TRIANGULAR WITT GROUPS.
PART I: THE 12-TERM LOCALIZATION EXACT SEQUENCE.

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Abstract. To a short exact sequence of triangulated categories with duality, we associate a long exact sequence of Witt groups. For this, we introduce *higher Witt groups* in a very algebraic and explicit way. Since those Witt groups are 4-periodic, this long exact sequence reduces to a cyclic 12-term one. Of course, in addition to higher Witt groups, we need to construct connecting homomorphisms, hereafter called *residue homomorphisms*.

Introduction.

The reader is expected to have some interest in the *usual* Witt group, as defined for schemes by Knebusch in [9, definition p. 133]. This Witt group is obtained by considering symmetric vector bundles (up to isometry) modulo the bundles possessing a *lagrangian* - that is a maximal totally isotropic sub-bundle. This being said, the present article is maybe more about triangulated categories than about symmetric forms and that might possibly make it generalizable to invariants other than the Witt group.

For the time being, it is not my goal to compute any explicit Witt group of a scheme by means of the techniques that I introduce. Nevertheless, computational applications do exist and are my obvious motivation. They will be treated in forthcoming articles, by myself and by others.

It is a very natural question, given an invariant of schemes, to try to understand its behavior with respect to localization. In the case of the Witt group functor, remarkable work has been done on this question ever since the beginning, at least over rings. The answer usually takes the form of an exact sequence relating (Witt groups of) "global data", "localized data" and "torsion data" – by which I mean data that vanishes after localization, like torsion modules. Any such exact sequence will be called *a localization sequence*. An overview of this would take us too far afield but the motivated reader is recommended to enjoy the famous contributions of Wall [16], Karoubi [8], Pardon [11], Ranicki [12 & 13] or Fernández-Carmena [6], the latter going out of the affine world into the world of schemes, at least in dimension 2. A localization sequence for Witt groups over arbitrary schemes and open subschemes is unknown, even when the global scheme is affine. If some over-optimistic mathematician maintains the contrary in your presence, just suggest to him that he tries his favorite techniques against the following very first and easy localization question (in the affine and local and regular case and with the smallest possible closed subset removed):

Problem. Let R be a noetherian local regular domain of Krull dimension n in which 2 is a unit. Let us denote by $\mathfrak{m}$ its maximal ideal, by $k = R/\mathfrak{m}$ its residue field and by $U = \mathrm{Spec}(R) \setminus \{\mathfrak{m}\}$ the *punctured spectrum* of R. When is the restriction map $\mathrm{W}(R) \to \mathrm{W}(U)$ an isomorphism? Recall that it is wrong for $n = 1$. Recall that it is true for $n = 2$ (see [5]) and for $n = 3$ (see [10]).

Solution. It is an isomorphism in dimension $n \not\equiv 1$ modulo 4 and when $n \equiv 1$ modulo 4, one has a short exact sequence:

$$0 \longrightarrow \mathrm{W}(R) \longrightarrow \mathrm{W}(U) \longrightarrow \mathrm{W}(k) \longrightarrow 0.$$

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The proof of this rather unexpected fact will appear in a forthcoming article as a *collateral damage* of more general results. For the moment, it could be regarded as a motivation and should convince us that the question of localization for Witt groups is far from being solved, even in the affine case!

Ideally, the present article would have contained:

(1) The basic definitions *adapting* the theory of symmetric forms to triangulated categories.
(2) The inclusion of the classical theory into this triangular framework introduced in (1), upgrading the *adaptation* into a *generalization*. That is, given a scheme X, the proof that the Witt group of some suitable triangulated category associated to X coincides with the usual Witt group of X.
(3) The use of this new triangular Witt group to produce a general theorem for localization.
(4) Applications.

Also ideally, this article would have been 20 pages long. This is not the article you have in your hands. Since it turned out that applications were so extensive that they need separate treatment, point 4 of the above list is to be made independent. Part of it will be a joint work with Charles Walter. Under the name *Triangular Witt group (TWG)*, I have then regrouped the three first points above-mentioned and, still being over the 20 pages, I cut the latter into two parts. The second part establishes point 2 of the above list. More precisely, in *TWG, Part II* (see [3, Theorem 4.7]), I prove that the Witt group of the derived category of bounded complexes of locally free coherent $\mathcal{O}_X$-modules is the same as the usual Witt group of X, for any scheme X in which 2 is invertible. Less precisely, the classical theory of Witt groups is merged into the theory of triangular Witt groups. As well, I defer until *TWG II* any discussion concerning how the new objects introduced here (like higher Witt groups we shall soon come to) can be linked, in special cases, with already existing constructions (like formations, e.g.). See also Remark 1.19.

So, for the present article, we are left with points 1 and 3 of my ideal list. An *aficionado* would kindly object that point 1 is already contained in [1] and [2]. Agreeing with him, I reduced this part to a bare minimum (§ 1) but made slight modifications in introducing the notion of *the cone of a symmetric (degenerated) morphism* and in numbering the *shifted Witt groups* in a more convenient way. *De facto*, this article is then mainly point 3 of the list: the general treatment of localization for Witt groups in the triangular framework.

What is then to be gained in treating localization with triangulated categories instead of classical exact categories of locally free coherent modules? Three answers should be made:

(1) First of all, it makes better sense, from the conceptual point of view. As we shall see, given a functor of abelian groups, $F : \mathcal{C} \to \mathcal{A}b$, defined on a friendly category $\mathcal{C}$ of triangulated categories, localization takes the following form. To any *short exact sequence of triangulated categories*

$$0 \longrightarrow J \xrightarrow{j} K \xrightarrow{q} L \longrightarrow 0,$$

F associates a very short exact sequence: $F(J) \xrightarrow{F(j)} F(K) \xrightarrow{F(q)} F(L)$ (*i.e.* F is exact in the middle) and F possesses right and left old-style derived functors $(F^i)_{i\in\mathbb{Z}}$ (or δ-functors) fitting in a long exact sequence:

$$\cdots \longrightarrow F^{-1}(K) \longrightarrow F^{-1}(L) \xrightarrow{\partial^{-1}} F(J) \longrightarrow F(K) \longrightarrow F(L) \xrightarrow{\partial^0} F^1(J) \longrightarrow F^1(K) \longrightarrow \cdots$$

By a short exact sequence of triangulated categories, $0 \to J \xrightarrow{j} K \xrightarrow{q} L \to 0$, I mean that $L = S^{-1}K$ is a localization of K (for some class S of morphisms) and that J is the *kernel* of this localization, that is the full subcategory of K on the objects M such that $q(M) \simeq 0$. We are going to establish such a localization exact sequence when F is W the Witt functor and where $\mathcal{C}$ is the category of triangulated categories with duality (see Theorem 5.2), assuming that these categories satisfy an enriched form of the octahedron axiom suggested by Beilinson-Bernstein-Deligne in [4]. Such a nice formulation would not be possible if one tried to deal strictly with the above-mentioned exact categories of locally free coherent modules, being clear that one can always disguise some piece of derived categories in a complicated *ad hoc* costume, without using the triangular vocabulary.

(2) When one expects a result of the kind described here above, there is a point in constructing those *higher and lower Witt groups* $(\mathrm{W}^i)_{i\in\mathbb{Z}}$ and the connecting homomorphisms ∂^i in a reasonably complicated way. In the triangular framework, we shall define first *shifted* Witt groups, W^i, by

merely replacing the used duality $\#$ by a shifted one $T^i \circ \#$. Those functors will turn out to be the good ones. Their definition is perfectly trivial and requires neither meta-construction – the category of triangulated categories $\mathcal{C}$ of (1) is just a bad memory now – nor topology – these are not, as presented here, the homotopy groups of some topological construction.

(3) *A posteriori*, studying with Charles Walter the Gersten conjecture for Witt groups, I understood that this formulation is remarkably *complete* in the sense that you do not need to introduce a different construction each time you come across a new category from which you want to get Witt groups. For instance, you can think of a filtration of triangulated categories, say $0 = K^n \subset K^{n-1} \subset \cdots \subset K^1 \subset K^0$ and imagine that you are interested in the Witt group of K^0. Then, using iteratively the localization sequence, you can reach a homological (or spectral) description of $\mathrm{W}(K^0)$ in terms of the Witt groups of the quotients K^i/K^{i+1}. Doing that, you will encounter Witt groups of a lot of quotients K^i/K^j, $j > i$, which may be far from having any classical significance, even if $\mathrm{W}(K^0)$ and $\mathrm{W}(K^i/K^{i+1})$ had! Those quotients K^i/K^j will be plain triangulated categories with duality and their Witt group(s) will be the one(s) defined in § 1 hereafter. The machinery is an all-terrain one.

Let us detail a little bit the content of this article.

For those who are not familiar with triangulated categories, section 0 has been included. I tried to keep it as short as possible and this cannot replace any classical and complete presentation as Verdier's original one (see [14]) or as many more recent ones (among which, Weibel's [17, chapter 10]).

Let me say once and for all that all over this paper 2 is supposed to be invertible. Generalization beyond this hypothesis are possible but with considerable additional work in some places. This is a distinct problem.

As already said, section 1 contains the definitions of what a triangulated category with duality is and how its Witt group is constructed. I tried to stay as close as possible to the classical definition involving metabolic spaces and lagrangians, because I expect my reader to have some intuitive ease with those notions. To avoid confusion with the usual notion when both will be considered simultaneously, the new metabolic spaces are generally called *neutral*. As we shall see in [3], a classical symmetric space (over a scheme if you want) will be trivial in the usual Witt group if and only if it is neutral in the derived category. This makes use of a remarkable property of triangulated categories with duality: *Witt-trivial spaces are neutral*. This result (Theorem 2.5) has a nice triangular proof which is presented in section 2.

Section 3 is the core of this article and of our pains. Its deals with the *sub-lagrangian construction*. In the classical framework, the sub-lagrangian construction allows you to get rid of a sub-lagrangian, that is a reasonable sub-object of a symmetric space on which the symmetric form is zero. By "get rid" I mean replace the symmetric space by a smaller space – a sub-quotient – which is Witt-equivalent to the first one and in which the sub-lagrangian has disappeared. In triangulated categories, there are no such notions as "sub-object" or "quotients" but the trivial ones obtained from decomposition into direct summands. Therefore, the way through consists in allowing many more sub-lagrangians – not even sub-objects. Since this question is technically sensitive, I included extensive introductions, explanations and comments in section 3. We shall see that the perfect result is not reached (and I do not know if it is true or not) but Theorem 3.22 is sufficient for the use we want to make of sub-lagrangians in the localization sequence. There I had to use a more complete (and more natural) axiom of the octahedron, as discussed in section 0.

In section 4, the scene is prepared for the final act. First, I explain what I mean by *localization of triangulated categories with duality*. Secondly, I give the construction of the *residue homomorphisms* that are the connecting homomorphisms of the long exact sequence of localization. Proving that they are well defined is a piece of work, that was missing in [1].

