Question 1. The following are standard properties of localisations which you should prove if you haven’t already.

1. Let $S$ be a multiplicative set of commutative ring $R$. Show there is a bijection between prime ideals of $R$ that don’t intersect $S$ and prime ideals of $R[S^{-1}]$.

2. Let $S \subseteq W$ be multiplicative sets in $R$. Show that $(R[S^{-1}])[W^{-1}] \cong R[W^{-1}]$.

Question 2. Given $f \in R$, let $D(f) = \text{Spec}(R) \setminus V(f)$. This is called a basic open set in the spectrum of $R$. Let $S$ be the set of all functions $g \in R$ that don’t vanish on $D(f)$. Show $S$ is multiplicatively closed and that $R_f = R[S^{-1}]$.

Question 3. The sets $D(f)$ are called basic open sets or distinguished open sets. Show that these form a basis for $\text{Spec}(R)$. Moreover, show that $\text{Spec}(R)$ is compact.

Question 4. Let $f_1, \cdots, f_n$ be such such that $D(f_i)$ cover the $\text{Spec}(R)$. Prove the following: (note $D(f) \cap D(g) = D(fg)$.

1. (identity) if $g, h \in R$ such that $g = h$ in $R_{f_i}$ for all $i$. Then $g = h$ in $R$.

2. (glueing) if $g_i$ in $R_{f_i}$ for all $i$ such that $g_i = g_j$ in $R_{f_i f_j}$ for $i \neq j$. Then there exists a $g \in R$ such that $g = g_i$ in $R_{f_i}$.

The previous exercise shows that the localisations form something called a Sheaf on the distinguished basis and so a sheaf on $\text{Spec}(R)$. The localisations $R_f$ should be viewed as the ring of functions over $D(f)$. 