqrt{1 - \sin^2(y)} \) since we have restricted the domain so that \( \cos(y) \) is positive. Hence altogether we have

\[
\frac{dy}{dx} = \frac{1}{\sqrt{1 - \sin^2(y)}} = \frac{1}{\sqrt{1-x^2}}.
\]