Yoneda lemma:

Let \( C \) be a locally small category and for \( A \in C \), \( h^A := \text{Hom}_C(A,-) \) and \( F : C \to \text{Set} \) be functors. Then there is a natural isomorphism

\[
\text{Nat}(h^A, F) \cong F(A)
\]

given by \( \alpha \mapsto \alpha_A(\text{id}_A) \).

Proof: exercise!

A representation is a natural isomorphism \( h^A \to F \), and so by Yoneda, these are associated to some element of \( F(A) \). What are they?

Definition: A universal element of the functor \( F : C \to \text{Set} \) is a pair \( (A, u) \) where \( A \in C \) and \( u \in F(A) \) such that for any other pair \( (B, v) \) where \( B \in C, v \in F(B) \), there exists a unique morphism \( \varphi : A \to B \) such that \( v = F(\varphi)(u) \).

\[
\begin{array}{ccc}
A & \xrightarrow{u} & F(A) \\
\downarrow{\varphi} & & \downarrow{F(\varphi)} \\
B & \xrightarrow{v} & F(B)
\end{array}
\]

Stupid example: Consider the functor \( h^Z : \text{Ab} \to \text{Set} \). Then \( (Z, \text{id}_Z) \) is a universal element for this functor since suppose we have another pair \( (G, \varphi) \). Suppose \( \varphi \in \text{Hom}_{\text{Ab}}(Z, G) \).
we have another pair \((G, \eta)\). \(\eta \in \text{Hom}_{\text{Ab}}(\mathbb{Z}, G)\) and we have \(\eta = \eta \circ \text{id}_{\mathbb{Z}} = h^2(\eta)(\text{id}_{\mathbb{Z}}).

The point is, representations correspond to universal elements.

**Lemma**: Let \(F : C \rightarrow \text{Set}\) be a functor, then \(\text{is a bijection}
\begin{array}{ccc}
\{\text{natural transformation}\} & \rightarrow & \{\text{universal elements}\} \\
h^A & \rightarrow & F \\
& \alpha & \\
\end{array}
\)

and this bijection is given by the map \(\alpha \mapsto (A, \alpha_A(id_A))\)

\(\alpha\) in the Yoneda Lemma.

**Proof**: Suppose \(\alpha : h^A \rightarrow F\) is a natural transformation. Then for each \(\varphi \in \text{Hom}(A, B)\), by naturality, we have
\[
\begin{array}{ccc}
\text{Hom}(A, A) & \rightarrow & F(A) \\
\downarrow h^A(\varphi) & & \downarrow F(\varphi). \\
\text{Hom}(A, B) & \rightarrow & F(B) \\
\alpha & \rightarrow & \alpha_B
\end{array}
\]

If \(\alpha\) is an isomorphism, then \(\alpha_B\) is a bijection and so for any \(\psi \in F(B)\), there exists \(\varphi' \in \text{Hom}(A, B)\) such that \(\alpha_B(\varphi') = \psi\). By (1), we have \(\psi = \alpha_B(\varphi') = \alpha_B(\varphi' \circ \text{id}_A) = F(\varphi')(\alpha_A(id_A))\). Hence \((A, \alpha_A(id_A))\) is a universal element.

Conversely, suppose \((A, \alpha_A(id_A))\) is a universal element. Then for any pair \((B, \psi)\), there exists a unique \(\varphi : A \rightarrow B\) such that \(\psi = F(\varphi)(\alpha_A(id_A))\)
\[\alpha_B(\psi) \text{ by (1)}\]
- \alpha_{B}(Y) \quad \text{by (1)}

Hence, by uniqueness, \( \alpha_{B} \) is a bijection and as \( B \) arbitrary, \( \alpha \) is an isomorphism.

**Useful Result:** Given a locally small category \( C \) and \( A \in C \), the functor \( \text{Hom}_{C}(A, -) \) preserves limits.

**Question 1.** Let \( G \) (not necessarily abelian) be a group and consider the functor \( F: \text{Ab} \to \text{Set} \) given by \( F(-) = \text{Hom}_{\text{Grp}}(G, -) \). Is this functor representable?

