Problem 1

Problem 2

The points and line on the xy-plane looks like:

- points to meet
- points to avoid

Not to scale

We build our function piecewise and deal with this in sections.

For $-\infty < x \leq 1$, there are no difficulties and take $f(x) = 0$ here.

For $1 \leq x \leq 2$, we need to avoid the line $S_1$, and connect the points $P_1, P_2$. We can do this in a bunch of ways. We will
Start by taking the polynomial that connects these points:
\[ y = (x-1)(x-2). \]
This however does intersect the line \( S_1 \) as \( x=1.5 \Rightarrow y = -\frac{1}{4} \in S_1 \).
Hence we will scale this by a number large enough so that the peak of the polynomial avoids the line, i.e., we will pick a scaling factor of 16, so the polynomial becomes \( y = 16(x-1)(x-2) \).
Then we have at \( x=1.5 \Rightarrow y = -9 \) and so this avoids the line \( S_1 \).

For \( 2 \leq x < +\infty \), we must find an equation that connects the point \( P_2, P_3 \) and avoids the line \( S_2 \). Since the only way to avoid \( S_2 \) is by going through the point \( Q=(3,1) \). We could use a similar strategy as before to fit a polynomial through these three points, or alternatively, notice that each point is 1 unit distant from the point \( (3,0) \). Hence they fit on the upper semicircle \( y = \sqrt{1 - (x-3)^2} \), \( 2 \leq x \leq 3 \).

Putting this all together, we have the function:
\[
f(x) = \begin{cases} 
0 & \text{if } -\infty < x \leq 1, \\
16(x-1)(x-2) & \text{if } 1 < x \leq 2, \\
\sqrt{1 - (x-3)^2} & \text{if } 2 < x \leq 3, \\
0 & \text{if } 3 < x < +\infty.
\end{cases}
\]
The points and line on the xy-plane looks like:

- point to meet
- point to avoid

$f(x)$.

Not to scale

Problem 3

(a) $y^2 = x^2$

Not a function.
One approach to this question is to draw $y = x^2 - 4$ first and then convert each $y$-value to its reciprocal. This turns to asymptotes, and $a \to 1/a$.

Rearranging gives $4x^2 + y^2 = 9$. This is almost the circle $x^2 + y^2 = 9$ but $x$ has been swapped with $2x$. This has the effect of scaling the graph in the $x$-direction by $1/2$. 
Problem 4

(a) Given two points \((x_0, y_0), (x_1, y_1)\), the equation that joins them is given by:

\[
y - y_0 = \frac{y_1 - y_0}{x_1 - x_0} (x - x_0).
\]

Hence, plug in 1, 1+\(a\) into \(f\) to get the y-coords:

\[
f(1) = 9,
\]

\[
f(1+a) = (1+a)^2 + (1+a) + 2 = a^2 + 2a + 1 + 1 + a + 2 = a^2 + 3a + 4.
\]

The equation becomes:

(with \(x_0 = 1, y_0 = 4, x_1 = 1+a, y_1 = a^2 + 3a + 4\))

\[
y - 4 = \frac{a^2 + 3a + 4 - 4}{1 + a - 1} \cdot (x - 1)
\]

\[
y = \frac{a^2 + 3a}{a} \cdot (x - 1) + 4
\]

secant line:

\[
y = (a + 3)(x - 1) + 4 \quad \text{as} \quad a > 0
\]

(b) The slope of the above line is \(a + 3\). Hence the slope of a function is

\[
m(a) = a + 3.
\]
(c) We get:
\[ m(1) = 4 \]
\[ m(0.1) = 3.1 \]
\[ m(0.01) = 3.01 \]

This gets closer and closer to 3.

(d) Just as above, \( m(0) = 3 \). Geometrically, I expect this to be the slope of the tangent line at \( x = 1 \).

(e) If we take \((x_0, y_0) = (1, 4)\) in the equation for a line we get:

\[ y - y_0 = \frac{y - y_0}{x - x_0} (x - x_0) \]

\[ y - 4 = \frac{0}{0} (x - 1) \]

This doesn't make sense as \( \frac{0}{0} \) doesn't make sense.

(f) We get from (c) that we expect the tangent line to have slope 3. Hence the tangent line should be:

\[ y - 4 = 3(x - 1) \]

\[ y = 3x + 1 \]