The main result (Theorem 5.2), and its proof form the closing section 5. The proof is in three parts, one for each triangulated category involved in the localization: the central one, the localized one and the kernel. Each time, one has to prove exactness of the long sequence at the corresponding Witt group. Exactness at the localized category was established in [1, Theorem 5.17]. This is the only place where we need an external reference, the rest of the article being mostly self-contained. Exactness at the kernel category is another simple corollary of the fact that Witt-trivial spaces are neutral (Theorem 2.5). The main point is exactness at the central category, or, in the above-used vocabulary, exactness in the middle of the Witt group functor. This needs sub-lagrangian Theorem 3.22.

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0. Background on triangulated categories and Axiom TR 4$^+$.

Here is a little *baise-en-ville* of the triangulated mathematician. Details can be found in [14] or in [17], for instance. Let K denote a triangulated category. Recall that K is additive and endowed with a *translation* additive functor $T : K \to K$ which is an automorphism. We have triangles in K :

$$(\Delta) \qquad\qquad A \xrightarrow{u} B \xrightarrow{v} C \xrightarrow{w} T(A)$$

and we choose among them a collection of *exact* triangles which replace the exact sequences of the abelian (or exact) framework. We recall quickly the axioms those exact triangles should satisfy and then, we stress some very elementary and useful techniques.

(TR 1) For any A, the following triangle is exact : $A \xrightarrow{1} A \longrightarrow 0 \longrightarrow T(A)$. Any morphism $u : A \to B$ fits into an exact triangle Δ. If Δ is exact, so is any isomorphic triangle.

In particular, any triangle obtained from Δ by changing the sign of two of the three morphisms is again exact. It is wrong with an odd number of changes.

(TR 2) Rotation. The triangle (Δ) is exact if and only if the following triangle is exact :

$$B \xrightarrow{v} C \xrightarrow{w} T(A) \xrightarrow{-T(u)} T(B).$$

(TR 3) Morphism. If the left square of the following diagram is commutative :

$$\begin{tikzcd}
A \arrow[r, "u"] \arrow[d, "f"'] & B \arrow[r, "v"] \arrow[d, "g"'] & C \arrow[r, "w"] \arrow[d, dashed, "\exists\, h"'] & T(A) \arrow[d, "T(f)"] \\
A' \arrow[r, "u'"'] & B' \arrow[r, "v'"'] & C' \arrow[r, "w'"'] & T(A')
\end{tikzcd}$$

and if both lines are exact triangles, there exists a morphism $h : C \to C'$ such that the above diagram commutes. This h is not unique.

An additive category with an automorphism of translation and a collection of exact triangles satisfying (TR1), (TR2) and (TR3) is sometimes called *pre-triangulated*. Before making it a plain triangulated category, we shall try to understand the sentence "exact triangles replace exact sequences". What follows is true in any pre-triangulated category, that is, can be proved from (TR1), (TR2) and (TR3) only.

To explain how exact triangles replace exact sequences, recall that necessarily the composition of two consecutive morphisms in Δ is trivial: $v\,u = 0$, $w\,v = 0$ but also $T(u)\,w = 0$, $u\,T^{-1}(w) = 0$ and so on. Let us focus on the middle morphism v. Then (A, u) is a *weak kernel* of v in the sense that $v\,u = 0$ and that any morphism $f : X \to B$ such that $v\,f = 0$ factors (non uniquely) through A:

$$\begin{tikzcd} A \arrow[r, "u"] & B \arrow[r, "v"] & C \arrow[r, "w"] & T(A). \\ & X \arrow[ul, dashed, "\exists\,\tilde f"] \arrow[u, "f"'] \arrow[ur, "v\,f = 0"'] & & \end{tikzcd}$$

As well, $\big(T(A), w\big)$ is a *weak cokernel* of v: for $g : C \to Y$, we have $g\,v = 0$ if and only if g factors (non uniquely) through $T(A)$:

$$\begin{tikzcd} A \arrow[r, "u"] & B \arrow[r, "v"] \arrow[dr, "g\,v = 0"'] & C \arrow[r, "w"] \arrow[d, "g"'] & T(A) \arrow[dl, dashed, "\exists\,\bar g"] \\ & & Y. & \end{tikzcd}$$

Another very common trick is the following. Given a morphism of exact triangles:

$$\begin{tikzcd} A \arrow[r, "u"] \arrow[d, "f"'] & B \arrow[r, "v"] \arrow[d, "g"'] & C \arrow[r, "w"] \arrow[d, "h"'] & T(A) \arrow[d, "T(f)"'] \\ A' \arrow[r, "u'"'] & B' \arrow[r, "v'"'] & C' \arrow[r, "w'"'] & T(A') \end{tikzcd}$$

if two among f, g and h are isomorphisms so is the third (analogous to the Five Lemma). This is an easy consequence of the following also useful remark: Given any endomorphism of an exact triangle of the form:

$$\begin{tikzcd} A \arrow[r, "u"] \arrow[d, "0"'] & B \arrow[r, "v"] \arrow[d, "0"'] & C \arrow[r, "w"] \arrow[d, "k"'] & T(A) \arrow[d, "0"'] \\ A \arrow[r, "u"'] & B \arrow[r, "v"'] & C \arrow[r, "w"'] & T(A), \end{tikzcd}$$

we have necessarily $k^2 = 0$. This, in turn, is an easy consequence of the above remarks about weak kernels and cokernels: $k\,v = 0$ forces $k = \bar{k}\,w$ for some $\bar{k} : T(A) \to C$ and then $k^2 = \bar{k}\,w\,k = 0$.

This implies that the triples (C, v, w) such that Δ is an exact triangle (for $u : A \to B$ fixed) are all isomorphic. Any such triple (or sometimes only the object C) is called *the cone* of u and will be written as $\mathrm{Cone}(u)$. The cone plays weakly but simultaneously the role of the kernel and the cokernel. It contains essentially all the *homological* information about u.

A very important question is then to know how to compare the cone of a composition $u' \circ u$ with the cones of u' and u. The answer is the famous axiom of the octahedron, which says: there is a *good* exact triangle linking the cone of the composition of two morphisms and the cones of these morphisms. More precisely, it says:

(TR 4) Composition: The Octahedron Axiom. Given any two composable morphisms $u : X \to Y$ and $v : Y \to Z$, let $w := v \circ u$; given any exact triangles over those morphisms (*i.e.* choosing any cone for u, v and w):

$$X \xrightarrow{u} Y \xrightarrow{u_1} U \xrightarrow{u_2} T(X), \qquad Y \xrightarrow{v} Z \xrightarrow{v_1} V \xrightarrow{v_2} T(Y)$$

$$\text{and} \qquad X \xrightarrow{w} Z \xrightarrow{w_1} W \xrightarrow{w_2} T(X)$$

there exist morphisms $f : U \to W$ and $g : W \to V$

$$
\begin{tikzcd}
 & & Z \arrow[dl, "v_1"] \arrow[ddr, "w_1"] & \\
Y \arrow[urr, "v"] \arrow[ddr, "u_1"'] & V \arrow[l, "v_2"', "{[+1]}"] \arrow[dd, "T(u_1)\circ v_2", "{[+1]}"'] & & \\
 & & X \arrow[uu, "w"'] \arrow[ull, "u"] & W \arrow[ull, "\exists\, g"'] \arrow[l, "w_2"', "{[+1]}"] \\
 & U \arrow[ur, "u_2", "{[+1]}"'] \arrow[urr, "\exists\, f"'] & &
\end{tikzcd}
$$

such that (the above is *an octahedron* which means):

(1) the triangle $U \xrightarrow{f} W \xrightarrow{g} V \xrightarrow{T(u_1)\, v_2} T(U)$ is exact;
(2) $g\, w_1 = v_1$ and $w_2\, f = u_2$;
(3) $f\, u_1 = w_1\, v$ and $v_2\, g = T(u)\, w_2$.

Note that the faces of the octahedron are alternatively commutative faces (like the "generating" one: $w = v \circ u$) or exact triangles (like the three obtained out of the generating face and chosen at the beginning). In "exact" faces, the third morphism is of degree one, *i.e.* it does not really go from the visible start of the drawn arrow to its visible target but to the translation of this target. For this first presentation, those arrows were marked with a [+1] here above but this will not be the case later on, since they will always be at the same place.

Observe that there is an obvious iniquity in the above formulation: all morphisms are given an explicit triangle to live in, but the two compositions from Y to W and from W to $T(Y)$ appearing in condition (3). Applying the octahedron again to these compositions will convince the reader that the following condition is very natural.

(TR4$^+$) Enriched Octahedron Axiom. Given any two composable morphisms $u : X \to Y$ and $v : Y \to Z$, let $w := v \circ u$; given any exact triangles over those morphisms as in (TR 4) above, there exist morphisms $f : U \to W$ and $g : W \to V$ such that (1), (2) and (3) holds in (TR 4) and moreover such that the following triangles are exact:

$$(4)\quad Y \xrightarrow{\ s\ } W \xrightarrow{\begin{pmatrix} -w_2 \\ g \end{pmatrix}} T(X) \oplus V \xrightarrow{\begin{pmatrix} T(u) & v_2 \end{pmatrix}} T(Y) \quad \text{where } s := f\, u_1 = w_1\, v,$$

$$(5)\quad Y \xrightarrow{\begin{pmatrix} v \\ u_1 \end{pmatrix}} Z \oplus U \xrightarrow{\begin{pmatrix} w_1 & -f \end{pmatrix}} W \xrightarrow{\ t\ } T(Y) \quad \text{where } t := v_2\, g = T(u)\, w_2.$$

This enriched axiom is satisfied by the usual triangulated categories, in particular by the derived category of an exact category. Moreover, it passes to triangulated sub-categories and to localization. It was already considered by Beilinson, Bernstein and Deligne in [4, *Remarque* 1.1.13]. As suggested in *loco citato*, the triangles (4) and (5) are usually not mentioned because they are not used. We shall use them hereafter in the technical section 3, when we shall deal with sub-lagrangians.

The author is not aware of the existence of any pre-triangulated category not satisfying (TR4$^+$), and, *a fortiori*, of any triangulated category not satisfying (TR4$^+$). Nevertheless, it is reasonable to think that such examples exist.

1. The four Witt groups of a triangulated category with duality.

1.1. Once and for all. Let K denote a triangulated category and T be its *translation automorphism.* Here, we will always suppose that $\frac{1}{2} \in K$. This means that the abelian group $\mathrm{Hom}_K(A,B)$ is uniquely 2-divisible for all objects A and B in K.

1.2. Definition. Let $\delta = \pm 1$. An additive contravariant functor $\# : K \to K$ is said to be *δ-exact* if $T \circ \# = \# \circ T^{-1}$ and if, for any exact triangle

$$A \xrightarrow{u} B \xrightarrow{v} C \xrightarrow{w} T(A),$$

the following triangle is exact:

$$C^{\#} \xrightarrow{v^{\#}} B^{\#} \xrightarrow{u^{\#}} A^{\#} \xrightarrow{\delta \cdot T(w^{\#})} T(C^{\#})$$

where $(-)^{\#}$ trickily denotes $\#(-)$. Suppose, moreover, that there exists an isomorphism of functors $\varpi : \mathrm{Id} \xrightarrow{\sim} \# \circ \#$ such that $\varpi_{T(M)} = T(\varpi_M)$ and $(\varpi_M)^{\#} \circ \varpi_{M^{\#}} = \mathrm{Id}_{M^{\#}}$ for any object M of K. Then the triple $(K, \#, \varpi)$ is called a *triangulated category with δ-duality.* In case $\delta = 1$, we shall simply talk about a *duality* and in case $\delta = -1$, we shall use the word *skew-duality.*

1.3. Easy and important exercise. Let $(K, \#, \varpi)$ be a triangulated category with δ-duality ($\delta = \pm 1$).
(1) Let $n \in \mathbb{Z}$. Prove that $(K, T^n \circ \#, \varpi)$ is again a triangulated category with $\big((-1)^n \cdot \delta\big)$-duality.
(2) Prove that $(K, \#, -\varpi)$ is again a triangulated category with δ-duality. Same δ !

