Q1.

I'm going to look at right ideals instead as I don't want to deal with row vectors. The idea is that every right ideal of $M_n(R)$ is of the form $[P \ldots P]$ where $P \in R^n$ is a right submodule (take $R^n$ to be column vectors).

Now, suppose $I \leq M_n(R)$ is a right ideal. By acting on the right by diagonal matrices, we see that each column is a submodule of $R^n$. By acting via permut. matrices, we see that the columns must be the same submodule.

Hence, let $C(I)$ be the submodule given by a column. Let $S(P) = [P \ldots P]$ for some $P \in R^n$ submodule. It is clear we have $CS = \text{id}_{C(I)} \forall \text{column}$ and $SC = \text{id}_{C(I)} + P \cdot (e)$. Hence these are inverses.

The left ideal situation is the same except we take row instead.

The two sided ideals must then be the same two sided ideal of $R$ in each entry.
Assume \( I \) is nonzero. Any \( \mathbb{Z} \)-submodule of \( \mathbb{Z}^n \) is free of rank \( k \leq n \). In particular, they are in the form \( \mathbb{Z}v_1 + \cdots + \mathbb{Z}v_k \), where \( v_i \in \mathbb{Z}^n \) (column vectors).

From the previous question, we then conclude that any left ideal \( I \subseteq M_n(\mathbb{Z}) \) is of the form

\[
I = \sum_{i=1}^{k} (\mathbb{Z}v_1 \mathbb{Z}v_i \cdots \mathbb{Z}v_k)^T
\]

Let \( v_{ij} \) be the \( j \)th component of \( v_i \).

Then \( I \mathbb{Z}^n = \sum_{i=1}^{k} (\mathbb{Z}v_1 \cdots \mathbb{Z}v_i)^T \mathbb{Z}^n \)

\[
= \sum_{i=1}^{k} \left( \begin{array}{c}
\mathbb{Z} \gcd_j(v_{ij}) \\
\mathbb{Z} \gcd_j(v_{ij}) \\
\vdots \\
\mathbb{Z} \gcd_j(v_{ij}) \\
\end{array} \right)
\]

\[
= \left( \begin{array}{c}
\mathbb{Z} \gcd_j(v_{ij}) \\
\mathbb{Z} \gcd_j(v_{ij}) \\
\vdots \\
\mathbb{Z} \gcd_j(v_{ij}) \\
\end{array} \right)
\]

\[
= \gcd_{ij}(v_{ij}) \mathbb{Z}^n.
\]

Since \( I \) is nonzero, \( \gcd_{ij}(v_{ij}) \neq 0 \) and so

\[
\mathbb{Z}^n / \gcd_{ij}(v_{ij}) \mathbb{Z}^n \cong \mathbb{Z} / \gcd_{ij}(v_{ij}) \mathbb{Z} \] is finite.
Q3

We show $\alpha$ is finitely generated for $\mathbb{N}$.

Since $a b = A$, there exist $a, b \in \mathbb{N}$ with $\sum_{i=1}^{n} a_i b_i = 1$.

Then for all $a \in \mathbb{N}$, we have:

$$a = a(\sum_{i=1}^{n} a_i b_i) = \sum_{i=1}^{n} a_i b_i a_i$$

and $a \cdot \alpha = \alpha$, for all $i$. Hence $\sum_{i=1}^{n} a_i b_i$ gives a generating set for $\alpha$.

To show $\alpha$ is projective, we show it is a direct summand of $A^n$. Since $\mathcal{S}$ is a simple module,

$$0 \rightarrow \ker f \rightarrow A^n \xrightarrow{f} \alpha \rightarrow 0$$

define a $A$-hom $g: \alpha \rightarrow A^n$ by $g(a) = \sum_{i=1}^{n} a_i b_i e_i$.

where $e_i$ is the idempotent that corresponds to the $i$th factor of $A^n$.

We then have $f g(a) = \sum_{i=1}^{n} a_i b_i a_i = a$, and so the sequence splits and we are done.
Q4

Let $x_1, \ldots, x_n$ be a minimal set of generators for $M$.

There exists a SES of $R$-modules

$$0 \rightarrow N \rightarrow R^n \rightarrow M \rightarrow 0$$

Since $M$ projective, $R^n = M \oplus N$.

