Q1  a) \( V(y(a)) \subseteq V(a) \subseteq V(\sqrt{a}) \) clear. 

If \( P \) prime such that \( \sqrt{a} \in P \), then \( a \in \sqrt{a} = P \). 
so \( a \in P \). Also we must have \( \sqrt{a} \in P \) since suppose \( x \in \sqrt{a} \). Then \( x^k a \) for some \( n \). Then \( x^k a \in P \) so prime. Hence \( V(\sqrt{a}) \subseteq V(\sqrt{a}) \).

b) Every ideal contains \( 0 \), so \( \text{Spec}(R) = V(0) \). 
Since prime ideals are proper, none contain \( 1 \). \( \Rightarrow V(1) = \emptyset \).

c) \( P \in V(\mathbb{U}E_i) \iff \mathbb{U}E_i \in P \)

\( \iff E_i \in P \) for all \( i \)
\( \iff P \in \text{V}(E_i) \) for all \( i \).
\( \iff P \in \cap \text{V}(E_i) \).

d) \( V(a) \cup V(b) \subseteq V(\gcd(a,b)) \) clear.

Support \( P \in \text{V}(\gcd(a,b)) \), so \( \gcd(a,b) \in P \) and suppose \( a \not\in P \) and \( b \not\in P \). Let \( x \not\in P \), \( y \not\in P \).
Then \( xy \not\in \gcd(a,b) \) so \( xy \not\in P \Rightarrow x \not\in P \) or \( y \not\in P \). A contradiction.
Q2 If \( f \) nilpotent, then let \([p] \in \text{spec} R\). Since \( f^n = 0 \) e P for some \( n \), \( f \notin P \) and so \( f([p]) = 0 \). Hence everywhere zero.

Conversely, suppose \( f \) is not nilpotent so the suggested set \( S \) does not intersect \( 0 \).

Let \( I \) be the set of all ideals that don't intersect \( S \). It is not empty as \( (0) \) is an ideal and every chain has an upper bound under inclusion by taking unions. Hence \( I \) contains a maximal element \( \mathfrak{m} \).

Let \( m \) be this element, we use ideal quotients here. Suppose \( xy \in \mathfrak{m} \) but \( xy \notin \mathfrak{m} \). Then \( (\mathfrak{m}; x) \supseteq \mathfrak{m} \) and larger. Hence \( f^n (\mathfrak{m}; x) \supseteq (\mathfrak{m}; x) \) for some \( n \). Similarly, \( (\mathfrak{m}, f^n) \supseteq \mathfrak{m} \) and larger. Hence, \( \mathfrak{m} \in \mathfrak{m} \) such that \( f^n \mathfrak{m} \in \mathfrak{m} \) a contradiction.
Q3 \ Spec(R/a) \text{ is the subspace } V(a) \subseteq \Spec(R).

This follows via bijecton between ideals and quotient sets.

\ Spec(R) = \Spec(R/N)

Q4 It is enough to show that \( IV(J) = \sqrt{J} \).
That is, the intersection of all primes that contain \( J \) is its radical. This follows from Q2 and the bijection between ideals after quotienting by \( J \).

Now, let \( C \) be a closed set of \( \Spec R \).
So \( C = V(I) \) for some \( I \). Then we have
\[ IV(C) = IV(V(I)) = IV(\sqrt{I}) = I(C) \]

So \( IV(C) = I(C) \)

Any \( \sqrt{V(I)} = V(I) \) for radical \( I \).

This gives us a bijection as required. Note \( I(C) \) is radical, \( V(I) \) closed so \( IV \) and \( VI \) coincide.
Note: $R$ not nec. commutative in this $Q$.

As if $R = \bigoplus_{i=1}^{n} R_i$. Then let $e_i$ be the identity of $R_i$.

Then $e_i$ are the central idempotents that sum to 1.

Conversely, we let $R_i = Re_i$ and since $e_i$ central, this is a subring. Let $\bigoplus_{i=1}^{n} R_i \to R$ be the map determined by the inclusion $R_i \to R$ (sum is coproduct). Since $\sum e_i = 1$, this is easily seen to be surjective. Note that if $re_i = se_j$,

(i) then apply $e_j$ on left gives $re_i e_j = se_j^2$.

$\Rightarrow 0 = se_j$ as orthogonal. Essentially the same argument shows that $(R_i + R_i + \ldots + R_i + R_i + \ldots + R_n) \cap R_i = 0$ for all $i$ and so $\bigoplus_{i=1}^{n} R_i \to R$ is injective.