Crash course on representation theory.

Let \( V \) be a \((d,1)\) vector space over field \( k \). A (linear) representation of finite group \( G \) is a homomorphism \( \rho : G \rightarrow GL_k(V) \). This is the same thing as a \( G \)-action on \( V \) such that \( g \cdot (v+w) = g \cdot v + g \cdot w \) and \( g \cdot (\lambda v) = \lambda (g \cdot v) \) for all \( \lambda \in k \).

**Question 1.** For each of the following representations \( \rho : G \rightarrow GL(V) \), describe what the matrices \( \rho(g) \) look like.

(a) \( V \) is a one dimensional vector space. Note, \( G \) is finite.

(b) Let \( X \) be a \( G \)-set. Then the action of \( G \) on \( X \) extends linearly to an action on \( F(X) \). Take \( V = F(X) \) and \( \rho(g) \) is given by \( \rho(g)(x) = g \cdot x \) on the basis \( x \in X \).

(a) Let \( g \in G \), then exists \( n \) s.t. \( g^n = 1 \). Then
\[
\rho(g)^n = \rho(g^n) = \rho(1) = I.
\]
Since \( V = k^x \) in this case, it follows \( \rho(g) \) are scalar matrices.

(b) \( X \) forms a basis after some ordering, and \( \rho(g) \) are the permutation matrices.

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A bunch of definitions:

1. If \( \rho : G \rightarrow GL(V) \) and \( \rho' : G \rightarrow GL(W) \) are two representations of \( G \), then \( \rho \oplus \rho' : G \rightarrow GL(V \oplus W) \) is the representation given by
The representation given by

\[(\rho \oplus \rho')(g) = \rho(g) + \rho'(g)\]

2. A subspace \(W \leq V\) is \(G\)-invariant if \(\forall g \in G\)
\[\rho(g)(W) \subseteq W.\]
Then \(\rho\) induces two representations.
one on \(W\) given by \(\rho\) restricted, and one on the quotient \(V/W\) given by \(\rho_{\| W}(g)(v+W) = \rho_{\| W}(g)(v) + W\).

3. A representation of \(G\), \(\rho: G \to GL(V)\) is irreducible if \(V\) has no nontrivial proper \(G\)-invariant subspace.
Otherwise reducible.

4. A representation \(\rho\) is completely reducible if for all \(G\)-invariant subspaces \(W\), there exists another \(G\)-invariant subspace \(W'\) such that \(W \oplus W' = V\).

5. Given two representations \(\rho_1: G \to GL(V), \rho_2: G \to GL(W)\)
a \(G\)-linear map \(\varphi: V \to W\) is a linear transformation such that \(\varphi(\rho_1(g)v) = \rho_2(g)\varphi(v)\) for all \(g \in G, v \in V\).
**Question 2.** We will prove Maschke’s theorem: Let \( k \) be a field such that \(|G|\) does not divide the order of \( k \). Then any \( k \)-representation \( \rho : G \to GL(V) \) is completely reducible. (If you don’t know what characteristic is, take \( k \) to be \( \mathbb{R} \) or \( \mathbb{C} \)).

(a) Let \( W \) be a \( G \)-invariant subspace of \( V \) and \( \pi : V \to V \) any projection onto \( W \). Define the following map \( \pi' : V \to W \) by

\[
\pi'(v) = \frac{1}{|G|} \sum_{g \in G} \rho(g)\pi(\rho(g^{-1})v).
\]

Show that \( \pi' \) is also a projection onto \( W \). A projection on \( W \) is a linear transformation such that \( \pi^2 = \pi \) and \( \text{im} (\pi) = W \).

(b) Show that \( \pi' \) is a \( G \)-linear map.

(c) Show that there exists a \( G \)-invariant subspace \( W' \) such that \( W \oplus W' = V \). That is, \( V \) is completely reducible.

\[ (a) \text{ Let } v \in V, \text{ then } \pi'(v) \in W \text{ since } \pi(\rho(g^{-1})v) \in W. \]

Hence \( \pi'(v) \leq W \). Let \( v \in W \). Then

\[
\pi'(v) = \frac{1}{|G|} \sum_{g \in G} \rho(g) \pi (\rho(g^{-1})v) \\
= \frac{1}{|G|} \sum_{g \in G} \rho(g) \rho(g^{-1})v \\
= \frac{1}{|G|} \sum_{g \in G} v = v.
\]

Hence \( \pi'(v) \leq W \) and \( \pi'(v) \in W \). Let \( v \in W \). Then \( \pi^2 = \pi \) and \( \text{im} \pi = W \).