Let \( G^{ab} = G/[G, G] \). I claim \( F \) is representable by \( G^{ab} \). For any abelian group \( H \), and morphism \( \varphi: G \to H \), this uniquely factors through the quotient \( q: G \to G^{ab} \).

\[
\begin{array}{ccc}
G & \xrightarrow{\varphi} & H \\
\downarrow q & & \downarrow q_{H} \\
G^{ab} & \xrightarrow{\text{quotient map}} & H
\end{array}
\]

Hence, let \( \alpha_{H}: \text{Hom}(G^{ab}, H) \to \text{Hom}(G, H) \), \( \alpha_{H}(h) = h \circ q_{H} \) is a bijection and defines a natural isomorphism. Hence \( F \) is representable.

**Note:** \( q: G \to G^{ab} \) is a universal element for \( F \).
Question 2. Consider the functor \( F : \text{Grp} \to \text{Set} \) given by \( F(G) = \{ g \in G \mid g^2 = e \} \). Is this functor representable? What about the functor \( \text{tor} : \text{Grp} \to \text{Set} \) given by \( G[\text{tor}] = \{ g \in G \mid g^n = e \text{ for some } n \} \)?

Let \( \alpha : \text{Hom}_{\text{Grp}}(\mathbb{Z}/2\mathbb{Z}, G) \to F(G) \)

\[ \varphi \mapsto \varphi(1) \]

this is easily seen to be bijective and define a natural transformation. Hence representable. Note, the universal element \((\mathbb{Z}/2\mathbb{Z}, 1)\).

Now, suppose \( \text{tor} \) was representable. Then there exists a universal element \((G, g)\) such that there exists a \( \varphi : G \to \mathbb{Z}/n\mathbb{Z} \) such that \( F(\varphi) : G[\text{tor}] \to \mathbb{Z}/n\mathbb{Z} \) with \( F(\varphi)(g) = 1 \) if \( \varphi(g) = 1 \). Hence \( n \) must divide the order of \( g \). Since, in arbitrary, no such universal element exists.

Question 3. Fix nonempty sets \( Y, Z \) and consider the contravariant functor \( F : \text{Set} \to \text{Set} \) given by \( F(X) = \text{hom}(X, Y) \sqcup \text{hom}(X, Z) \). Is this functor representable?

Suppose it is representable. Then \( F(-) \cong \text{Hom}(-, S) \) for some \( S \in \text{Set} \). Then

\[ \text{Hom}(\ast \times 3, S) \cong \text{Hom}(\ast \times 3, X) \sqcup \text{Hom}(\ast \times 3, Y) \]

\[ S = X \cup Y. \]

But then \( F(\ast \times 2) \) are hom with image either completely in \( X \) or \( Y \). While \( \text{Hom}(\ast \times 2, S) \) can be either.
in $X$ or $Y$. While $\text{Hom}(\mathbb{1}, \mathbb{2}, s)$ can be either.

Question 4. Prove that representable functors preserve limits.

Suppose $\mathcal{C}$ is locally small and let $A \in \mathcal{C}$. Let $J$ be index set and $F : J \to \mathcal{C}$ a diagram. Suppose we have a limiting cone $\nu : \lim F \to F$. Apply $\text{Hom}(A, -)$ which gives us a cone.

$$\text{Hom}_\mathcal{C}(A, \lim F) \xrightarrow{\nu_i} \text{Hom}(A, F_i)$$

$$\Rightarrow \quad X \xrightarrow{\tau_i} \text{Hom}(A, F_i)$$

Suppose we have a cone $\tau : X \to \text{Hom}(A, F_i)$. Then for each $\alpha \in X$, $\tau_i \alpha : A \to F_i$ and $\nu i$ a cone. Hence exist unique $h : A \to \lim F$ such that $\tau_i \alpha = \nu_i h \alpha$ for all $i \in J$.

This gives a unique map $h : X \to \lim F$ such that $\tau_i = \nu_i h$.

Hence $\lim (A, F_i) \simeq \text{Hom}(A, \lim F)$.