1.4. Notions using only the additive structure. Let $(K, \#, \varpi)$ be a triangulated category with δ-duality ($\delta = \pm 1$). In particular, K is an additive category with duality. A *symmetric space* is a pair (P, φ) such that P is an object in K and $\varphi : P \xrightarrow{\sim} P^{\#}$ is a *symmetric form*, meaning that $\varphi^{\#} \circ \varpi_P = \varphi$. Orthogonal sum and isometries are defined as usual. A *skew-symmetric form* $\varphi : P \xrightarrow{\sim} P^{\#}$ verifies $\varphi^{\#} \circ \varpi_P = -\varphi$ or, in other words, is a symmetric form in $(K, \#, -\varpi)$. Pay attention : the sign $\delta = \pm 1$ has nothing to do *a priori* with symmetry and skew-symmetry.

1.5. Remark. Our K is more than an additive category : we have the cone data. If u is symmetric morphism (*i.e.* $u^{\#} = u$ *without* assuming u to be an isomorphism), then the cone of u is more than the homological information about u, it also carries a symmetric structure. Let us make this more precise.

This is a central point of our study : do not skip this part ! It will be useful for the definitions of *metabolic* spaces and of the connecting homomorphisms in the long exact sequence of localization.

1.6. Theorem. *Let $\delta = \pm 1$. Let $(K, \#, \varpi)$ be a triangulated category with δ-duality. Suppose that $\frac{1}{2} \in K$. Let A be an object of K and $u : A \to A^{\#}$ be a symmetric morphism, that is $u^{\#} \circ \varpi_A = u$.*
(1) *Choose any exact triangle over u :*

$$A \xrightarrow{u} A^{\#} \xrightarrow{u_1} C \xrightarrow{u_2} T(A).$$

Then there exists an isomorphism ψ such that the diagram :

$$(\Gamma)\qquad \begin{tikzcd} A \arrow[r, "u"] \arrow[d, equal, "\delta \cdot \varpi_A"'] & A^{\#} \arrow[r, "u_1"] \arrow[d, equal] & C \arrow[r, "u_2"] \arrow[d, "\psi", "\simeq"'] & T(A) \arrow[d, equal, "\delta \cdot T(\varpi_A)"] \\ A^{\#\#} \arrow[r, "\delta \cdot u^{\#}"'] & A^{\#} \arrow[r, "-T(u_2^{\#})"'] & T(C^{\#}) \arrow[r, "T(u_1^{\#})"'] & T(A^{\#\#}) \end{tikzcd}$$

commutes and such that

$$T\big(\psi^{\#}\big) \circ \varpi_C = (-\delta) \cdot \psi.$$

(2) *Other choices of (C, u_1, u_2) and ψ satisfying (1) give an isometric space (C, ψ).*

1.7. Proof. Choose an exact triangle like in (1). Then dualize it using definition 1.2 to get:

$$C^{\#} \xrightarrow{u_1^{\#}} A^{\#\#} \xrightarrow{u^{\#}} A^{\#} \xrightarrow{\delta \cdot T(u_2^{\#})} T(C^{\#})$$

and rotate it to put $u^{\#}$ in the first place. Make an even number of sign changes to get the second line of (Γ). Now the symmetry hypothesis on u can be expressed as the commutativity of the left square of (Γ). By the third axiom of triangulated categories, one can complete it with some morphism ψ satisfying:

$$\begin{cases} \psi\, u_1 = -T(u_2^{\#}) \\ T(u_1^{\#})\,\psi = \delta \cdot T(\varpi_A)\, u_2. \end{cases}$$

Apply $T \circ \#$ to those equations and compose on the right the first line with ϖ_C and the second with $\varpi_{A^{\#}}$. Use definition of $\varpi : \mathrm{Id} \simeq \# \circ \#$ to find that

$$-\delta \cdot T(\psi^{\#}) \circ \varpi_C$$

can replace ψ in the diagram (Γ). Since $\frac{1}{2} \in K$, we can replace ψ by $\frac{1}{2}\big(\psi - \delta \cdot T(\psi^{\#}) \circ \varpi_C\big)$. This morphism is necessarily an isomorphism (*confer* §0) and satisfies (1).

Let us prove (2). Part (1) says that ψ is symmetric in $\big(K, T \circ \#, (-\delta) \cdot \varpi\big)$ which is a triangulated category with the $(-\delta)$-duality

$$* := T \circ \#.$$

Pay attention that "$(-\delta)$-duality" does not come from $(-\delta) \cdot \varpi$ but from Exercise 1.3 part 1 for $n = 1$!

Suppose that we made another choice of ψ, call it χ, satisfying (1). Then $\chi^{-1}\psi$ fits in an endomorphism of the triangle over u:

$$\begin{tikzcd} A \arrow[r, "u"] \arrow[d, equal] & A^{\#} \arrow[r, "u_1"] \arrow[d, equal] & C \arrow[r, "u_2"] \arrow[d, "\chi^{-1}\psi", "\simeq"'] & T(A) \arrow[d, equal] \\ A \arrow[r, "u"'] & A^{\#} \arrow[r, "u_1"'] & C \arrow[r, "u_2"'] & T(A). \end{tikzcd}$$

By § 0, $\chi^{-1}\psi = 1 + k$ with $k^2 = 0$. It is easy to check that $\chi\, k = k^*\, \chi$. Therefore,

$$(1 + \frac{1}{2}k)^*\, \chi\, (1 + \frac{1}{2}k) = \chi\, (1 + \frac{1}{2}k)^2 = \chi\, (1 + k) = \psi.$$

That is, χ and ψ are isometric. It is even easier to prove that the isometry class of (C, ψ) does not depend on the choice of the exact triangle. ♮

1.8. Definition. As suggested by the above result and by its proof, it is convenient to make the following convention. Let $(K, \#, \varpi)$ be a triangulated category with δ-duality for $\delta = \pm 1$. Then the *translated* (or *shifted*) structure of triangulated category with $(-\delta)$-duality is

$$T\big(K, \#, \varpi\big) := \big(K,\, T \circ \#\,,\, (-\delta) \cdot \varpi\,\big).$$

1.9. Remark and definition. In other words, if $\#$ is a duality, we consider $-\varpi$ as the identification $\mathrm{Id} \xrightarrow{\sim} (T \circ \#)^2$. But, if $\#$ is a skew-duality, then we use ϖ again as the identification $\mathrm{Id} \xrightarrow{\sim} (T \circ \#)^2$. This construction is invertible and we can define: for any triangulated category with δ-duality $(K, \#, \varpi)$,

$$T^{-1}\big(K, \#, \varpi\big) := \Big(K,\, T^{-1} \circ \#\,,\, (+\delta) \cdot \varpi\,\Big)$$

which is a structure of triangulated category with $(-\delta)$-duality. Check that this is really the inverse construction! If confusion with signs occurs, redo Exercise 1.3.

For example, if $(K, \#, \varpi)$ is a triangulated category with duality (*i.e.* $\delta = +1$) then $T^n(K, \#, \varpi) = \big(K, T^n \circ \#, (-1)^{\frac{n(n+1)}{2}} \cdot \varpi\big)$ is a triangulated category with $(-1)^n$-duality, for all $n \in \mathbb{Z}$.

With this terminology, Theorem 1.6 can be rephrased as: *The cone of a symmetric morphism for* $(K, \#, \varpi)$ *inherits a symmetric form for the translated duality* $T(K, \#, \varpi)$. *This symmetric form is well defined up to isometry.* This justifies the following definition.

1.10. Definition. Let $(K, \#, \varpi)$ be a triangulated category with δ-duality (for $\delta = \pm 1$). Let A be an object in K and $u : A \to A^{\#}$ be a symmetric morphism ($u^{\#} \circ \varpi_A = u$). The *cone* of (A, u) is defined to be the symmetric space

$$\operatorname{Cone}(A, u) := (C, \psi)$$

fitting in the diagram (Γ) of Theorem 1.6. This is a space for the shifted duality $T(K, \#, \varpi)$. It is well defined up to (non-unique) isometry.

1.11. Remark. Consider $\#$ as the translated duality of $T^{-1} \circ \#$. Theorem 1.6 allows us to construct trivial forms for $\#$ starting with symmetric morphisms u for that *previous* skew-duality $T^{-1} \circ \#$. These are the forms for the duality $\#$ which we are going to ignore in the Witt group. In [1] and [2], we chose to call these forms "neutral" because we also had to deal with classical metabolic forms and we wanted to avoid the confusion.

1.12. Definition. Let $(K, \#, \varpi)$ be a triangulated category with δ-duality ($\delta = \pm 1$). A symmetric space (P, φ) is *neutral* (or *metabolic*) if there exist an object A and a morphism $u : A \to T^{-1}(A^{\#})$ such that

$$T^{-1}(u^{\#}) \circ (\delta \cdot \varpi_A) = u \qquad \text{and} \qquad (P, \varphi) = \operatorname{Cone}(A, u).$$

It is easy to convince oneself that this is the natural generalization of Knebusch's definition of *metabolic forms* (see [9]). Compare with [2, *définition* 2.18, p. 32] and the subsequent remarks or with Lemma 2.1 (2) below for the "optical" approach to the analogy.

1.13. Definition. Let $(K, \#, \varpi)$ be a triangulated category with δ-duality ($\delta = \pm 1$). We define the *Witt monoid* of K to be the monoid of isometry classes of symmetric spaces endowed with the orthogonal sum. We write it as

$$\operatorname{MW}(K, \#, \varpi).$$

Obviously, since this cone-construction is additive and since neutrality is preserved by isometry, the set of isometry classes of neutral spaces form a well defined sub-monoid of $\operatorname{MW}(K, \#, \varpi)$ that we shall call

$$\operatorname{NW}(K, \#, \varpi).$$

The quotient monoid is a group (for quotient of abelian monoids, see [2, *remarque* 2.27, p. 35] if necessary), called the *Witt group* of $(K, \#, \varpi)$ and written as

$$\operatorname{W}(K, \#, \varpi) = \frac{\operatorname{MW}(K, \#, \varpi)}{\operatorname{NW}(K, \#, \varpi)}.$$

If (P, φ) is a symmetric space, we write $[P, \varphi]$ for its class in the corresponding Witt group. We say that two symmetric spaces are *Witt-equivalent* if their classes in the Witt groups are the same.

Referring to Exercise 1.3 and definitions 1.8 and 1.9, we define

$$\operatorname{W}^n(K, \#, \varpi) := \operatorname{W}\big(T^n(K, \#, \varpi)\big)$$

for all $n \in \mathbb{Z}$. These are the *shifted* Witt groups of $(K, \#, \varpi)$.

