Week 6

Chain complexes of R-modules

A chain complex $C_\cdot$ of $R$-modules is a family $\{C_n\}_{n \in \mathbb{Z}}$ of $R$-modules, together with maps $d_i : C_n \to C_{n-1}$ such that $d_i \circ d_i = 0$.
- $d_i$ are called differentials
- $Z_n = \ker d_i$ the $n$-cycles
- $B_n = \text{im} d_{i+1}$ the $n$-boundaries
- $H_n(C_\cdot) = Z_n/B_n$ the $n$-th homology module of $C_\cdot$

- A morphism of chain complexes $f : C_\cdot \to D_\cdot$ is a family of maps $f_i : C_i \to D_i$ and $df_i = f_{i+1}d_i$.

This forms the category of $R$-module complexes, $\text{Ch}(\text{mod-}R)$.

Question 1:

(a) basis $(V_1, V_2 - V_1, V_3 - V_2, \ldots, V_n - V_{n-1})$.

Now, since $T$ is connected, there exists a path from $V_i$ to $V_j$, say by edge $e_{i1}, \ldots, e_{ip}$.

Then $d(e_{i1} + \ldots + e_{ip}) = V_i - V_j$
Now suppose we have $d(e_i) = V_k - V_j$.

If $V_k = V_j$, then $V_k - V_j = -(V_j - V_k)$.

If $V_k \neq V_j$, then $V_k - V_j = V_k - V_i - (V_j - V_i)$.

Hence, it follows that $d(\Sigma_r e_j) = \frac{1}{m} \sum_{j=2}^{n} V_j (V_j - V_1)$.

Hence $\operatorname{im} \phi = \operatorname{span} (V_2 - V_1, \ldots, V_n - V_1)$. Hence it follows that $H_0(C_\ast) = RV_1$ is free of rank 1.

Now, consider $R = \mathbb{Z}$. Hence $\ker \phi$ is free of rank $m - n + 1$ and so the general case follows.

Snake lemma: If we have exact rows:

$A \rightarrow B \rightarrow C \rightarrow 0$

Then exhibit long exact sequence:

ker $h$ \rightarrow ker $g$ \rightarrow ker $f$ \rightarrow coker $h$
Question 2

we have

\[
\begin{array}{c}
A_n/B_n(A_\cdot) \rightarrow B_n/B_n(B_\cdot) \rightarrow C_n/B_n(C_\cdot) \rightarrow 0 \\
A_\downarrow \quad d_\downarrow \quad d_\downarrow \\
0 \rightarrow Z_{n-1}(A_\cdot) \rightarrow Z_{n-1}(B_\cdot) \rightarrow Z_{n-1}(C_\cdot)
\end{array}
\]

if the rows are exact, then snake lemma gives us

\[
\begin{array}{c}
H_n(A_\cdot) \rightarrow H_n(B_\cdot) \rightarrow H_n(C_\cdot) \rightarrow H_{n-1}(A_\cdot) \\
\overbrace{H_{n-1}(B_\cdot) \rightarrow H_{n-1}(C_\cdot)}
\end{array}
\]

which is what we want. we can show the rows are exact via snake lemma.

\[
\begin{array}{c}
0 \rightarrow A_n \rightarrow B_n \rightarrow C_n \rightarrow 0 \\
\downarrow d \quad \downarrow d \quad \downarrow d \\
0 \rightarrow A_{n-1} \rightarrow B_{n-1} \rightarrow C_{n-1} \rightarrow 0
\end{array}
\]

gives

\[
\begin{array}{c}
0 \rightarrow Z_n(A_\cdot) \rightarrow Z_n(B_\cdot) \rightarrow Z_n(C_\cdot) \\
\overbrace{Z_{n-1}(A_\cdot) \rightarrow Z_{n-1}(B_\cdot) \rightarrow Z_{n-1}(C_\cdot)}
\end{array}
\]

exact.
Tor functors.

For $R$-module $M$, a free resolution is an exact sequence

$$
\cdots \to R^{\oplus_2} \to R^{\oplus_1} \to R^{\oplus_0} \to M \to 0
$$

throw $M$ away, and tensor with $R$-module $N$ to obtain chain complex $M \otimes_R N$:

$$
\cdots \to N^{\oplus_2} \to N^{\oplus_1} \to N^{\oplus_0} \to 0.
$$

(tensors commute with colimit as left adjoints)

define $\text{Tor}_i^R(M, N) := H_i(M \otimes_R N)$

the does not depend on choice of free resolution.

Question 3:

we have $N^{\oplus_2} \to N^{\oplus_2} \to M \otimes_R N \to 0$ exact

since $- \otimes_R N$ is right exact.

hence $\text{Tor}_0^R(M, N) = M \otimes_R N$. 

Q4.

Essentially, we have

\[ 0 \rightarrow R^{\oplus s'} \rightarrow R^{\oplus s} \rightarrow R^{\oplus s''} \rightarrow 0 \]
\[ \downarrow \quad \downarrow \quad \downarrow \]
\[ 0 \rightarrow M' \rightarrow M \rightarrow M'' \rightarrow 0 \]

after applying $\otimes N$, the rows of the resolution stay exact since free modules are projective. This gives us a SES

\[ 0 \rightarrow N^{\oplus s'} \rightarrow N^{\oplus s} \rightarrow N^{\oplus s''} \rightarrow 0 \]

and the corresponding LES in homology is what we want.

Question 5.

Submodules of a free module are free over a PID.

Hence we have a resolution

\[ 0 \rightarrow R^{\oplus i} \rightarrow R^{\oplus i} \rightarrow M \rightarrow 0 \]

and the rest follows
Question 6

\[ 0 \rightarrow (r) \rightarrow R \rightarrow R/(r) \rightarrow 0 \quad (\text{as } R\text{-module}) \]

given \( n \)

\[ \text{Tor}_1^R(R, N) \rightarrow \text{Tor}_1^R(R/(r), N) \rightarrow (n) \otimes_R N \rightarrow \]

\[ \rightarrow R \otimes_R N \rightarrow R/(r) \otimes_R N \rightarrow 0 \]

Note, \( \text{Tor}_1^R(R, N) = 0 \), and so \( \text{Tor}_1^R(R/(r), N) \)

is isomorphic to \( \ker((n) \otimes_R N \rightarrow R \otimes_R N) \) which

is exactly the \( r \)-torsion of \( n \).