Let $R$ be a unital commutative ring. There is a topological space that we can view the ring $R$ as a natural set of functions on. We will construct it in the first question, but first some definitions:

**Definition 1.** A proper ideal $I \subset R$ is prime if whenever $ab \in I$ then either $a \in I$ or $b \in I$.

**Definition 2.** The spectrum of a ring $R$, denoted by $\text{Spec}(R)$, is the set of all prime ideals.

**Definition 3.** The radical of an ideal $a$ is given by $r(a) = \{ r \in R | r^n \in a \text{ for some } n \}$.

**Question 1.** Let $R$ be a ring. For any subset $S \subseteq R$ let

$$V(S) = \{ x \in \text{Spec}(R) | S \subseteq x \}.$$ 

These are sometimes called the vanishing sets. Show the following:

(a) Let $E \subseteq R$ and $a$ the ideal generated by $E$. Show that $V(E) = V(a) = V(r(a))$

(b) $V(0) = \text{Spec}(R)$ and $V(1) = \emptyset$.

(c) For an arbitrary collection of subset $E_i$ of $R$ we have that

$$V(\bigcup E_i) = \bigcap V(E_i)$$

(d) For two ideals $a, b$ we have $V(a \cap b) = V(a) \cup V(b)$

Observe that these show that the vanishing sets form the closed sets of a topology on $\text{Spec}(R)$ called the Zariski topology.

We can view elements $f \in R$ as functions on $\text{Spec}(R)$ by considering it’s value on $P \in \text{Spec}(R)$ as $f(P) = f \pmod{P}$. So the Vanishing sets are exactly the points it is zero. Unlike normal functions, a function being everywhere zero is not enough for it to be zero as the next exercise shows us.

**Question 2.** Show that $f \in R$ is everywhere zero on $\text{Spec}(R)$ if and only if $f$ is nilpotent. *Hint:* Consider the multiplicative set $S$ which conatins 1 and all powers of some element $f$. One can use zorn’s to show that the set of all ideals not intersecting $S$ must have a maximal and one can then show this is prime.

**Question 3.** Given a ring $R$ and an ideal $a$. How are the spectrums $\text{Spec}(R)$ and $\text{Spec}(R/a)$ related? What about $\text{Spec}(R)$ and $\text{Spec}(R/N)$ where $N$ is the nilradical(set of all nilpotents)?

**Question 4.** Question 2 can be generalized into what’s sometimes called the nullstellensatz for general rings. For a subset $S \subseteq \text{Spec}(R)$ let

$$I(S) = \cap_{P \in S} P$$

i.e, all the elements that vanish on $S$. Show that

$$VIV(J) = V(r(J))$$

for any ideal $J$. Moreover, show that $V$ and $I$ form a bijective correspondence between closed subsets and radical ideals.

**Question 5.** Professor Merkurjev asked me to explain the following, which I feel is a nice little exercise:

Show that a ring $R$ is a product of $n$ rings if and only if there exists $n$ central orthogonal idempotents that sum to 1. (that is central elements $e_i$ such that $e_i^2 = e_i$ and $e_ie_j = 0$ for $i \neq j$).