Week 3 Notes.

Reminder:
Sylow theorems (Application of group actions)

Sylow 1: Given a group $G$ and a prime divisor $p$ of $|G|$. There exists a sylow $p$-subgroup of $G$.

Sylow 2: For each prime $p$, the sylow $p$-subgroup of $G$ are conjugate to each other.

Sylow 3: If $|G| = p^nm$ where $p$ prime, $n \geq 0$ and $p \nmid m$. Let $n_p$ be the number of sylow $p$-subgroup of $G$.

Then
1) $n_p | m$
2) $n_p \equiv 1 \pmod{p}$
3) $n_p = |G : N_G(P)|$ when $P \in \text{Syl}_p(G)$

It’s useful to realize that all of these can be proven via group actions:

- $S1$: Via a $p$-subgroup $P$ on set $G / H$ via left mult.
- $S2$: $P$-sylow subgroup $P$ on set $G / P$ via left mult.
- $S3$: $P$-sylow subgroup $P$ on set $G / H$ via left mult.

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Question 1

We have the factorization $18 = 2 \cdot 3^2$.

From Sylow 3 we have that $n_3 \equiv 1 \mod 3$ and $n_3 | 2$. Hence $n_3 = 1$ and so there exists a normal subgroup of order 9.

Question 2

We observe that we may assume

Note $P \leq N_G(P)$. Hence we have $N_G(P)/P$ a well defined quotient group. Now $H_P/P \leq N_G(P)/P$ and by the second isomorphism theorem we have $|H_P/P| = [H/HNP]$.

Hence it follows by Lagrange's theorem that $|H/HNP| | N_G(P)/P |$ and $|H/HNP| \leq |H|$. As $p \nmid |N_G(P)/P|$ and $H$ is $p$-group. It follows that $H/HNP$ is trivial and so $H \leq P$.

Alternatively, if you are willing to accept that every $p$-subgroup is contained in a Sylow $p$-subgroup $Q$, then Sylow 2 implies there exists $g \in G$
Then there exists a Sylow $p$-subgroup $Q$ of $N_G(P)$ that must have the same order as $P$. By Sylow 2, there exist $g \in N_G(P)$ such that $gPg^{-1} = Q$, but as normalizer $P = Q$.

**Lemma:** Let $H$ be a $p$-subgroup of finite group $G$ such that $H$ isn't a Sylow $p$-subgroup. Then there exists $P \in \text{Syl}_p(G)$ such that $H \leq P$.

**Proof:** First observe that since $hgh^{-1}gH = gH \subseteq g^{-1}hgeH$, it follows that $N_G(H) = \bigcup_{g \in G} gHg^{-1} = (G/H)^H$. The union of all cosets fixed by $H$ by left multiplication. Hence we have that since $|G/H| |N_G(H)/H| \equiv |(G/H)^H| \equiv |G/H| \bmod p$.

Since $G$ not Sylow, $|G/H|$ is divisible by $p$ and so $|N_G(H)/H| \equiv 0 \bmod p$. Hence, by Cauchy's theorem and lattice theorem, there exists $p$-subgroup $H' \leq G$ such that $H \leq H'$.
Question 3

Suppose \( Q \in \textrm{Syl}_p(G) \) is such that \( p \mid Q = q \) for all \( p \in P \). Then \( P \leq N_G(Q) \) and by the previous \( Q \) we have \( P \leq Q \). Hence \( P = Q \).

Question 4

The action gives us a group homomorphism \( \phi : G \to \text{Sym}(\text{Syl}_p(G)) \cong S_{np} \). Since \( G \) is simple, this mapping must be injective and so by Lagrange's theorem \( |G| / np \).

Question 5

Consider the left action of \( G \) on the left cosets of a proper subgroup \( H \). Let \( n = [G:H] \).
Since \( G \) is simple, we have an injective homomorphism \( G \to S_n \). We want to show \( 10 \mid n \).

Now, since \( G \) has an element of order 23, \( S_n \) must contain a cycle of order 23. Since an order of a permutation is equal to the least common multiple of cycle lengths, and \( 23 \cdot 3 = 7 \cdot 10 \), we see that \( n \) must be at least \( 23 \cdot 3 \cdot 7 = 10 \).
Question 6

We first check that $N_G(P)$ acts on $X^p$.

Let $y \in X^p$, i.e., $gy = y$ for all $g \in P$. We want to show that $g'y \in X^p$ for $g' \in N_G(P)$.

Now, $g''g'g'y = y$ since $g''g' \in P$ for $g' \in N_G(P)$, $g \in P$.

Hence $gg'y = g'y$ and so $g'y \in X^p$.

Now, suppose $y \in X^p$. We already have $x \in X^p$ since $P \leq G_x$. It is sufficient to show that there exists $g \in N_G(P)$ such that $gx = y$.

Since $G \triangleright X$ is transitive, there exists $g \in G$ such that $gx = y$. Then we have $Gy = gGxg^{-1}$.

and so $g^{-1}Pg \leq G_x$ since $P \leq Gy. (y \in X^p)$.

Hence by Sylow 2, there exists $h \in G_x$ such that $h^{-1}g^{-1}Pgh = P \Rightarrow ghe \in N_G(P)$ and we have

$ghx = gx = y$. \hfill \square$