Week 7

Burnside’s Lemma

Given a G-set X, we have
\[ \left| X/G \right| = \frac{1}{\left| G \right|} \sum_{g \in G} \left| X^g \right| . \]

Useful corollary: Given a transitive G-set X with at least 2 elements, then exists an element \( g \in G \) such that \( X^g = \emptyset \).

Proof: Suppose we had \( \left| X^g \right| \geq 1 \) for all \( g \in G \). Then
\[ \frac{1}{\left| G \right|} \sum_{g \in G} \left| X^g \right| \geq \frac{1}{\left| G \right|} \left( \sum_{g \in G} \left| X^g \right| \right) \]
\[ \geq \frac{1}{\left| G \right|} \left( \left| X \right| \left( 2 + \left| G \right| \right) \right) \]
\[ > 1 = \left| X/G \right| \] as transitive.

Hence we have a contradiction.

Question 1. Let \( G \) be a finite group and \( H < G \) a proper subgroup. Show that \( \cup_{g \in G} gHg^{-1} \neq G \). Bonus question: What happens when \( G \) is infinite?
Let $\Delta$ be the set of all subgroups of $G$ conjugate to $H$. Then $\Delta$ is a transitive $G$-set and suppose $Ug \leq gHg^{-1} = G$. Then for all $g \in G$, there exists $h \in G$ such that $g = hHh^{-1}$. Then we have $g = hHh^{-1} = hHh^{-1}$ and so $|\Delta| \geq 1$ for all $g \in G$. This contradicts the above corollary.

When $G$ is infinite, this is not true. Take $T \leq \text{GL}_n(\mathbb{C})$ subgroup of upper triangular matrices. The Jordan-normal form implies that $T$'s conjugates cover $\text{GL}_n(\mathbb{C})$.

**Question 2.** Show that for a finite group $G$ and proper subgroup $H$, there exists a conjugacy class of $G$ that does not intersect $H$.

Let $g \in G \setminus \bigcup \gamma \leq gHg^{-1}$. This exists by the previous problem. Suppose there exists a $h \in G$ such that $h \in H \implies g = hHh^{-1}$. But this cannot be true. Hence $C_g \cap H = \emptyset$.

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Use of Burnside in counting.

Suppose we want to count the number of ways
Suppose we want to count the number of ways we can colour the edges of a square with two colours up to rotation.

\[ \begin{array}{c c c c}
 & & \text{are considered the same} \\
\end{array} \]

Let \( \mathcal{X} \) be the set of all configurations of colourings and \( \sigma \in S_4 \) acts on this set by \( 90^\circ \) clockwise rotation. Hence we want to count the number of orbits \( |\mathcal{X}/S_4| \), so we can use Burnside for this, and instead count the number of configurations that are fixed by elements of \( S_4 \).

<table>
<thead>
<tr>
<th>element of ( S_4 )</th>
<th>number fixed</th>
<th>squares left fixed</th>
</tr>
</thead>
<tbody>
<tr>
<td>( e )</td>
<td>16</td>
<td>all</td>
</tr>
<tr>
<td>( \sigma )</td>
<td>2</td>
<td>all edges same colour.</td>
</tr>
<tr>
<td>( \sigma^2 )</td>
<td>9</td>
<td>opposite edges same colour</td>
</tr>
<tr>
<td>( \sigma^3 )</td>
<td>2</td>
<td>all edges same colour</td>
</tr>
</tbody>
</table>

\[ |\mathcal{X}/G| = \frac{1}{4} (16 + 2 + 9 + 2) = \frac{29}{4} = 6. \]

\( \) (colourings up to rotation).
Question 3. (Useful results for counting) Let $G$ be a group and $X$ a $G$-set. For $x, y \in G$, show that

(a) if $x$ and $y$ are conjugate then $|X^x| = |X^y|$.
(b) if $x$ and $y$ generate the same subgroup then $|X^x| = |X^y|$.

