MATH 210C: HOMEWORK 5

Problem 40. Recall that the nilradical of a commutative ring A is the set of all nilpotent elements, that is, all $x \in A$ such that $x^n = 0$ for some $n \in \mathbb{N}$.

(a) Prove that the nilradical is the intersection of all prime ideals, and hence itself an ideal. Prove that the nilradical is always contained in the Jacobson radical.
(b) Let $A = k[X_1, \dots, X_n]/I$ be a finitely generated k-algebra for k a field. Prove that its nilradical and its Jacobson radical coincide.
(c) Let $A = R[[X]]$, the formal power series ring of a Noetherian commutative ring R. Compute the nilradical and the Jacobson radical of A, and conclude that these ideals are not equal in general.

Problem 41. If A is a noncommutative ring, we can still consider the subset of nilpotent elements of A. Prove that this subset is generally neither closed under addition nor under left or right multiplication by elements of A. Further, give an example of two nilpotent elements whose product is no longer nilpotent.

Problem 42. Let A be a ring with nilradical N. Suppose that every ideal I not contained in N contains a nonzero idempotent. Prove that $\mathrm{Jac}(A) = N$.

Problem 43. Compute the Jacobson radical of $\mathbb{Z}$.

Problem 44. Compute the Jacobson radical of $\mathbb{Z}/24\mathbb{Z}$. Generally, what is the Jacobson radical of $\mathbb{Z}/n\mathbb{Z}$?

Problem 45. Let $e \in A$ be a central idempotent, i.e. it belongs to the center of A and satisfies $e^2 = e$. Let M be a left A-module.

(a) Show that eM is a submodule of M, and show that this fails in general if e is not central.
(b) Show that $M = eM \oplus (1-e)M$, and show that this fails in general if e is not idempotent.
(c) Give an explicit equivalence between the category of A-modules and the product of the categories of A_1- and of A_2-modules, where $A_1 = eA = Ae = eAe$ and $A_2 = (1-e)A$.
(d) Explain how the existence of such an idempotent corresponds uniquely to a splitting of A as a product of two rings, and explain the above correspondence on ideals.

Problem 46. Let A be a commutative ring. Prove that the subsets $S \subset \operatorname{Spec} A$ which are both closed and open have the form $S = V(eA)$ for e an idempotent element. Then prove that the following are equivalent:

(a) A has no idempotents other than 0 and 1.
(b) $\operatorname{Spec} A$ is connected.
(c) Every finitely generated projective R-module has constant rank.

Problem 47. Let A be a (not necessarily commutative) ring, and let $N \subset A$ be nil ideal, that is, every $x \in N$ is nilpotent. Prove that every idempotent element $\bar{e} \in A/N$ has an idempotent lift $e \in A$.

Problem 48. Let $\mathcal{A}$ be an additive category. The radical of $\mathcal{A}$, $\operatorname{rad} \mathcal{A}$, is defined to be the class of arrows $f : A \to B$ such that $1_A - gf : A \to A$ is a retraction for any $g : B \to A$.

(a) Prove that $\operatorname{rad} \mathcal{A}$ is closed under addition of morphisms.
(b) Prove that if $(f : A \to B) \in \operatorname{rad} \mathcal{A}$ and $u : B \to X$ and $v : Y \to A$ are any morphisms, prove that $u \circ f$ and $f \circ v$ are in $\operatorname{rad} \mathcal{A}$.

Any class of arrows satisfying the above conditions is called an ideal in $\mathcal{A}$.

Problem 49. Prove that $(f : A \to B) \in \operatorname{rad} \mathcal{A}$ if and only if $1_A - gf$ is an isomorphism for any $g : B \to A$.

Problem 50. Let $\mathcal{A}$ be the category of left R-modules for a ring R. For any R-module $M \in \mathcal{A}$, prove that

$$\operatorname{rad} \mathcal{A} \cap \operatorname{End}_R(M, M) = \operatorname{Jac}(\operatorname{End}_R(M, M)).$$

Problem 51. Let $\mathcal{A}$ be the category of left R-modules, and consider the representable functors $M_X \colon \mathcal{A}^{\mathrm{op}} \to \mathbf{Ab}$ given by $Y \mapsto \operatorname{Hom}_R(Y, X)$. Prove that for any $f \in \operatorname{rad} \mathcal{A}$ and any simple projective R-module X, $M_X(f) = 0$. (This also holds if M_X is simple as a functor $\mathcal{A}^{\mathrm{op}} \to \mathbf{Ab}$, in the obvious sense.)

Problem 52. Show that for any $g : Y \to X$, we have a morphism $M_g : M_Y \to M_X$ and whose image $\operatorname{im}(M_Y \to M_X)$ is a subobject of M_X, in the notation of the previous exercise. Find examples of g where X is simple, and the resulting image submodule of M_X is neither zero nor M_X.

Problem 53. Compute the radical of the category of vector spaces over any field k.