**Week 10**

**Definition 1.** Given a finite extension $E/F$, we can view an element $x \in E$ as a $F$-linear transformation on $E$ via multiplication. The determinant of this transformation is called the norm and we denote it by $N_{E/F}(x)$. The trace is called the trace and denoted $tr_{E/F}(x)$.

**Question 1.** Given a simple algebraic extension $F(\alpha)/F$ with $\alpha$ having minimal polynomial $x^n + a_{n-1}x^{n-1} + \cdots + a_0$ over $F$. Prove that

$$tr_{F(\alpha)/F}(\alpha) = -a_{n-1} \quad \text{and} \quad N_{F(\alpha)/F}(\alpha) = (-1)^n a_0.$$

**Question 2.** Let $E/F$ be a finite extension and $\alpha \in E$. Assume that $[E : F(\alpha)] = r$. Prove that

$$tr_{E/F}(\alpha) = rtr_{F(\alpha)/F}(\alpha) \quad \text{and} \quad N_{E/F}(\alpha) = N_{F(\alpha)/F}(\alpha)^r.$$

**Question 3.** Suppose $E/F$ is a finite Galois extension. Show that for $\alpha \in E$ we have that

$$N_{E/F}(\alpha) = \prod_{\sigma \in Gal(E/F)} \sigma(\alpha), \quad tr_{E/F}(\alpha) = \sum_{\sigma \in Gal(E/F)} \sigma(\alpha).$$

At this point it is a good idea to read Keith Conrad’s notes ”Linear independence of characters” for a good introduction to Hilbert’s 90 theorem and basic kummer theory where the trace and norm find applications. Originally I wanted to go through this but realised it would take too long.

**Question 4.** Suppose we have an irreducible polynomial $f \in \mathbb{Q}[X]$ of degree $p$, where $p$ is prime. Assume that $f$ has $p-2$ real roots and 2 nonreal, complex roots. What is that Galois group of $f$?

**Question 5.** (Fall ’16) Let $f \in F[X]$ be an irreducible polynomial of prime degree over a field $F$, and let $K/F$ be the splitting field of $f$. Prove there is an element in the Galois group of $K/F$ permuting cyclically all the roots of $f$ in $K$.

**Question 6.** (Spring ’18) let $\alpha \in \mathbb{C}$ and suppose that $[\mathbb{Q}(\alpha) : \mathbb{Q}]$ is finite and coprime to $n!$ for some integer $n > 0$. Show that $\mathbb{Q}(\alpha^n) = \mathbb{Q}(\alpha)$.