1.14. Proposition. *Let $(K, \#, \varpi)$ be a triangulated category with duality ($\delta = \pm 1$). The translation functor $T : K \to K$ induces isomorphisms:*

$$\operatorname{W}(K, T^n \circ \#, \varpi) \xrightarrow{\sim} \operatorname{W}(K, T^{n+2} \circ \#, \varpi)$$

for all $n \in \mathbb{Z}$. In particular, we have the four-periodicity:

$$\operatorname{W}^n(K) \xrightarrow{\sim} \operatorname{W}^{n+4}(K).$$

1.15. Proof. See [1, Proposition 1.20] or [2, *proposition* 2.38, p. 37] or do it as an easy exercise. To see the periodicity, recall that definition 1.8 gives $T^2(K, \#, \varpi) = (K, T^2 \circ \#, -\varpi)$. Therefore the isomorphism of the proposition has to be used twice to reach $(K, T^4 \circ \#, +\varpi)$ which is $T^4(K, \#, \varpi)$. $\sharp$

1.16. Remark. Some people like to call symmetric forms for $(K, \#, -\varpi)$ *skew-symmetric*, considering that $(K, \#, \varpi)$ is somehow fixed. Using the previous proposition and our convention for the translation of a duality, the Witt group of skew-symmetric forms is nothing but $\mathrm{W}^2(K, \#, \varpi)$.

On a given triangulated category with a $(+1)$-duality $\#$, there is essentially one *associated* skew-duality, namely $T \circ \#$. In terms of Witt groups, it means that we have two Witt groups for the given duality and two Witt groups for *the* associated skew-duality, since in both cases one can consider symmetric or skew-symmetric forms. Then we keep four Witt groups:

$\mathrm{W}(K) \quad = \mathrm{W}(K, \#, \varpi)$ the *natural* Witt group of symmetric forms.

$\mathrm{W}^2(K) \quad \cong \mathrm{W}(K, \#, -\varpi)$ the *natural* Witt group of skew-symmetric forms.

$\mathrm{W}^1(K) \quad = \mathrm{W}(K, T \circ \#, -\varpi)$ the *associated* Witt group of skew-symmetric forms, using *the* skew-duality associated to $\#$.

$\mathrm{W}^3(K) \quad \cong \mathrm{W}(K, T \circ \#, \varpi)$ the *associated* Witt group of symmetric forms.

1.17. Example. Let X be a scheme. Consider $\mathrm{D}^{\mathrm{b}}_{\mathrm{lf}}(X)$ the derived category of bounded complexes of locally free $\mathcal{O}_X$-modules of finite rank. This is naturally endowed with a structure of triangulated category with duality by localizing the usual duality on the exact category of locally free $\mathcal{O}_X$-modules of finite rank. Its Witt group is the same as the usual one, as we shall establish in part II of this series. This generalizes to any exact category with duality and we shall prove it in this generality.

1.18. Remark. It is interesting to notice that there exists some kind of a Witt group for finitely generated (or coherent) modules also. To understand this, note that under certain hypotheses $\mathrm{D}^{\mathrm{b}}_{\mathrm{coh}}(X)$ is endowed an essentially unique duality as explained in [7, chapter V, pp. 252-301]. In other words, this is a G_0 (or K_0') type of Witt group. Even for rings, it is quite unusual, the point being that any coherent module becomes reflexive in the derived category $\mathrm{D}^{\mathrm{b}}_{\mathrm{coh}}(X)$, which is pretty *wrong* in the abelian category of coherent modules. But this will be another story.

1.19. Remark. As could be said at any step in this article, similar things were done before, in special cases, that could be re-interpreted in triangular terms if the link between the usual Witt group and the Witt group of some derived category was done. But, for rings in which 2 is invertible, this link was established in [1, Theorem 4.29]. Let me here mention two relations between results of section 1 and existing literature that were kindly brought to my attention by the referee, whom I partially quote hereafter.

First, concerning the cone construction of Theorem 1.6, for which two special cases have been around for a long time:

(1) If L is an integral lattice in a rational symmetric form, with dual lattice, then there is defined a symmetric linking pairing on the finite abelian group $L^{\#}/L$. This was the starting point of Karoubi [8], Pardon [11] and Ranicki [13].
(2) The boundary of an n-dimensional symmetric complex C is an $(n-1)$-dimensional Poincaré complex ∂C, following Ranicki [12, p. 141].

Secondly, the 4-periodicity for the higher Witt groups has a collection of antecedents in the already mentioned literature. Let me refer for instance to 4-periodicity in the quadratic L-groups $L_*(R)$ of Wall [16], which was interpreted in terms of Poincaré duality chain complexes in [12].

2. Neutrality, stable neutrality and Witt-triviality.

From now on, we shall identify any object with its double dual, that is consider $\varpi = \mathrm{Id}$.

2.1. Lemma. *Let $(K, \#)$ be a triangulated category with duality. Let (P, φ) be a symmetric space. The following conditions are equivalent:*

(1) *(P, φ) is neutral;*

(2) *there exist L, α and w such that the triangle:*

$$T^{-1}(L^{\#}) \xrightarrow{\quad w \quad} L \xrightarrow{\quad \alpha \quad} P \xrightarrow{\quad \alpha^{\#}\varphi \quad} L^{\#}$$

is exact and such that $T^{-1}(w^{\#}) = w$;

(3) *there exists an exact triangle:*

$$T^{-1}(M^{\#}) \xrightarrow{\quad \nu_0 \quad} L \xrightarrow{\quad \nu_1 \quad} P \xrightarrow{\quad \nu_2 \quad} M^{\#}$$

and an isomorphism $h : L \xrightarrow{\sim} M$ such that the following diagram commutes:

$$\begin{tikzcd}
T^{-1}(M^{\#}) \arrow[r, "\nu_0"] \arrow[d, "T^{-1}(h^{\#})"', "\simeq"] & L \arrow[r, "\nu_1"] \arrow[d, "\simeq"', "h"] & P \arrow[r, "\nu_2"] \arrow[d, "\simeq"', "\varphi"] & M^{\#} \arrow[d, "\simeq"', "h^{\#}"] \\
T^{-1}(L^{\#}) \arrow[r, "T^{-1}(\nu_0^{\#})"'] & M \arrow[r, "\nu_2^{\#}"'] & P^{\#} \arrow[r, "\nu_1^{\#}"'] & L^{\#}.
\end{tikzcd}$$

2.2. Proof. Using definition 1.12 for $\delta = +1$ and diagram (Γ) of Theorem 1.6 (recall that $T^{-1}\#$ is a skew-duality), it is obvious that condition (1) and (2) are equivalent. Clearly (2) implies (3) with $h = \mathrm{Id}$.

Suppose now that there exists a diagram like in (3). The triangle

$$T^{-1}(L^{\#}) \xrightarrow{\nu_0 \circ T^{-1}(h^{\#})^{-1}} L \xrightarrow{\quad \nu_1 \quad} P \xrightarrow{\quad h^{\#} \circ \nu_2 \quad} L^{\#}$$

is isomorphic to the exact triangle of (3) and is therefore exact. Let $w = \nu_0 \circ T^{-1}(h^{\#})^{-1}$ and $\alpha = \nu_1$. Then the triple (L, α, w) satisfies (2). $\sharp$

2.3. Definition. Given a neutral symmetric space (P, φ), a triple (L, α, w) satisfying condition (2) of the above lemma is called a *lagrangian* of (P, φ). In particular, one has $\alpha^{\#}\varphi\,\alpha = 0$.

Given any symmetric space (P, φ), a pair (L, α) is called a *sub-lagrangian* of (P, φ) if $\alpha : L \to P$ is a morphism such that $\alpha^{\#}\varphi\,\alpha = 0$. We shall deal with sub-lagrangians in section 3.

A lagrangian is clearly a sub-lagrangian. See also Remark 3.2.

2.4. Definitions and remark. A symmetric space (P, φ) is said to be *stably neutral* if there exists a neutral space (R, ψ) such that

$$(P, \varphi) \perp (Q, \chi) \simeq (R, \psi) \perp (Q, \chi)$$

for some symmetric space (Q, χ). This implies that $(P, \varphi) \perp$ some neutral space is neutral (by adding $(Q, -\chi)$ on both sides). The latter is the same as saying that $[P, \varphi] = 0$ in $\mathrm{W}(K)$, which we shall call *Witt-triviality.* Obviously neutral spaces are stably neutral and therefore Witt-trivial. The converse is wrong in the *usual* framework (there exist non-metabolic spaces which are zero in the Witt group) but is true in the *triangulated* framework.

2.5. Theorem. *Let K be a triangulated category with duality. Suppose that $\frac{1}{2} \in K$. Any Witt-trivial symmetric space is neutral.*

2.6. Proof. Let (P,φ) be a Witt-trivial space. By the above remark, there exists a space (Q,χ) such that the space $(P,\varphi)\perp(Q,\chi)\perp(Q,-\chi)$ is neutral. Since $\frac{1}{2} \in K$, $(Q,\chi)\perp(Q,-\chi)$ is isometric to $(Q \oplus Q^{\#}, \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix})$ and this means that the space

$$(P \oplus Q \oplus Q^{\#}, \begin{pmatrix} \varphi & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{pmatrix})$$

is neutral. By condition (2) of Lemma 2.1 applied to the above space, there exists an exact triangle:

$$T^{-1}(L^{\#}) \xrightarrow{w} L \xrightarrow{\begin{pmatrix} a \\ b \\ c \end{pmatrix}} P \oplus Q \oplus Q^{\#} \xrightarrow{(\,a^{\#}\varphi \quad c^{\#} \quad b^{\#}\,)} L^{\#}$$

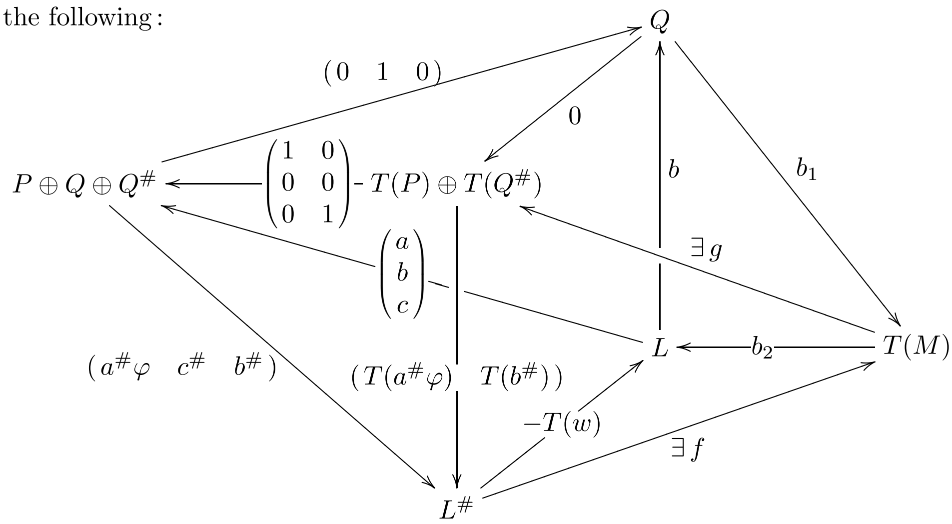
with $w = T^{-1}(w^{\#})$ for some morphisms a, b and c as above. Chose an exact triangle containing b:

$$L \xrightarrow{b} Q \xrightarrow{b_1} T(M) \xrightarrow{b_2} T(L)$$

(we choose to call $T(M)$ the cone of b). Now, apply the octahedron axiom to the relation

$$(\,0 \quad 1 \quad 0\,) \cdot \begin{pmatrix} a \\ b \\ c \end{pmatrix} = b.$$

This gives the following:



for some morphisms $f : L^{\#} \to T(M)$ and $g : T(M) \to T(P) \oplus T(Q^{\#})$ such that the above diagram is an octahedron (meaning that faces are alternatively commutative or exact triangles and that the two ways from the left to the right (respectively the two ways from the right to the left) coincide). *Confer* section 0 for the octahedron axiom.

