Question 1. Show that the order of a finite field $F$ cannot be divisible by two distinct primes.

Question 2. Show for a field $F$ the following are equivalent.

1. There exists $\alpha_1, \cdots, \alpha_n \in F$ such that the only subfield of $F$ containing all the $\alpha_i$’s is $F$ itself.
2. $F$ is the fraction field of $\mathbb{Z}[x_1, \cdots, x_n]/p$ for some prime ideal $p$.

Question 3. Let $a/b \in \mathbb{Q}$. Show that $\cos(\frac{a}{b}\pi)$ is algebraic over $\mathbb{Q}$.

Question 4. Given a commutative ring $B$ and subring $A$, we say that an element of $x \in B$ is integral over $A$ if there exists a monic polynomial $f(t) \in A[t]$ such that $f(x) = 0$. Note that if $A$ and $B$ are fields, then this is the same definition for algebraic over. Show that the following are equivalent:

1. $x$ is integral over $A$;
2. The subring $A[x]$ of $B$ generated by $A$ and $x$ is a finitely generated $A$-module;
3. There exists a subring $C$ of $B$ containing $A[x]$ and which is finitely generated as an $A$-module;
4. There exists a finitely generated $A$-submodule $M$ of $B$ such that $xM \subseteq M$ and $M$ is faithful over $A[x]$ (That is the map given by the action $A[x] \to Hom(M)$ is injective).

Question 5. Show that any automorphism on the real numbers is identity.