MATH 210C: HOMEWORK 4

Problem 31. If A is a simple ring, show that $M_n(A)$ is as well.

Problem 32. Show that a commutative semisimple ring is a finite direct product of fields.

Problem 33. Give an example of a simple ring R that is not simple as a left R-module. Which rings R do satisfy this property?

Problem 34. Give an example of a ring A and a semisimple A-module M such that the ring homomorphism $A \to \mathrm{End}_B(M)$ is not onto, where $B = \mathrm{End}_A(M)$.

Problem 35. Let k be a field, and consider the k-algebra $W = k[X, \partial_X]$ generated by one variable and the differential operator ∂_X.

(a) Show that $W \cong k[X,Y]/\langle YX - XY = 1\rangle$ as a k-algebra.
(b) Show that W is simple but not artinian.

Problem 36. Let $W = k[X, Y]$ be the Weyl algebra over a field k of characteristic zero. Let $M = k[T]$.

(a) Show that M has a natural structure of a simple W-module.
(b) Compute $B = \mathrm{End}_W(M)$.
(c) Describe the homomorphism $W \to \mathrm{End}_B(M)$.

Problem 37. Let $\mathrm{Jac}(A)$ denote the Jacobson radical of a ring A.

(a) Reprove that $\mathrm{Jac}(A)$ is equivalently the intersection of all maximal right ideals, the intersection of all maximal left ideals, or the set
$$\{a \in A : 1 + xa \in A^\times \text{ for all } x \in A\}$$
(b) Show that $\mathrm{Jac}(A)$ need not be the intersection of all maximal two-sided ideals.
(c) Prove that the Jacobson radical of a semisimple ring is zero, but show by example that the converse may not hold.

Problem 38. Let p be a prime, and let $R = \mathbb{F}_p[S_3]$ be the group algebra of the finite field $\mathbb{F}_p$ on the symmetric group on 3 elements. Compute the Jacobson radical of R (your answer will depend on p).

Problem 39. Let A be a ring, and consider the subring $R \subset M_n(A)$ of upper triangular matrices. Compute the Jacobson radical of R. What is the quotient $R/\mathrm{Jac}(R)$?