Crash course on representation theory

**Question 1.** For each of the following representations $\rho : G \rightarrow GL(V)$, describe what the matrices $\rho(g)$ look like.

(a) $V$ is a one dimensional vector space. Note, $G$ is finite.

(b) Let $X$ be a $G$-set. Then the action of $G$ on $X$ extends linearly to an action on $F(X)$. Take $V = F(X)$ and $\rho(g)$ is given by $\rho(g)(x) = g \cdot x$ on the basis $x \in X$.

**Question 2.** We will prove Maschke’s theorem: Let $k$ be a field such that $|G|$ does not divide the order of $k$. Then any $k$-representation $\rho : G \rightarrow GL(V)$ is completely reducible. (If you don’t know what characteristic is, take $k$ to be $\mathbb{R}$ or $\mathbb{C}$)

(a) Let $W$ be a $G$-invariant subspace of $V$ and $\pi : V \rightarrow V$ any projection onto $W$. Define the following map $\pi' : V \rightarrow W$ by

$$\pi'(v) = \frac{1}{|G|} \sum_{g \in G} \rho(g) \pi(\rho(g^{-1})v).$$

Show that $\pi'$ is also a projection onto $W$. A projection on $W$ is a linear transformation such that $\pi^2 = \pi$ and $\text{im}(\pi) = W$.

(b) Show that $\pi'$ is a $G$-linear map.

(c) Show that there exists a $G$-invariant subspace $W'$ such that $W \oplus W' = V$. That is, $V$ is completely reducible.

**Question 3.** Let $G = \{1, x, x^2\}$ be the cyclic group of order 3 and define a complex representation $\rho : G \rightarrow GL(\mathbb{C}^3)$ by $\rho(x)(z_1, z_2, z_3) = (z_2, z_3, z_1)$. Find the irreducible $G$-invariant subspaces of $\mathbb{C}^3$. (There will only be three of them for reasons, what’s special about linear operators over complex numbers?)

**Question 4.** We will prove Schur’s lemma (Some version of it at least). Consider a irreducible complex representation $\rho : G \rightarrow V$. Let $\phi : V \rightarrow V$ be a $G$-linear map. Show there exists a scalar $\lambda \in \mathbb{C}$ such that $\phi(v) = \lambda v$ for all $v \in V$.

**Question 5.** Let $G$ be an abelian group. Prove all irreducible complex representations of $G$ are one-dimensional.