Nilpotent and Solvable Groups

We will assume that all groups $G$ are finite. There are a bunch of equivalent definitions for nilpotent groups, but we will use this one.

**Definition.** A group $G$ is nilpotent if it has a finite central series. That is, a normal series
\[ \{e\} = G_0 \triangleleft G_1 \triangleleft \cdots \triangleleft G_n = G \]
such that $G_{i+1}/G_i \leq Z(G/G_i)$ or equivalently $[G, G_{i+1}] \leq G_i$.

Note, a normal series of $G$ is one where each subgroup is normal in $G$. Compare this to a subnormal series where they are just normal in the next consecutive group. We have the following theorem

**Theorem 1.** The following a equivalent
(a) Group $G$ is nilpotent
(b) If $H$ is a proper subgroup of $G$, then $H$ is a proper subgroup of $N_G(H)$. That is, normalizers grow.
(c) Every Sylow subgroup is normal.
(d) $G$ is the direct product of Sylow subgroups.
(e) If $d$ divides $|G|$, then $G$ contains a normal subgroup of order $d$.

While we won’t go through the proof of this, it’s a good exercise to prove this theorem.

**Question 1.** (Why the heck are they called nilpotent groups anyway?) Consider the map $\text{adj}_g : G \to G$ given by $\text{adj}_g(x) = [g, x]$. This is called the adjoint map. Show that if $G$ is nilpotent, then there exists some $n$ such that $\text{adj}_g^n(x) = e$ for all $x, g \in G$.

**Question 2.** Consider the quaternion group
\[ Q_8 = \langle -1, i, j, k \mid i^2 = j^2 = k^2 = ijk = -1, (-1)^2 = 1 \rangle. \]
Is this nilpotent?

**Question 3.** Prove that $G$ is nilpotent if and only if $G/Z(G)$ is nilpotent.

**Question 4.** Prove that the dihedral group
\[ D_n = \langle a, b \mid a^n = b^2 = 1, bab = a^{-1} \rangle \]
is nilpotent if and only if $n = 2k$ for $k \in \mathbb{N}$.

**Definition.** A group $G$ is solvable if it has a subnormal series with each factor group abelian.

**Question 5.** The derived series of a group $G$ is the normal series
\[ G = G^{(0)} \triangleright G^{(1)} \triangleright \cdots \]
where $G^{(i+1)} = [G^{(i)}, G^{(i)}]$. Show that a group is solvable if and only if the derived series eventually terminates at $\{e\}$.
**Question 6.** Show that nilpotent groups are solvable. Can you think of an example of a solvable group that is not nilpotent?

**Question 7.** Show that all groups of order $< 60$ are solvable. *Hint: $A_5$ is the smallest simple nonabelian group.*