Group Actions.

A group action of $G$ on a set $A$ is a map from $G \times A \rightarrow A$ satisfying
1) $g_1 \cdot (g_2 \cdot a) = (g_1 g_2) \cdot a$ for all $g_1, g_2 \in G$ and $a \in A$, and
2) $1 \cdot a = a$ for all $a \in A$.

It is not hard to see that each $g \in G$ acts on $A$ as a permutation of the elements, so an equivalent way to define a group action is as a group homomorphism $\varphi: G \rightarrow \text{Sym}(A)$.

For $a \in A$, the stabilizer of $a$ in $G$ is the subgroup
\[ G_a := \{ g \in G \mid g \cdot a = a \} \]
(warm up)

$G \cdot a = \{ g \cdot a \mid g \in G \}$ is the orbit of $a$ under $G$. This is an equivalence class. (warm up)

**Question 1**

$G_a$ is a group: $e \in G_a$ so nonempty let $g, h \in G_a$. then
$(gh)^{-1} \cdot a = (gh)^{-1} \cdot (g \cdot a) = g^{-1} \cdot (h^{-1} \cdot g \cdot a) = g^{-1} \cdot e = G_a$ and so subgroup.

$G \cdot a$ is an equivalence class. In particular, we can define an equivalence relation on $A$ by $a \sim b \iff$ exists $g \in G$ s.t. $g \cdot a = b$.

- reflexive: clear as $1 \cdot a = a$
- symmetric: if $a \sim b \iff g \cdot a = b \iff a = g^{-1} \cdot b \iff b \sim a$
- transitive: if $a \sim b$ and $b \sim c$ then $\exists g, h \in G$ s.t. $g \cdot a = b, h \cdot b = c$ so $h \cdot g \cdot a = h \cdot b = c \iff a \sim c$. 
Question 2

Proof. Let us define a mapping \( \varphi : G \to G \cdot x \) given by \( g \mapsto g \cdot x \).

By definition, this is surjective. Now, given \( g, h \in G \), we have that

\[
\varphi(g) = \varphi(h) \iff g \cdot x = h \cdot x
\]

\[
\iff h^{-1}g \cdot x = x
\]

\[
\iff h^{-1}g \in C_x.
\]

ie, \( \varphi(g) = \varphi(h) \) if and only if \( g \) and \( h \) are in the same left coset of \( G \cdot x \). Hence we get a bijective mapping \( \varphi : G/G_x \to G \cdot x \).

Hence it follows that \( |G|/|G_x| = |G \cdot x| \).

Orbit-stabilizer forms the basis of a lot of important results, we will go through a few of these.

Question 3

we have that \( \sum_{g \in G} |x^g| = \sum \{ (g, x) \in G \times X \mid g \cdot x = x \} = \sum_{x \in X} |G_x| \)

and so by orbit-stabilizer

\[
\sum_{g \in G} |x^g| = \sum_{x \in X} |G_x| = |X| \sum_{x \in X} \frac{1}{|G_x|} = |G| |X/G|
\]

Question 4

note \( C_G(x_i) = \text{centralizer of } x_i \) in \( G = \{ g \in G \mid g x_i g^{-1} = x_i \} \).

\( G \) acts on itself via conjugation: \( g \cdot h = ghg^{-1} \). Under this action, \( C_G(x_i) \) is the stabilizer subgroup of \( x_i \), and so by orbit-stabilizer

\( |G : C_G(x_i)| = \text{size of the orbit of } x_i \) under \( G \). Hence, as orbit
\( |G: C_{G}(x)| = \text{size of the orbit of } x \text{ under } G \). Hence, as orbit partitions the set, the result follows.

**Question 5**

From orbit-stabilizer, as \( |G| = |G.x|/|G_{x}| \), we see that orbit sizes must divide the order of the group. Since \( x \in X^{c} \) are precisely the elements with trivial orbits, the result follows.

**Question 6**

Let \( X = \{ (g_{1}, \ldots, g_{p}) \in G^{p} \mid g_{1} \cdots g_{p} = e \} \). Observe that we have the first \( p-1 \) entries freely determined out of \( G \) and then \( g_{p} \) fixed. Hence \( |X| = |G|^{p-1} \). Now, consider the action \( C_{p} \times X \) via permuting entries.

Now, the orbits under this have either size 1 or \( p \) (from orbit-stabilizer).

From the previous question, we see that \( |X^{G}| \equiv 0 \pmod{p} \) and \( |X^{G}| \neq 0 \) since \( (e, \ldots, e) \in X \) and is fixed under the action, we see then must be an element \( g \in C_{p} \) s.t. \( (g, \ldots, g) \in X \). Hence \( g^{p} = e \)  \( \Box \)