In particular, the condition $\begin{pmatrix} 1 & 0 \\ 0 & 0 \\ 0 & 1 \end{pmatrix} \cdot g = T(\begin{pmatrix} a \\ b \\ c \end{pmatrix}) \cdot b_2$ forces g to be $\begin{pmatrix} T(a)b_2 \\ T(c)b_2 \end{pmatrix}$. Thus the above octahedron reduces to the existence of a morphism $f : L^{\#} \to T(M)$ such that:

(1) the following triangle is exact:

$$L^{\#} \xrightarrow{f} T(M) \xrightarrow{\begin{pmatrix} T(a)b_2 \\ T(c)b_2 \end{pmatrix}} T(P) \oplus T(Q^{\#}) \xrightarrow{(\,T(a^{\#}\varphi) \quad T(b^{\#})\,)} T(L^{\#})$$

(2) $b_2\, f = -T(w)$;
(3) $f\, c^{\#} = b_1$.

It will appear that $(M, a{\circ}T^{-1}(b_2), m)$ is a lagrangian of (P,φ) for a suitable morphism $m : T^{-1}(M^{\#}) \to M$ (more or less). Therefore we baptize:

$$\alpha := a \circ T^{-1}(b_2) \;:\; M \longrightarrow P.$$

The reluctant reader can motivate himself by checking that $\alpha^{\#}\varphi\,\alpha = 0$. Let us consider the following identity: $(\, a^{\#}\varphi \quad b^{\#}\,) \cdot \begin{pmatrix} 0 \\ 1 \end{pmatrix} = b^{\#}$. Apply to this the octahedron axiom again to find a morphism $m : T^{-1}(M^{\#}) \to M$ such that:

(4) the following triangle is exact:

$$T^{-1}(M^{\#}) \xrightarrow{m} M \xrightarrow{\alpha} P \xrightarrow{\alpha^{\#}\varphi} M^{\#}$$

(5) $m\,b_2^{\#} = -T^{-1}f$

(6) $c\,T^{-1}(b_2)\,m = -b_1^{\#}$.

To make (M,α,m) a lagrangian of (P,φ) we still need $T^{-1}(m^{\#}) = m$. So let us consider the triangles over m and over $T^{-1}(m^{\#})$. To lighten a little bit the notations, we shall write

$$* = T^{-1} \circ \#.$$

Dualizing the exact triangle (4), we get the second line of the following diagram, whose first line is simply exact triangle (4):

$$(7) \qquad \begin{tikzcd} M^* \arrow[r, "m"] \arrow[d, equal] & M \arrow[r, "\alpha"] \arrow[d, dashed, "\exists\, 1+x"', "\simeq"] & P \arrow[r, "\alpha^{\#}\varphi"] \arrow[d, "\varphi"', "\simeq"] & M^{\#} \arrow[d, equal] \\ M^* \arrow[r, "m^*"'] & M \arrow[r, "\varphi\alpha"'] & P^{\#} \arrow[r, "\alpha^{\#}"'] & M^{\#} \end{tikzcd}$$

and by axiom (TR 3) of triangulated categories, the above morphism of exact triangles can be completed. The third morphism is called $1 + x$ because we are going to prove that x is nilpotent. This is quite a technical computation. Let us do it step by step, very carefully. Directly from (7), we get:

(8) $x\,m = m^* - m$

(9) $\alpha\,x = 0$.

From (8), we immediately have $m^* x^* = (x\,m)^* = (m^* - m)^* = m - m^* = -x\,m$. Call this

$$(10) \qquad m^* x^* = -x\,m.$$

Now, compose (5) on the right with c^* and use (3) to obtain

$$(11) \qquad m\,b_2^{\#}\,c^* = -T^{-1}(b_1).$$

Apply $*$ to this last equality, use (6) and replace m^* by $(1+x)\,m$, which is allowed by (7), to obtain:

$$(12) \qquad c\,T^{-1}(b_2)\,x\,m = 0.$$

Dualize this last relation, use (10) and then use (11) to replace $m\,b_2^{\#}c^*$ to get:

$$(13) \qquad x\;T^{-1}(b_1) = 0.$$

Now we are going to use the symmetry of the w given at the beginning to deduce some information about x. Compose relation (5) on the left by $T^{-1}(b_2)$ and use (2) to obtain that

$$T^{-1}(b_2)\,m\,b_2^{\#} = -T^{-1}(b_2)\,T^{-1}(f) = w.$$

Since $w = w^*$, the left hand side of the above equation is also $*$-symmetric. Use this and replace m^* by $(1+x)\,m$, to obtain $T^{-1}(b_2)\,x\,m\,b_2^\# = 0$. But then (5) allows us to replace $m\,b_2^\#$ and to have

$$T^{-1}(b_2)\,x\,T^{-1}(f) = 0. \tag{14}$$

We are almost done. Recalling the definition of $\alpha = a\,T^{-1}(b_2)$, composing (9) with m on the right gives $a\,T^{-1}(b_2)\,x\,m = 0$. Since, on the other hand (12) insures us that $c\,T^{-1}(b_2)\,x\,m = 0$, we have

$$\begin{pmatrix} a\,T^{-1}(b_2) \\ c\,T^{-1}(b_2) \end{pmatrix} x\,m = 0.$$

Since (1) is an exact triangle, there exists, by considerations of § 0 on weak cokernels, a morphism $y : M^* \to L^*$ such that

$$xm = T^{-1}(f)\,y. \tag{15}$$

From relation (13) and the exact triangle over b that we chose at the very beginning of the proof, there exists a morphism $z : L \to M$ such that

$$x = z\,T^{-1}(b_2). \tag{16}$$

Now we are going to compute x^3m using the above relations. By (15) and (16), we have

$$x^3m = x \cdot x \cdot (xm) = z\,T^{-1}(b_2) \cdot x \cdot T^{-1}(f)\,y.$$

But (14) insures that the composition of the three morphisms in the middle is zero. Thus we have established $x^3m = 0$. Of course, from (9), we have that $\alpha\,x^3 = 0$. In other words, x^3 makes the following diagram commute:

$$\begin{tikzcd} M^* \arrow[r, "m"] \arrow[d, "0"'] & M \arrow[r, "\alpha"] \arrow[d, "x^3"] & P \arrow[r, "\alpha^\# \varphi"] \arrow[d, "0"] & M^\# \arrow[d, "0"] \\ M^* \arrow[r, "m"'] & M \arrow[r, "\alpha"'] & P \arrow[r, "\alpha^\# \varphi"'] & M^\#. \end{tikzcd}$$

It is an easy exercise (*confer* § 0) to show that such endomorphisms have trivial square:

$$x^6 = 0.$$

Hence, $h := 1 + \dfrac{1}{2}\,x$ is an automorphism of M. Relation (10) says that $m^*x^* = -x\,m$. This and the fact that $m^* = (1+x)\,m$ insures that $m^*\,h^* = h\,m$. In other words, the following diagram commutes (since, of course, we still have $\alpha\,h = \alpha$ by (9)):

$$\begin{tikzcd} M^* \arrow[r, "m"] \arrow[d, "h^*"', "\simeq"] & M \arrow[r, "\alpha"] \arrow[d, "h"', "\simeq"] & P \arrow[r, "\alpha^\# \varphi"] \arrow[d, "\varphi"', "\simeq"] & M^\# \arrow[d, "h^\#"', "\simeq"] \\ M^* \arrow[r, "m^*"'] & M \arrow[r, "\varphi\alpha"'] & P^\# \arrow[r, "\alpha^\#"'] & M^\#. \end{tikzcd}$$

Lemma 2.1 insures that (P, φ) is neutral. $\sharp$

2.7. Remark. The same proof goes through for a skew-duality.

2.8. Exercise. Establish how a stably metabolic space in the usual sense becomes neutral in the derived category. The above proof gives an explicit way to find the lagrangian. Of course, the latter will not be a complex concentrated in degree 0.

3. The sub-lagrangian construction.

In this section, we fix the triangulated category K and its duality $\#$. The case of a skew-duality can be treated as well and is left to the reader.

3.1. Sub-lagrangians and orthogonal. The classical theory of symmetric spaces as well as some technical problems we shall encounter in the subsequent sections lead us to the following questions.

Consider a symmetric space (P,φ). Suppose we have a sub-lagrangian (L,ν_1), *i.e.* L is an object and $\nu_1 : L \to P$ is a morphism such that

$$\nu_1^{\#}\varphi\,\nu_1 = 0.$$

Here are natural questions:

(1) Can we define $L^{\perp}$, the orthogonal of L?
(2) Can we map $L \to L^{\perp}$?
(3) Can we endow $\mathrm{Cone}(L \to L^{\perp})$ with a structure of symmetric space Witt-equivalent to (P,φ)?

The first mental step is to renounce to the constraint for L or for $L^{\perp}$ to be sub-objects of P, as well as L to be a sub-object of $L^{\perp}$. In triangulated categories, it would definitely be a too strong hypothesis since any monomorphism (*i.e.* any morphism α such that $\alpha\,\beta = 0$ forces $\beta = 0$) is a split inclusion as a direct summand. In other words, this would be the same as forgetting the triangulated structure and focusing only on the additive one.

