MATH 210C: HOMEWORK 1

Problem 1. Let k be a field and let $R = k[[X]]$. Let M be a finitely generated R-module. Let us denote by $n = \dim_k(\bar{M})$ the dimension of $\bar{M} := M/XM$. Show that M is generated by n elements as an R-module.

Problem 2. Let $S \subset A$ be a multiplicative subset and let M, N be A-modules. Construct a natural homomorphism

$$S^{-1}\operatorname{Hom}_A(M, N) \to \operatorname{Hom}_{S^{-1}A}(S^{-1}M, S^{-1}N).$$

Show that this is not an isomorphism in general but that it is one when M is finitely presented.

Problem 3. Show that a sequence $L \to M \to N$ is exact (at M) if and only if the induced sequence $L_{\mathfrak{p}} \to M_{\mathfrak{p}} \to N_{\mathfrak{p}}$ is exact at every $\mathfrak{p} \in \operatorname{Spec}(A)$.

Problem 4. Show that, for an A-module P the following are equivalent:

(a) P is finitely presented and $P_{\mathfrak{p}}$ is a free module for every $\mathfrak{p} \in \operatorname{Spec} A$.
(b) P is a finitely generated projective A-module.

Problem 5. Prove that any unique factorization domain is integrally closed (in its field of fractions).

Problem 6. Let $d \in \mathbb{Z}$ be a squarefree integer, and consider $\mathbb{Q}(\sqrt{d})/\mathbb{Q}$ (for any determination of the square root). Prove that $\alpha \in \mathbb{Q}(\sqrt{d})$ is integral if and only if the trace and norm of α are integers.

Problem 7. Let $K/\mathbb{Q}$ be a number field. Prove that any integral element of K has trace and norm in $\mathbb{Z}$, but find a counterexample to show that the converse is generally false.

Problem 8. Let $d \in \mathbb{Z}$ and $c = \frac{\sqrt{d}-1}{2} \in \mathbb{C}$ (for any determination of the square root). Show that c is integral over $\mathbb{Z}$ if and only if $d \equiv 1 \bmod 4$. [Hint: Compute $c^2 + c$.]

Problem 9. Describe the map $\operatorname{Spec}\mathbb{Z}[i] \to \operatorname{Spec}\mathbb{Z}$ induced by the natural homomorphism $\mathbb{Z} \to \mathbb{Z}[i]$.

Problem 10. Let $A = \mathbb{R}[X, Y]/\langle Y^2 - X^2(X + 1)\rangle = \mathbb{R}[x, y]$ and $B = \mathbb{R}[Z]$.

(a) Show that A is a domain.

(b) Show that there is a ring homomorphism $f : A \to B$ such that $f(x) = Z^2 - 1$ and $f(y) = Z(Z^2 - 1)$.

(c) Prove that there exist two closed points in $\operatorname{Spec} B$ with the same image in $\operatorname{Spec} A$ under $f^* : \operatorname{Spec} B \to \operatorname{Spec} A$.

(d) Consider the ring homomorphism $g : B \to \operatorname{Frac}(A)$ given by $g(Z) = \frac{y}{x}$. Show that $g \circ f$ is the inclusion of A in its field of fractions.

(e) Show that $\operatorname{Frac}(f) : \operatorname{Frac}(A) \to \operatorname{Frac}(B)$ exists and that $\operatorname{Frac}(f) \circ g$ is the inclusion of B into its field of fractions.

(f) Deduce that $\operatorname{Frac}(A) = \operatorname{Frac}(B)$ and that B is the integral closure of A.

(g) Draw a picture of the Zariski spectra in this situation.

Problem 11. Let A be as above, and consider $\mathbb{R}[x] \subset A$.

(a) Show that $\mathbb{R}[x] \cong \mathbb{R}[X]$ and that this extension is integral.

(b) Describe the preimage of $\langle x - 2 \rangle$ under the induced map $\operatorname{Spec} A \to \operatorname{Spec} \mathbb{R}[x]$.

(c) Draw a picture of the Zariski spectra in this situation.