(a) If $a \in X^y$ and $x = hyh^{-1}$. Then $ha \in X^x$ since $x \cdot ha = hyh^{-1} \cdot ha = ha$. Hence we get a map $f_h : X^y \to X^x$ by $a \mapsto ha$ which has an inverse by $h^{-1}$! Therefore a bijection and so $|X^x| = |X^y|$.

(b) If $a \in X^y$, then as $y = x^n$ for some $n$, so $a \in X^y \Rightarrow X^x \leq X^y$ and by symmetry we get equality.

Question 4. Use Burnside’s lemma to answer the following counting problem. Let $n$ be an even number and suppose we have $n$ indistinguishable balls and put them into 3 indistinguishable jars. How many ways can we do this?

$$\Sigma = \{ (a, b, c) \in \mathbb{Z}^3 \geq 0 \mid ab + bc = n^2 \}$$ 
and we have

$S_3 \cong \Sigma$ given by $(12) \cdot (a, b, c) = (b, a, c)$
and $(123) \cdot (a, b, c) = (c, a, b)$.

We want to count $\Sigma / S_3$ and so by previous question we want to figure out fixed points for a representative for each conjugacy class.
A representative for each conjugacy class

<table>
<thead>
<tr>
<th>Element in conjugacy</th>
<th># fixed</th>
<th>what gets fixed</th>
</tr>
</thead>
<tbody>
<tr>
<td>( e )</td>
<td>( \binom{n+2}{2} )</td>
<td>all of them</td>
</tr>
<tr>
<td>(12)</td>
<td>( \frac{n}{2}+1 )</td>
<td>( (a,a,b) ) for ( a = 0, \ldots, \frac{n}{2} )</td>
</tr>
<tr>
<td>(123)</td>
<td>0</td>
<td>since ( n ) even, can't have one of form ((a,a,a)).</td>
</tr>
</tbody>
</table>

Hence Burnside gives us:

\[
\left| \frac{X}{S^3} \right| = \frac{1}{6} \left( \binom{n+2}{2} + 3 \left( \frac{n}{2}+1 \right) \right).
\]

**Question 5.**

(a) Let \( X \) be a finite \( G \)-set with \( |G| = p^n \) for some prime \( p \) and \( p \) does not divide \( |X| \). Show there exists an element \( x \in X \) such that \( gx = x \) for all \( g \in G \).

(b) Let \( V \) be a \( d \)-dimensional vector space over \( \mathbb{Z}_p \) and let \( G \subset GL_d(\mathbb{Z}_p) \) be a group such that \( |G| = p^n \). Show that there exists a nonzero vector \( v \in V \) such that \( g \cdot v = v \) for all \( g \in G \).

Note, you don't need Burnside to do these questions.

(a) Suppose no such element \( x \in X \) exists. Then the size of every orbit is divisible by \( p \) (all orbits are nontrivial and orbit-stabilizer implies that there order divides \( p^n \)).

Since \( |X| = \Sigma |\Theta_i| \) where \( \Theta_i \) are the orbits, we have \( |X| \equiv 0 \mod p \), but this is a contradiction.

(b) Take \( x = V \setminus \{e\} \). Then \( |X| = p^d - 1 \equiv -1 \mod p \).
(b). Take $\overline{X} = V \setminus \{0\}$. Then $|\overline{X}| = p^{d-1} \equiv -1 \mod p$. Hence apply the previous part.

**Question 6.** Prove the Frattini argument: Let $G$ be a finite group and $H \triangleleft G$. Suppose $P$ is a Sylow $p$-subgroup of $H$. Then $G = HN_G(P)$.

Since $H \triangleleft G$, we have $G \triangleleft \text{Syl}_p(H)$ transitivity. But by Sylow Theorems, $H \triangleleft \text{Syl}_p(H)$ also acts transitively. In particular, for all $g \in G$, there exists $h \in H$ such that $h^g P g^{-h} = P$

$\Rightarrow h^g \in N_G(P) \Rightarrow g \in HN_G(P) \Rightarrow G = HN_G(P)$