In the classical case, $L^{\perp}$ is defined to be the kernel of $\nu_1^{\#}\circ\varphi$. Since we know that $T^{-1}\big(\mathrm{Cone}(u)\big)$ is a weak kernel of u, the natural analogous to $L^{\perp}$ would be here

$$T^{-1}\big(\mathrm{Cone}(\nu_1^{\#}\circ\varphi)\big)$$

which is the same as $\big(\mathrm{Cone}(\nu_1)\big)^{\#}$ as can be deduced from exactness of $\#$ (*confer* definition 1.2). Therefore, we prefer to introduce directly the following exact triangle (setting, if you prefer, $M = \mathrm{Cone}(\nu_1)^{\#} = L^{\perp}$):

$$(\Delta) \qquad T^{-1}(M^{\#}) \xrightarrow{\nu_0} L \xrightarrow{\nu_1} P \xrightarrow{\nu_2} M^{\#}$$

in which M is already the orthogonal of L. Then, dualizing the above triangle, we get:

$$(\Delta^{\#}) \qquad T^{-1}(L^{\#}) \xrightarrow{T^{-1}(\nu_0^{\#})} M \xrightarrow{\nu_2^{\#}} P^{\#} \xrightarrow{\nu_1^{\#}} L^{\#}.$$

This allows us to understand the more precise definition of *the orthogonal of* $L \xrightarrow{\nu_1} P$:

$$(L,\nu_1)^{\perp} := (M, \varphi^{-1}\nu_2^{\#}).$$

3.2. Remark. Triangle Δ and the above definition of the orthogonal give the following reformulation of Lemma 2.1 into a vaguer but more conceptual maxim: *a lagrangian is a sub-lagrangian which is equal to its orthogonal.*

3.3. Definition. We are going to construct a morphism of triangles between Δ and $\Delta^{\#}$ of 3.1. Its first part is given by $\varphi : P \to P^{\#}$. Since $\nu_1^{\#}\circ(\varphi\,\nu_1) = 0$ and since $\Delta^{\#}$ is exact, there exists a morphism $\alpha : L \to M$ such that $\varphi\,\nu_1 = \nu_2^{\#}\,\alpha$ (see below). The third axiom of triangulated categories insures us of the existence of a third morphism $\beta : M^{\#} \to L^{\#}$ such that the following diagram commutes:

$$\begin{tikzcd}
T^{-1}(M^{\#}) \arrow[r, "\nu_0"] \arrow[d, dashed, "T^{-1}(\beta)"'] & L \arrow[r, "\nu_1"] \arrow[d, "\alpha"'] & P \arrow[r, "\nu_2"] \arrow[d, "\simeq"', "\varphi"] & M^{\#} \arrow[d, dashed, "\exists\beta"] \\
T^{-1}(L^{\#}) \arrow[r, "T^{-1}(\nu_0^{\#})"'] & M \arrow[r, "\nu_2^{\#}"'] & P^{\#} \arrow[r, "\nu_1^{\#}"'] & L^{\#}.
\end{tikzcd}$$

Of course, we would prefer to have $\beta = \alpha^{\#}$. This is easy to obtain. It suffices to define $\eta_0 := \frac{1}{2}(\alpha + \beta^{\#})$ and to check, using the above diagram, that the following diagram commutes:

$$(\omega)\qquad \begin{tikzcd} T^{-1}(M^{\#}) \arrow[r, "\nu_0"] \arrow[d, "T^{-1}(\eta_0^{\#})"'] & L \arrow[r, "\nu_1"] \arrow[d, "\eta_0"'] & P \arrow[r, "\nu_2"] \arrow[d, "\varphi", "\simeq"'] & M^{\#} \arrow[d, "\eta_0^{\#}"] \\ T^{-1}(L^{\#}) \arrow[r, "T^{-1}(\nu_0^{\#})"'] & M \arrow[r, "\nu_2^{\#}"'] & P^{\#} \arrow[r, "\nu_1^{\#}"'] & L^{\#}. \end{tikzcd}$$

For a fixed choice of Δ, we shall call a *good morphism* $\eta_0 : L \to L^{\perp}$ any morphism $\eta_0 : L \to M$ such that the above diagram (ω) commutes. We have proved the following proposition:

3.4. Proposition. *Let (P, φ) be a symmetric space and (L, ν_1) be a sub-lagrangian of this space. Let*

$$T^{-1}(M^{\#}) \xrightarrow{\nu_0} L \xrightarrow{\nu_1} P \xrightarrow{\nu_2} M^{\#}$$

be any choice of an exact triangle containing ν_1, then there exists a good morphism $\eta_0 : L \longrightarrow L^{\perp}$. $\sharp$

3.5. The questions. Now that we have sent any sub-lagrangian into its orthogonal, it is natural, as in the classical case, to consider the analogue of $L^{\perp}/L$, namely to choose an exact triangle over η_0. We display such a triangle in the second column of the following diagram in which the first column is the dual of the second:

$$(\Omega)\qquad \begin{tikzcd} T^{-1}(M^{\#}) \arrow[r, "\nu_0"] \arrow[d, "T^{-1}(\eta_0^{\#})"'] & L \arrow[r, "\nu_1"] \arrow[d, "\eta_0"] & P \arrow[r, "\nu_2"] \arrow[d, "\varphi", "\simeq"'] & M^{\#} \arrow[d, "\eta_0^{\#}"] \\ T^{-1}(L^{\#}) \arrow[r, "T^{-1}(\nu_0^{\#})"'] \arrow[d, "\eta_2^{\#}"'] & M \arrow[r, "\nu_2^{\#}"'] \arrow[d, "\eta_1"] & P^{\#} \arrow[r, "\nu_1^{\#}"'] & L^{\#} \\ Q^{\#} \arrow[d, "\eta_1^{\#}"'] & Q \arrow[d, "\eta_2"] & & \\ M^{\#} \arrow[r, "T(\nu_0)"'] & T(L). & & \end{tikzcd}$$

We shall refer several times to this diagram in the sequel. We now ask the following question:

Question A. *Given a commutative diagram Ω with exact lines and columns, can we construct a symmetric form $\psi : Q \xrightarrow{\sim} Q^{\#}$ such that (P, φ) and (Q, ψ) are Witt-equivalent?*

More precisely, we could also ask ψ to fit in the above diagram. As an exercise, the reader should try to solve the problem for $L = 0$. Even in this very case, it is immediate that we cannot expect any form making the above diagram commute to be Witt-equivalent to (P, φ). It appears also from this example that one cannot simply complete the above diagram using the axiom (TR3). The induced morphism $Q^{\#} \to Q$ might very well not be an isomorphism and can even be zero. For these reasons, it is important to consider the morphism:

$$s := \nu_2\, \varphi^{-1} \nu_2^{\#} \,:\, M \to M^{\#}.$$

This is the symmetric map induced by φ on $M = L^{\perp}$ (which is sent into P through $\varphi^{-1}\nu_2^{\#}$):

$$(\varphi^{-1}\nu_2^{\#})^{\#}\, \varphi\, (\varphi^{-1}\nu_2^{\#}) = \nu_2\, \varphi^{-1} \nu_2^{\#} = s.$$

This morphism will play a central role hereafter. For some reasons, the form ψ we shall be able to construct and that will fit in Ω will actually give the opposite of $[P,\varphi]$ in the Witt group. Of course, this answers also the question since we can replace ψ by $-\psi$. Therefore, when we consider a form $\psi : Q \xrightarrow{\sim} Q^{\#}$ "making the diagram (Ω) commutative", we shall also ask for:

$$\eta_1^{\#}\psi\,\eta_1 = -\nu_2\,\varphi^{-1}\nu_2^{\#} \qquad (= -s).$$

We are then lead to ask the following second, quite natural question:

Question B. *Given a commutative diagram $\bar\Omega$ as below with exact lines and columns:*

$$(\bar\Omega)\qquad \begin{tikzcd} T^{-1}(M^{\#}) \arrow[r, "\nu_0"] \arrow[d, "T^{-1}(\eta_0^{\#})"'] & L \arrow[r, "\nu_1"] \arrow[d, "\eta_0"] & P \arrow[r, "\nu_2"] \arrow[d, "\simeq"', "\varphi"] & M^{\#} \arrow[d, "\eta_0^{\#}"] \\ T^{-1}(L^{\#}) \arrow[r, "T^{-1}(\nu_0^{\#})"'] \arrow[d, "\eta_2^{\#}"'] & M \arrow[r, "\nu_2^{\#}"'] \arrow[d, "\eta_1"] & P^{\#} \arrow[r, "\nu_1^{\#}"'] & L^{\#} \\ Q^{\#} \arrow[d, "\eta_1^{\#}"'] & Q \arrow[l, "\psi = \psi^{\#}"', "\simeq"] \arrow[d, "\eta_2"] & & \\ M^{\#} \arrow[r, "T(\nu_0)"'] & T(L) & & \end{tikzcd}$$

such that

$$\eta_1^{\#}\psi\,\eta_1 = -\nu_2\,\varphi^{-1}\nu_2^{\#}$$

can we conclude that (P,φ) and $(Q,-\psi)$ are Witt-equivalent?

and therefore the intermediate question:

Question C. *Given a commutative diagram Ω as in Question A, can we complete it into a diagram $\bar\Omega$ as in Question B?*

We shall see that with some restriction on η_0, we can give a positive answer to those questions. First of all, we establish that there is at most one solution to Question C (Theorem 3.8). To do that, we start by proving quite a weird lemma, that we shall use later again.

3.6. Lemma. *Assume that we have the following mysterious and commutative diagrams (one for f and one for g):*

$$\begin{tikzcd} A \arrow[r, "u"] \arrow[d, "\alpha"'] & B \arrow[r, "v"] \arrow[d, "\beta"'] & C \arrow[r, "w"] & T(A) \arrow[d, "T(\alpha)"] \\ A' \arrow[r, "u'"'] & B' \arrow[r, "v'"'] & C' \arrow[r, "w'"'] \arrow[u, "f,g"'] & T(A') \end{tikzcd}$$

and assume that the two lines are exact triangles. Suppose that f is an isomorphism and that

$$w\,f\,v' = w\,g\,v'.$$

Then g is also an isomorphism and, more precisely, there exists $x : C' \to C'$ such that

(1) $g = f\,(1+x)$
(2) $x^3 = 0$.

3.7. Proof. Since f is an isomorphism, it suffices to define $x = f^{-1}g - 1$ to have $g = f\,(1+x) = f + f\,x$. We have to check that $x^3 = 0$.

From the commutativity of the diagrams (middle square), we have $v = g\,v'\,\beta = (f + f\,x)\,v'\,\beta = v + f\,x\,v'\,\beta$ and therefore $f\,x\,v'\,\beta = 0$, *i.e.* (f being an isomorphism and using commutativity again) $x\,f^{-1}v = 0$. From the exact triangle over v, we deduce that there exists $\bar{x} : T(A) \to C'$ such that

$$(1) \qquad x = \bar{x}\,w\,f.$$

From the other commutativities (right square), we obtain in the same way that $T(\alpha)\,w\,f = T(\alpha)\,w\,g = T(\alpha)\,w\,(f + f\,x)$ and thus $T(\alpha)\,w\,f\,x = 0$, *i.e.* $w'\,x = 0$. From the exact triangle over w', we deduce the existence of a morphism $\tilde{x} : C' \to B'$ such that

$$(2) \qquad x = v'\,\tilde{x}.$$

Now, we also assumed that $w\,f\,v' = w\,g\,v'$ which forces

$$(3) \qquad w\,f\,x\,v' = w\,(g - f)\,v' = 0.$$

Now, using (1) and (2), it is easy to conclude that

$$x^3 = \bar{x}\,w\,f\,x\,v'\,\tilde{x} \overset{(3)}{=} 0.$$

Of course, this implies that $(1 + x)$ is an isomorphism and the result. $\sharp$

3.8. Theorem. *Let $(K, \#)$ be a triangulated category with duality containing $\frac{1}{2}$. Consider a commutative diagram Ω with exact rows and columns like in 3.5. Suppose that there exist two symmetric forms $\psi, \psi' : Q \to Q^\#$ fitting in diagram $\bar{\Omega}$ and such that $\eta_1^\#\,\psi\,\eta_1 = \eta_1^\#\,\psi'\,\eta_1$. Then ψ and ψ' are isometric.*

3.9. Proof. Apply the above lemma to the diagram obtained by the two first columns of $\bar{\Omega}$ with $f = \psi$ and $g = \psi'$. The hypotheses about ψ and ψ' are exactly the ones of the lemma. Therefore, there exists $x : Q \to Q$ such that $x^3 = 0$ and $\psi' = \psi\,(1 + x)$. Now, from the symmetry of ψ and ψ', one immediately gets $x^\#\,\psi = \psi\,x$. Consider the isomorphism ($x^3 = 0\,!$)

$$h := 1 + \frac{1}{2}\,x - \frac{1}{8}\,x^2 \quad : Q \overset{\sim}{\to} Q.$$

Note that $h^\#\,\psi = \psi\,h$ and that $h^2 = 1 + x$. Therefore:

$$h^\#\,\psi\,h = \psi\,h^2 = \psi\,(1 + x) = \psi'$$

which is the announced result. $\sharp$

3.10. Remark. A last problem makes the story a little bit more complicated. In real life, the (good) morphism $\eta_0 : L \to L^\perp$ is usually chosen by the opponent! (*confer* Lemma 5.4). Of course, it might happen that some choices of η_0 allow a sub-lagrangian construction and others don't. This illustrates the usual non-uniqueness problem in triangulated categories (this problem was evacuated in the special case of [1, Theorem 3.11, p. 110] by assuming $\mathrm{Hom}_K(L, T^{-1}L^\#) = 0$, but we can't do any such hypothesis here). Anyway, we shall use several such good morphisms $\eta_0 : L \to L^\perp$, some better than others, and it will be useful to have the following momentary terminology.