(b) We want to show that \( \pi'(\rho(g)v) = \rho(g)\pi'(v) \).

\[
\pi'(\rho(g)v) = \frac{1}{|G|} \sum_{h \in G} \rho(h) \pi (\rho(h^{-1})\rho(g)v) \\
= \frac{1}{|G|} \sum_{h \in G} \rho(gh) \pi (\rho(h^{-1})v)
\]

since \( h \mapsto gh \) is a bijection.
\[ \begin{aligned}
&= \frac{1}{|G|} \sum_{h \in G} \rho(h) \pi'(\rho(h^{-1})v) \\
&= \rho(g)\pi'(v).
\end{aligned} \]

(3) Let \( W = \ker \pi' \). Then we want to show that \( W \) is \( G \)-invariant. Suppose \( g \in G \), \( v \in W \)
\[ \pi'(\rho(g)v) = \rho(g)\pi'(v) = 0. \]
Hence \( G \)-invariant.
We now show \( W \cap W' = \{0\} \).
\[ \begin{aligned}
&\text{Let } v \in W \cap W', \text{ then } \pi'(v) = 0 \text{ since } v \in W \\\n&\quad \quad \pi'(v) = 0 \quad \text{since } v \in W' \\
&\quad \quad \quad \Rightarrow \quad v = 0. \\
&\quad \quad \quad \Rightarrow \quad W \cap W' = \{0\}. 
\end{aligned} \]

Now, let \( v \in V \). Then \( v = \pi'(v) + v - \pi'(v) \).
\[ \pi'(v) \in W \text{ and } v - \pi'(v) \in W' \text{ since } \pi'(v - \pi'(v)) = \pi'(v) - \pi'(v) = 0. \]

**Question 3.** Let \( G = \{1, x, x^2\} \) be the cyclic group of order 3 and define a complex representation \( \rho: G \to GL(\mathbb{C}^3) \) by \( \rho(x)(z_1, z_2, z_3) = (z_2, z_3, z_1) \). Find the irreducible \( G \)-invariant subspaces of \( \mathbb{C}^3 \). (There will only be three of them for reasons, what’s special about linear operators over complex numbers?)

\[ \rho(x) = \begin{pmatrix}
0 & 1 & 0 \\
0 & 0 & 1 \\
1 & 0 & 0 \\
\end{pmatrix} \text{ with respect to the standard basis.} \]

Moreover, \( \rho(x)^3 = I \) so has eigenvalue \( 1, \omega, \omega^2 \) where \( \omega \) is cube root of unity.
we find the corresponding eigenvalues
\[
\begin{pmatrix}
-1 & 1 & 0 \\
0 & -1 & 1 \\
1 & 0 & -1
\end{pmatrix}
\sim
\begin{pmatrix}
0 & 1 & -1 \\
0 & -1 & 1 \\
1 & 0 & -1
\end{pmatrix}
\sim
\begin{pmatrix}
0 & 1 & 0 \\
0 & -1 & 1 \\
1 & 0 & -1
\end{pmatrix}
\]
Hence \( \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix} \) is an eigenvector.
\[
\begin{pmatrix}
-w & 1 & 0 \\
0 & -w & 1 \\
1 & 0 & -w
\end{pmatrix}
\sim
\begin{pmatrix}
0 & w^2 - w & 0 \\
0 & -w & 1 \\
1 & 0 & -w
\end{pmatrix}
\sim
\begin{pmatrix}
0 & 0 & 0 \\
0 & 1 & -w^2 \\
0 & 0 & 0
\end{pmatrix}
\]
so \( \begin{pmatrix} w^2 \\ 1 \\ 1 \end{pmatrix} \) is \( w \)-eigenvector. 

Similarly, \( \begin{pmatrix} w^2 \\ 1 \\ 1 \end{pmatrix} \) is \( w^2 \)-eigenvector. 

Hence we have the decomposition \( \mathbb{C}^3 = \mathbb{C} \langle \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix} \rangle \oplus \mathbb{C} \langle \begin{pmatrix} w^2 \\ 1 \\ 1 \end{pmatrix} \rangle \oplus \mathbb{C} \langle \begin{pmatrix} w^2 \\ 1 \\ 1 \end{pmatrix} \rangle \)

we can easily see that each of these eigenspace are \( \mathbb{C} \)-inv.

\textbf{Question 4.} We will prove Schur's lemma (some version of it at least). Consider an irreducible complex representation \( \rho : G \to \mathbb{C} \). Let \( \phi : \mathbb{C} \to V \) be a \( G \)-linear map. Show there exists a scalar \( \lambda \in \mathbb{C} \) such that \( \phi(g \cdot x) = \lambda \phi(x) \) for all \( x \in V \).

If \( \phi \) is a \( G \)-invariant subspace, if \( \phi \) is a \( G \)-linear map.

- if \( y = \phi(x) \), then for \( g \in G \), \( \rho(g)y = \rho(g)\phi(x) = \phi(\rho(g)x) \).

Hence \( \phi \) is a \( G \)-invariant subspace. If \( \phi(x) = 0 \), then

\( \phi(\rho(g)x) = \rho(g)\phi(x) = 0 \).
Consider \( \psi = \phi - \lambda I \). For some \( \lambda \in \mathbb{C} \), \( \psi \) has nontrivial kernel. Since \( V \) is irreducible, \( \psi = 0 \) on \( V \).

Hence \( \phi = \lambda I \).

**Question 5.** Let \( G \) be an abelian group. Prove all irreducible complex representations of \( G \) are one-dimensional.

Suppose \( \rho : G \to GL(V) \) is an irreducible complex representation. For \( g \in G \), let \( \phi_g : V \to V \) be given by \( \phi_g(v) = \rho(g)v \). Since \( G \) is Abelian, this determines a \( G \)-linear map

\[
\phi_g(\rho(h)v) = \rho(g)\rho(h)v = \rho(h)\phi_g(v).
\]

Hence Schur's lemma tells us that \( \phi_g(v) = \lambda_g v \) for some scalar \( \lambda_g \in \mathbb{C} \). Since this holds for all \( G \), we conclude \( V \) is 1-dimensional.