Question 1

Let \( n \) be the order of 1 in \( F \). (Abelian additive structure). \( n \) must be prime since if \( n = km \), then \( 0 = n \cdot 1 = (km) \cdot 1 = (k \cdot 1)(m \cdot 1) \) and so we have zero division which can't happen as \( F \) a field.

\( n \) is called the characteristic of \( F \), usually denoted \( \chi(F) \).

Now, suppose \( p = \chi(F) \) and there exist another distinct prime \( q \) s.t. \( q \parallel F \).

Then by Cauchy's Lemma, there exists a \( x \in F \) \( x \neq 0 \) s.t. \( q \cdot x = 0 \). We also have that \( p \cdot x = (p \cdot 1) \cdot x = 0 \cdot x = 0 \). By Bezout's Lemma, as \( p, q \) distinct primes, there exist integers \( a, b \in \mathbb{Z} \) s.t. \( ap + bq = 1 \). Then \( x = 1 \cdot x = (a \cdot p) \cdot x + (b \cdot q) \cdot x = 0 \) a con trichion.
Question 2

(1)⇒(2)

We have a ring map $\mathbb{Z}[x_1, ..., x_n] \rightarrow F$ given by mapping $x_i \mapsto x_i$.

This factors to an injective map by 1st isomorphism:

$\overline{\phi}: \mathbb{Z}[x_1, ..., x_n]/\ker \phi \rightarrow F$.

Since this is injective, $\mathbb{Z}[x_1, ..., x_n]/\ker \phi$ is an IDP and so $\ker \phi$ is prime.

The universal property of localisation of rings gives

$$\begin{array}{ccc}
\Phi & \Phi \\
\downarrow & \downarrow \\
\mathbb{Z}[x_1, ..., x_n]/\ker \phi & F
\end{array}$$

Now, $\phi$ is an injection, since it is an integral domain and so $\overline{\phi}$ is injective. Moreover, as the $d_i$ generate $F$, it is surjective.
Suppose we have an isomorphism:

$$\phi: K(\mathbb{Z}[x_1, \ldots, x_n]/\mathfrak{p}) \to F.$$ 

Let $A = \mathbb{Z}[x_1, \ldots, x_n]/\mathfrak{p}$ and $\bar{x}_i$ the image of $x_i$ in $A$. I claim that $\bar{x}_1, \ldots, \bar{x}_n$ generate $K(A)$ as a field.

**Question 3**

Let $z = \cos \left( \frac{a}{b} \pi \right) + i \sin \left( \frac{a}{b} \pi \right)$

Then $z^b = \pm 1$ and so $z$ algebraic over $\mathbb{Q}$.

Then $\cos \left( \frac{a}{b} \pi \right) = \frac{z + z^{-1}}{2}$ and so algebraic over $\mathbb{Q}$.

**Note:** In general, if $\alpha$ algebraic over field $F$, then every element in $F(\alpha)$ algebraic over $F$. Since $|\#(\alpha)|$ is finite.
Question 9

(1) $\Rightarrow$ (2) clear

(2) $\Rightarrow$ (3) Take $C = A[x]$.

Moreover, since $A[x] \subseteq C$, this ensures $M$ faithful.

(3) $\Rightarrow$ (4) Take $M = C$. Since $A[x] \subseteq C$, this is enough to ensure that $x C \subseteq C$ by looking at the generator of $C$ (as $A$-module).

(4) $\Rightarrow$ (1) By generalized Cayley Hamilton, we have

\[ x^n + a_{n-1} x^{n-1} + \ldots + a_1 x + a_0 = 0 \quad \text{in} \quad \text{Hom}(M) \]

Since faithful, we have $x^n + a_{n-1} x^{n-1} + \ldots + a_1 x + a_0 = 0$ in $A[x]$. 
Question 5.

Suppose $\phi: \mathbb{R} \to \mathbb{R}$ is an automorphism.

If $x \in \mathbb{R}$ is positive, then $\sqrt{x}$ positive
and $\phi(x) = \phi((\sqrt{x})^2) > 0$. Hence $x > 0 \Rightarrow \phi(x) > 0$.

Now, if $x < y$, then $\phi(y-x) > 0 \Rightarrow \phi(x) < \phi(y)$.

If $x = \frac{a}{b}$ for some $a, b \in \mathbb{Z}$, then $\phi(x) = \phi\left(\frac{a}{b}\right) = \frac{\phi(a)}{\phi(b)}$

$= \frac{a}{b}$ since $\phi$ is fixed on $\mathbb{Z}$.

Hence $\phi\left(\frac{a}{b}\right) = \frac{a}{b}$.

Now, let $x \in \mathbb{R}$.

Then there exists a rational $q$, s.t. $x < q < \phi(x)$
and as $x < q$, $\phi(x) < \phi(q) = q$. This gives a
contradiction. Hence $\phi(x) = x$. We get a
similar contradiction from $\phi(x) < x$ and so
we conclude $\phi(x) = x$. 