The following is sometimes called Nakayama’s lemma. I prefer to call it gerneralized Cayley-Hamilton.

**Theorem 1.** Let $R$ be a commutative ring and $I$ an ideal. Let $M$ be a finitely generates $R$-module and $f$ an endormorphism of $M$. If $f(M) \subseteq IM$ then there exists an $n$ and $a_i \in I$ such that

$$f^n + a_{n-1}f^{n-1} + \cdots + a_1f + a_0 = 0 \text{ in } \text{End}(M)$$

**Question 1.** The following are all called Nakayama’s lemma. Prove all of them. Let $R$ be a commutative ring and $M$ a finitely generated $R$-module.

1. Let $I$ be an ideal of $R$ such that $IM = M$. Then there exists an $x \in R$ such that $x = 1$ in $R/I$ and $xM = 0$. An equivalent formulation to this which is easier to remember is that if $IM = M$ then there exists an element $i \in I$ such that $im = m$ for all $m \in M$.

2. Let $I$ be an ideal contained in $\text{Rad}(R)$. Then $IM = M$ implies that $M = 0$.

3. Let $N$ be submodule of $M$ and $I \subseteq \text{Rad}(R)$. Then if $M = IM + N$ then $M = N$.

4. Suppose that $R$ is a local ring with maximal ideal $m$ and residue field $F = R/m$. Suppose we have elements $x_i$ of $M$ such that their images in $M/mM$ form an $F$-basis. Then the $x_i$ generate $M$.

**Question 2.** Suppose $R$ is an integral domain which isn’t a field and let $F = R_{(0)}$. Show that $F$ cannot be a finitely generated $R$-module.

**Question 3.** Let $M$ be a finitely generated $R$-module for commutative ring $R$. Show that every surjective endomorphism is an isomorphism.

**Question 4.** Let $S$ be a multiplicative set of ring $R$. Show that the functor $S^{-1} : \text{mod}(R) \to \text{mod}(S^{-1}R)$ is exact. That is, maps short exact sequences to short exact sequences.

**Definition 1.** We call a property $P$ of $R$-modules local if a module $M$ has property $P$ if and only if $M_q$ has property $P$ for all prime ideals $q$.

**Question 5.** Show the following are local properties:

1. A module being trivial.
2. A $R$-homomorphism being injective.
3. A module being torsion free.

**Question 6.** Prove the following version of Nakayama’s Lemma: