MATH 210C: HOMEWORK 7

Problem 62. Let M be a (left) A-module. We say that M has *finite length* if there exists a finite filtration of submodules

$$M = M_0 \supsetneq M_1 \supsetneq \cdots \supsetneq M_n = 0$$

such that M_i/M_{i+1} is simple for all $i = 0, \ldots, n-1$.

(a) Prove that if we have a short exact sequence $L \rightarrowtail M \twoheadrightarrow N$ of A-modules, M has finite length if and only if L and N do.
(b) Prove that the class of finite length A-modules coincides with the smallest Serre subcategory of A-**Mod** containing every simple module.
(c) Prove that M has finite length if and only if M is both artinian and noetherian.

Problem 63. Let $\mathcal{A} \subset A$-**Mod** denote the subcategory of finite length modules. Prove that there is a unique (and well-defined) length function $\ell : \operatorname{ob} \mathcal{A} \to \mathbb{N}$ such that $\ell(M) = 1$ if M is simple and ℓ is additive on short exact sequences.

Problem 64. Deduce from Fitting's Lemma, that given any A-linear endomorphism $f : M \to M$ of a finite length A-module M, we have a decomposition of M as $M = M_0 \oplus M_1$ in such a way that f becomes 'diagonal' (that is, $f(M_i) \subseteq M_i$ for $i = 0, 1$) and f is nilpotent on M_0 and an isomorphism on M_1.

Problem 65. Describe all finite length modules over $\mathbb{Z}$. What is the length of $\mathbb{Z}/n\mathbb{Z}$?

Problem 66. Give an example of a finite length module that is not semisimple.

Problem 67.

(a) Prove that every finitely generated module over a semisimple ring has finite length.
(b) Give an example of a ring which is not semisimple yet all its finitely generated modules have finite length.

Problem 68. Give an example of a ring with only one unique simple module (up to isomorphism) but with two non-isomorphic modules of length 2.

Problem 69. Let A be ring. Prove that if $A/\operatorname{rad}(A)$ is a division ring, then A is local.

Problem 70. Let G be a p-group and let k be a field of characteristic p. Prove that $k[G]$ is local and determine its Jacobson radical.

Problem 71. Prove that every finitely generated projective A-module over a local ring A is free. Hint: Adapt the commutative case.

Problem 72. Let C_5 denote the cyclic group with 5 elements.

(a) Compute all irreducible representations of $\mathbb{Q}[C_5]$.
(b) Compute all irreducible representations of $\mathbb{R}[C_5]$.
(c) Compute all irreducible representations of $\mathbb{C}[C_5]$. Conclude that the representation theory of finite groups can vary over fields of characteristic zero.
(d) Repeat (a)-(c) for C_3 instead of C_5.

Problem 73. Compute all indecomposable representations of $\mathbb{F}_3[C_3]$, where $\mathbb{F}_3$ denotes the field with 3 elements.

Problem 74. Compute all indecomposable representations of $\mathbb{F}_3[C_4]$ and $\mathbb{F}_4[C_4]$.

Problem 75. Compute all irreducible representations of $\mathbb{R}[A_4]$ and $\mathbb{C}[A_4]$, where A_4 is the alternating group on 4 elements.