Apply the functor $- \otimes_R M$, we get

$$(R/m)^n = (M/mM) \oplus (N/mN)$$

$R/m$ is a field, and any basis of $M/mM$ pull back to a generating set of $M$ by Nakayama.

Hence by minimality and dimension, $N/mN = 0$.

Therefore, we conclude by Nakayama $N = 0$ and so $M = R^n$. 

\[\text{\$}\]
Q5.

Thus $u$ essentially the Artin-Tate Lemma.

Let $x_1, \ldots, x_n$ be the generators of $A$ as an $R$-algebra and $y_1, \ldots, y_n$ the generators of $A$ as a $B$-module.

Then $x_i = \sum_{j=1}^{n} b_{ij} y_j$ for some $b_{ij} \in B$

and $y_j y_k = \sum_{k=1}^{n} b_{kij} y_k$ for some $b_{kij} \in B$.

Let $B_0$ be the $R$-subalgebra of $A$ generated by $\{b_{ij} \mid b_{ij} \in B\}$. So $B_0$ is a finitely generated $R$-algebra and by Hilbert basis, $B_0$ is Noetherian as a ring.

(Note: $B_0 \subseteq B \subseteq \mathbb{Z}(A)$ and so everything is commutative here)

Observe that via the relations $(*), A$ is generated as a $B_0$-module with generators $y_i$, and so by Hilbert basis, $A$ is a Noetherian $B_0$-module.

Since $B$ is a $B_0$-submodule of $A$, it is finitely generated as a $B_0$-submodule and as $B_0$ is a finitely generated $R$-algebra, we conclude $B$ is a finitely generated $R$-algebra.
Question 6

Solution 1 (by Ben Spitz)

Let \( \{x_1, \ldots, x_m\} \) be a finite set of elements of \( B \) that generate \( B/I \) over \( A \). Since \( B \) is Noetherian, \( I \) is f.g. as a \( B \)-module. Let \( \{y_1, \ldots, y_n\} \) be generators of \( I \) as a \( B \)-module.

Then \( B = I + \sum_{i=1}^{m} A x_i \), so enough to show \( I \) f.g. over \( A \).

\[
I = \sum_{j=1}^{n} B i_j = \sum_{j=1}^{n} \left( I + \sum_{i=1}^{m} A x_i \right) i_j = \sum_{j=1}^{n} I i_j + \sum_{j=1}^{n} \sum_{i=1}^{m} A x_i i_j
\]

so enough to show each \( I i_j \) f.g. over \( A \).

we repeat this argument and it's then enough to show \( I i_j \) f.g. for all \( i, j \) to show \( I \) f.g. over \( A \).

Then enough to show \( I i_j k_i \) f.g. over \( A 

etc... eventually this must be zero since \( I \) is nilpotent and so by induction we are done.
solution 2

After seeing Ben's solution, I realised the following observation is what I was missing to make my argument work:

Claim: If I have two rings $A \leq C$ s.t. $C$ is $fg$ as an $A$-module, then all $fg$ $C$-modules $M$ are also $fg$ $A$-modules.

Proof: we have $C = \sum_{i=1}^{n} Aa_i$ for some $a_i$.

If $M$ is $f$ $C$-module with generators $x_1, \ldots, x_m$.

Then $M = \sum_{i=1}^{m} Cx_i = \sum_{i=1}^{m} \sum_{j=1}^{n} Aa_j x_i$. Hence $a_j x_i$ generate $M$ as an $A$-module.

Now, since $B$ noeth., $I$ is $f$ $g$ and as each element nilpotent, we conclude $I^n = 0$ for some $N$.

we then have a filtration of $A$-modules:

$0 = I^N \subseteq I^{N-1} \subseteq \ldots \subseteq I^1 \subseteq I^0 = B$.

As $B$ noeth., each $I^k$ is $f$ $g$ $B$-module and so $I^k/I^{k+1}$ are $f$ $g$ $B/I$-modules. By the above claim, we conclude $I^k/I^{k+1}$ are $f$ $g$ $A$-modules. Since the filtration is of finite length with $fg$ quotients
we conclude B is a f.g. A-module.