# MATH 210C: HOMEWORK 6

**Problem 53.** Exhibit an explicit $M_n(R), R$-bimodule which gives an equivalence between $R$-**Mod** and $M_n(R)$-**Mod** for any ring $R$ and any $n \in \mathbb{N}$.

**Problem 54.** Prove that the subring of $M_n(A)$ of upper triangular matrices is never Morita equivalent to $A$ for $n > 1$.

**Problem 55.** Let $A$ and $B$ be two rings and $G\colon B$-**Mod** $\to A$-**Mod** be a left-exact product preserving functor, which (therefore) admits a left adjoint $F$. Show that there exists a $B, A$-bimodule $X$ such that $G \cong \mathrm{Hom}_B(X, -)$. Describe $X$.

**Problem 56.** Let $A$ be a ring and let $P$ be a finitely generated projective right $A$-module which is a generator of $A$-**Mod** (as in Morita theory). Let $B = \mathrm{End}_A(P)$ and view $P$ as a $B, A$-bimodule. Show that there is an isomorphism of $A, B$-bimodules $\mathrm{Hom}_A(P, A) \cong \mathrm{Hom}_B(P, B)$.

**Problem 57.** Let $A$ be a ring, and let $P$ be a finitely generated projective right $A$-module. Prove that $P$ is a generator if and only if every simple right $A$-module $M$ is quotient of $P$.

**Problem 58.** Let $A$ be a ring, and let $P$ be a finitely generated projective right $A$-module. Prove that the following are equivalent:

(a) $\mathrm{Hom}_A(P, -)$ is faithful.
(b) Every right $A$-module $M$ is the quotient of a (possibly infinite) coproduct of copies of $P$.
(c) For every right $A$-module $M$ and every $m \in M$, there exist a family of homomorphisms $f_1, \ldots, f_n : P \to M$ and $x_1, \ldots, x_n \in P$ such that

$$\sum_{i=1}^{n} f_i(x_i) = m.$$

(d) The evaluation map $\varepsilon\colon P^\vee \otimes P \to A$ is surjective (recall Problem 27).

In this case $P$ is a generator of $A$-**Mod** in the sense of Morita theory.

**Problem 59.** If $A$ is a commutative ring with no nontrivial idempotents, prove that any (nonzero) finitely generated projective module is a generator.

**Problem 60.** Let $R$ be any ring, and let $A$ be the subring of $M_2(R)$ given by upper triangular matrices. Let $e$ be the idempotent element $\begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}$.

(a) Prove that $P = eA$ is a finitely generated projective right $A$-module.
(b) Prove that $P$ is a not generator.

**Problem 61.** Let $A$ be a ring with a unique maximal left ideal $\mathfrak{m}$.

(a) Prove that $A$ also has a unique maximal right ideal which coincides with $\mathfrak{m}$.
(b) Prove that $A$ has a unique left ideal if and only if the set of nonunits of $A$ forms an ideal (i.e. $A$ is a local ring).