3.11. Definition. Let (P, φ) be a symmetric space and (L, ν_1) be a sub-lagrangian of this space. For a fixed choice of an exact triangle

$$(\Delta) \qquad T^{-1}(M^\#) \xrightarrow{\nu_0} L \xrightarrow{\nu_1} P \xrightarrow{\nu_2} M^\#$$

we say that a good morphism $\mu_0 : L \to L^\perp$ (definition 3.3) is *very good* if moreover the triangle

$$M \xrightarrow{\nu_2\,\varphi^{-1}\nu_2^\#} M^\# \xrightarrow{\begin{pmatrix} T(\nu_0) \\ -\mu_0^\# \end{pmatrix}} T(L) \oplus L^\# \xrightarrow{\begin{pmatrix} T(\mu_0) & \nu_0^\# \end{pmatrix}} T(M)$$

is exact.

As already said before, the morphism $s = \nu_2\,\varphi^{-1}\nu_2^\# : M \to M^\#$ plays a central role and the above exact triangle will be useful in the forthcoming proofs. We shall now see that very good morphisms exist and later on, we shall see their use to tackle the sub-lagrangian questions.

3.12. Orientation. The end of section 3 goes as follows: We are going to establish a few lemmas about good and very good morphisms and we shall then state the first existence theorem (Theorem 3.17) which says that one can modify a good morphism into a very good morphism by keeping its cone ("$L^\perp/L$") up to stable isomorphism.

Then, we shall mention that, for very good morphisms, Question C of 3.5 has a positive answer (Remark 3.19). Finally, assuming that our triangulated category satisfies (TR4$^+$), we shall give a positive answer to Question B, again for very good morphisms only (Theorem 3.20). Since Question A comes from Question B + Question C, very good morphisms merit their name.

From now on, the symmetric space (P,φ) and its lagrangian (L,ν_1) is fixed. As well, we choose an exact triangle:

$$(\Delta)\qquad T^{-1}(M^\#)\xrightarrow{\nu_0}L\xrightarrow{\nu_1}P\xrightarrow{\nu_2}M^\#.$$

Recall also that $s:=\nu_2\,\varphi^{-1}\nu_2^\#\ :M\to M^\#$ and that

$$*:=T^{-1}\circ\#.$$

3.13. Lemma. *Adopt the notations of 3.12. Let $\eta_0\ :L\to L^\perp$ be a good morphism (definition 3.3). Then there exists a morphism $\lambda:L\to L^*$ such that*

(1) $\nu_0^*\,\lambda\,\nu_0=0$
(2) *the morphism $\mu_0:=\eta_0+\nu_0^*\,\lambda\ :L\to L^\perp$ is very good (definition 3.11).*

3.14. Proof. Let us give a name to the following diagram, which is commutative because η_0 is good:

$$(\omega)\qquad \begin{tikzcd} M^* \arrow[r, "\nu_0"] \arrow[d, "\eta_0^*"'] & L \arrow[r, "\nu_1"] \arrow[d, "\eta_0"] & P \arrow[r, "\nu_2"] \arrow[d, "\varphi", "\simeq"'] & M^\# \arrow[d, "\eta_0^\#"] \\ L^* \arrow[r, "\nu_0^*"'] & M \arrow[r, "\nu_2^\#"'] & P^\# \arrow[r, "\nu_1^\#"'] & L^\#. \end{tikzcd}$$

We are going to study exact triangles over $s=\nu_2\,\varphi^{-1}\nu_2^\#$.

Consider the octahedron axiom for the relation $\nu_1^\#\circ(\varphi\,\nu_1)=0$. We get the following picture, in which the reader should ignore for a while the isomorphism on the right (the exact triangles over $\nu_1^\#$ and over $\varphi\,\nu_1$ are obtained from Δ):

$$(3)\qquad \begin{tikzcd} & & & L^\# \arrow[dl, "\nu_0^\#"'] \arrow[ddrr, "\begin{pmatrix}0\\1\end{pmatrix}"] & & & \\ P^\# \arrow[urrr, "\nu_1^\#"] \arrow[ddrr, "-\nu_2\,\varphi^{-1}"'] & & T(M) \arrow[ll, "T(\nu_2^\#)"'] \arrow[dd, "-T(s)"] & & & & \\ & & & L \arrow[uu, "0"] \arrow[ulll, "\varphi\,\nu_1"] & & T(L)\oplus L^\# \arrow[ll, "(1\quad 0)"'] \arrow[ulll, "\exists(c\quad d)"'] \arrow[r, "\begin{pmatrix}1&0\\-e&1\end{pmatrix}", "\simeq"'] & T(L)\oplus L^\# \\ & & M^\# \arrow[ur, "T(\nu_0)"] \arrow[urrr, "\exists\begin{pmatrix}a\\b\end{pmatrix}"'] & & & & \end{tikzcd}$$

for some morphisms a,b,c,d. The relations involving the injection $\begin{pmatrix}0\\1\end{pmatrix}$ and the projection $(1\quad 0)$ do force $a=T(\nu_0)$ and $d=\nu_0^\#$. As well, the relation

$$b\circ\nu_2\overset{(3)}{=}-\nu_1^\#\varphi\overset{(\omega)}{=}-\eta_0^\#\circ\nu_2$$

insures the existence of a morphism $e : T(L) \to L^\#$ such that:

$$b = -\eta_0^\# + e\,T(\nu_0).$$

One can use the automorphism $\begin{pmatrix} 1 & 0 \\ -e & 1 \end{pmatrix}$ of $T(L)\oplus L^\#$ presented in the above picture (3) to replace b by $-\eta_0^\#$. The morphism c is modified into $c+\nu_0^\# e$, which still satisfies $T(\nu_2^\#)\,(c+\nu_0^\# e) = T(\nu_2^\#)\,c = T(\varphi\nu_1)$. We baptize $\beta := T^{-1}\big(c+\nu_0^\# e\big)$. The octahedron reduces then to the existence of an exact triangle:

$$M \xrightarrow{s} M^\# \xrightarrow{\begin{pmatrix} T(\nu_0) \\ -\eta_0^\# \end{pmatrix}} T(L)\oplus L^\# \xrightarrow{\begin{pmatrix} T(\beta) & \nu_0^\# \end{pmatrix}} T(M) \tag{4}$$

for some morphism $\beta : L \to M$ such that:

$$\nu_2^\# \beta = \varphi\,\nu_1. \tag{5}$$

Of course, the esthete would prefer to have $\beta = \eta_0$, in which case we say that η_0 is very good (see definition 3.11). This is not true in general but it appears that one can replace η_0 by some μ_0 such that the following triangle is exact:

$$M \xrightarrow{s} M^\# \xrightarrow{\begin{pmatrix} T(\nu_0) \\ -\mu_0^\# \end{pmatrix}} T(L)\oplus L^\# \xrightarrow{\begin{pmatrix} T(\mu_0) & \nu_0^\# \end{pmatrix}} T(M). \tag{6}$$

Namely, this μ_0 will be very good and will be constructed as $\mu_0 := \eta_0 + \nu_0^*\,\lambda$ for some morphism $\lambda : L \to L^*$ such that $\nu_0^*\,\lambda\,\nu_0 = 0$ as announced in the lemma.

From property (5) and from (ω), we deduce that $\nu_2^\#\beta = \varphi\,\nu_1 = \nu_2^\#\eta_0$. In other words, $\nu_2^\#(\beta-\eta_0) = 0$. Then using exact triangle $(\Delta^\#)$, we know that there exists a morphism $l : L \to L^*$ such that

$$\beta = \eta_0 + \nu_0^*\,l. \tag{7}$$

Dualizing exact triangle (4), recalling that $s^\# = s$, gives the following exact triangle:

$$M \xrightarrow{s} M^\# \xrightarrow{\begin{pmatrix} \beta^\# \\ T(\nu_0) \end{pmatrix}} L^\#\oplus T(L) \xrightarrow{\begin{pmatrix} -\nu_0^\# & +T(\eta_0) \end{pmatrix}} T(M).$$

Then using the isomorphism $\begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}$ from $L^\#\oplus T(L)$ to $T(L)\oplus L^\#$, we get another exact triangle. We display it as the second row of the following diagram, in which the first is simply (4):

$$\begin{tikzcd} M \arrow[r, "s"] \arrow[d, equal] & M^\# \arrow[r, "\begin{pmatrix} T(\nu_0) \\ -\eta_0^\# \end{pmatrix}"] \arrow[d, dashed, "\exists\, 1+h"] & T(L)\oplus L^\# \arrow[r, "\begin{pmatrix} T(\beta) & \nu_0^\# \end{pmatrix}"] \arrow[d, "\begin{pmatrix} 1 & 0 \\ T(l) & 1 \end{pmatrix}"] & T(M) \arrow[d, equal] \\ M \arrow[r, "s"'] & M^\# \arrow[r, "\begin{pmatrix} T(\nu_0) \\ -\beta^\# \end{pmatrix}"'] & T(L)\oplus L^\# \arrow[r, "\begin{pmatrix} T(\eta_0) & \nu_0^\# \end{pmatrix}"'] & T(M). \end{tikzcd} \tag{8}$$

Its right square commutes because of (7). Since it has exact rows, there exists a morphism $M^\# \to M^\#$ such that the entire diagram commutes and nobody can keep us from calling it $1+h$ for some morphism $h : M^\# \to M^\#$. We have the following relations:

(9) $h\,s = 0$ (10) $T(\nu_0)\,h = 0$ (11) $-\beta^\#(1+h) = T(l)\,T(\nu_0) - \eta_0^\#$.

Compose (11) on the right with ν_2, use (5) and (ω) to prove that:

$$\beta^{\#} h\, \nu_2 = 0. \tag{12}$$

From (9) and the exact triangle presented in the second line of (8), we deduce that

$$h = \bar{h} \begin{pmatrix} T(\nu_0) \\ -\beta^{\#} \end{pmatrix} \tag{13}$$

for some morphism $\bar{h} : T(L) \oplus L^{\#} \to M^{\#}$. Similarly, from (10) and exact triangle (Δ), we deduce that:

$$h = \nu_2\, \tilde{h} \tag{14}$$

for some morphism $\tilde{h} : M^{\#} \to P$.

Now, compute h^3 using (13) and (14):

$$h^3 = \bar{h} \begin{pmatrix} T(\nu_0) \\ -\beta^{\#} \end{pmatrix} \circ h \circ (\nu_2\, \tilde{h}) = \bar{h} \begin{pmatrix} T(\nu_0)\, h\, \nu_2 \\ -\beta^{\#}\, h\, \nu_2 \end{pmatrix} \tilde{h}.$$

It follows then immediately from (10) and (12) that $h^3 = 0$. Therefore $1 + \frac{1}{2} h : M^{\#} \xrightarrow{\sim} M^{\#}$ is an isomorphism.

But, dualizing (8) and using as before the isomorphism $\begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}$, we get the following morphism of the same exact triangles: $(1 + h^{\#}, 1, \begin{pmatrix} 1 & 0 \\ -l^{\#} & 1 \end{pmatrix})$. Taking the mean of this last morphism with the one of (8), we get the following morphism of exact triangles:

$$\begin{tikzcd}[column sep=large]
M \arrow[r, "s"] \arrow[d, "1+\frac{1}{2}h^{\#}"', "\simeq"] & M^{\#} \arrow[r, "\begin{pmatrix} T(\nu_0) \\ -\eta_0^{\#} \end{pmatrix}"] \arrow[d, "\simeq"', "1+\frac{1}{2}h"] & T(L) \oplus L^{\#} \arrow[r, "\begin{pmatrix} T(\beta) & \nu_0^{\#} \end{pmatrix}"] \arrow[d, "\begin{pmatrix} 1 & 0 \\ \frac{1}{2}(T(l) - l^{\#}) & 1 \end{pmatrix}"] & T(M) \arrow[d, "\simeq"', "1+\frac{1}{2}T(h^{\#})"] \\
M \arrow[r, "s"'] & M^{\#} \arrow[r, "\begin{pmatrix} T(\nu_0) \\ -\beta^{\#} \end{pmatrix}"'] & T(L) \oplus L^{\#} \arrow[r, "\begin{pmatrix} T(\eta_0) & \nu_0^{\#} \end{pmatrix}"'] & T(M).
\end{tikzcd} \tag{15}$$

Let us give names:

$$f := 1 + \frac{1}{2} h^{\#} : M \xrightarrow{\sim} M \qquad \text{and} \qquad \bar{l} := \frac{l - l^*}{2} = -\bar{l}^* : L \to L^*.$$

Pushing f up and $f^{\#}$ down in (15), we obtain the following morphism of exact triangles:

$$\begin{tikzcd}[column sep=large]
M \arrow[r, "s\, f^{-1}"] \arrow[d, equal] & M^{\#} \arrow[r, "\begin{pmatrix} T(\nu_0) \\ -\eta_0^{\#} \end{pmatrix}"] \arrow[d, equal] & T(L) \oplus L^{\#} \arrow[r, "T(f) \circ \begin{pmatrix} T(\beta) & \nu_0^{\#} \end{pmatrix}"] \arrow[d, "\begin{pmatrix} 1 & 0 \\ T(\bar{l}) & 1 \end{pmatrix}"] & T(M) \arrow[d, equal] \\
M \arrow[r, "(f^{\#})^{-1} s"'] & M^{\#} \arrow[r, "\begin{pmatrix} T(\nu_0) \\ -\beta^{\#} \end{pmatrix} \circ f^{\#}"'] & T(L) \oplus L^{\#} \arrow[r, "\begin{pmatrix} T(\eta_0) & \nu_0^{\#} \end{pmatrix}"'] & T(M).
\end{tikzcd}$$

From (9) and the definition of f, we have $s\, f = s$ and from (10) we have $T(\nu_0)\, f^{\#} = T(\nu_0)$. Use this in the above diagram to obtain the following one:

$$\begin{tikzcd}[column sep=large]
M \arrow[r, "s"] \arrow[d, equal] & M^{\#} \arrow[r, "\begin{pmatrix} T(\nu_0) \\ -\eta_0^{\#} \end{pmatrix}"] \arrow[d, equal] & T(L) \oplus L^{\#} \arrow[r, "\begin{pmatrix} T(\gamma) & \nu_0^{\#} \end{pmatrix}"] \arrow[d, "\begin{pmatrix} 1 & 0 \\ T(\bar{l}) & 1 \end{pmatrix}"] & T(M) \arrow[d, equal] \\
M \arrow[r, "s"'] & M^{\#} \arrow[r, "\begin{pmatrix} T(\nu_0) \\ -\gamma^{\#} \end{pmatrix}"'] & T(L) \oplus L^{\#} \arrow[r, "\begin{pmatrix} T(\eta_0) & \nu_0^{\#} \end{pmatrix}"'] & T(M)
\end{tikzcd} \tag{16}$$

where $\gamma := f\,\beta$. Note that the exact triangle of the first line gives: $0 = \begin{pmatrix} T(\gamma) & \nu_0^{\#} \end{pmatrix} \cdot \begin{pmatrix} T(\nu_0) \\ -\eta_0^{\#} \end{pmatrix} = T(\gamma\,\nu_0) - \nu_0^{\#}\eta_0^{\#}$ which implies

$$\gamma\,\nu_0 = \nu_0^*\eta_0^* \overset{(\omega)}{=} \eta_0\,\nu_0. \tag{17}$$

Now, since the third commutative square of (16) gives

$$\gamma = \eta_0 + \nu_0^*\bar{l} \tag{18}$$

we have, composing this last equation with ν_0 on the right and comparing with (17), that:

$$\nu_0^*\bar{l}\,\nu_0 = 0. \tag{19}$$

Consider now the following commutative diagram (direct check):

$$\begin{array}{ccccccc} M & \xrightarrow{s} & M^{\#} & \xrightarrow{\begin{pmatrix} T(\nu_0) \\ -\eta_0^{\#} \end{pmatrix}} & T(L)\oplus L^{\#} & \xrightarrow{\begin{pmatrix} T(\gamma) & \nu_0^{\#} \end{pmatrix}} & T(M) \\ \| & & \| & & \simeq\Big\downarrow \begin{pmatrix} 1 & 0 \\ \frac{1}{2}T(\bar{l}) & 1 \end{pmatrix} & & \| \\ M & \xrightarrow[s]{} & M^{\#} & \xrightarrow[\begin{pmatrix} T(\nu_0) \\ \frac{1}{2}T(\bar{l}\,\nu_0) - \eta_0^{\#} \end{pmatrix}]{} & T(L)\oplus L^{\#} & \xrightarrow[\begin{pmatrix} T(\gamma - \frac{1}{2}\nu_0^*\bar{l}) & \nu_0^{\#} \end{pmatrix}]{} & T(M) \end{array} \tag{23}$$

in which the first row (and therefore also the second) is an exact triangle (see 16). Define

$$\mu_0 := \eta_0 + \frac{1}{2}\nu_0^*\bar{l}.$$

Observe that (18) implies $\gamma - \frac{1}{2}\nu_0^*\bar{l} = \eta_0 + \frac{1}{2}\,\nu_0^*\bar{l} = \mu_0$ and that (here we use $\bar{l} = -\bar{l}^*$ which is immediate from the definition of $\bar{l}$):

$$\mu_0^{\#} = \eta_0^{\#} + \frac{1}{2}\,\bar{l}^{\#}T(\nu_0) = \eta_0^{\#} + \frac{1}{2}\,T(\bar{l}^*\,\nu_0) = \eta_0^{\#} - \frac{1}{2}\,T(\bar{l}\nu_0).$$

Replace these two last facts in the second line of (23) to obtain the announced exact triangle (6):

$$M \xrightarrow{s} M^{\#} \xrightarrow{\begin{pmatrix} T(\nu_0) \\ -\mu_0^{\#} \end{pmatrix}} T(L)\oplus L^{\#} \xrightarrow{\begin{pmatrix} T(\mu_0) & \nu_0^{\#} \end{pmatrix}} T(M)$$

guaranteeing that μ_0 is very good (condition (2) of the lemma). Define $\lambda = \frac{1}{2}\bar{l}$, so that μ_0 is $\eta_0 + \nu_0^*\lambda$ as required. Condition (1) requires $\nu_0^*\lambda\,\nu_0 = 0$, which is a direct consequence of (19). ♯

3.15. Lemma. *Adopt the notations of 3.12. Let $\eta_0 : L \to L^{\perp}$ be a good morphism. Then*

$$\mathrm{Cone}(\eta_0\,\nu_0) \simeq \mathrm{Cone}(\eta_0) \oplus \mathrm{Cone}(\nu_0).$$

3.16. Proof. Choose an exact triangle containing η_0: $L \xrightarrow{\eta_0} M \xrightarrow{\eta_1} \mathrm{Cone}(\eta_0) \xrightarrow{\eta_2} T(L)$. From the definition of good morphisms (see definition 3.3), we have $\varphi\,\nu_1 = \nu_2^{\#}\,\eta_0$. Therefore, we have $\varphi\,\nu_1\,T^{-1}(\eta_2) = \nu_2^{\#}\,\eta_0\,T^{-1}(\eta_2) = 0$. Since φ is an isomorphism, this forces:

$$\nu_1 \circ T^{-1}(\eta_2) = 0 \;:\; T^{-1}\big(\mathrm{Cone}(\eta_0)\big) \longrightarrow P \simeq \mathrm{Cone}(\nu_0)$$

Apply the octahedron axiom to this last relation and you immediately obtain the result of the lemma. ♯

3.17. Theorem – Existence of very good morphisms. *Let $(K, \#)$ be a triangulated category with duality containing $\frac{1}{2}$. Let (P, φ) be a symmetric space and (L, ν_1) be a sub-lagrangian of this space. Let*

$$T^{-1}(M^{\#}) \xrightarrow{\nu_0} L \xrightarrow{\nu_1} P \xrightarrow{\nu_2} M^{\#}$$

be any choice of an exact triangle containing ν_1 and let $\eta_0 : L \to L^{\perp}$ be a good morphism. Then there exists a very good morphism $\mu_0 : L \to L^{\perp}$ such that $\mathrm{Cone}(\eta_0)$ *and* $\mathrm{Cone}(\mu_0)$ *are stably isomorphic, namely:*

$$\mathrm{Cone}(\eta_0) \oplus P \simeq \mathrm{Cone}(\mu_0) \oplus P.$$

In particular, there exists a very good morphism $\mu_0 : L \to M = L^{\perp}$.

3.18. Proof. By Lemma 3.13, there exists $\lambda : L \to L^*$ such that $\mu_0 := \eta_0 + \nu_0^* \lambda$ is very good and such that $\nu_0^* \lambda \nu_0 = 0$ which implies

$$\mu_0 \nu_0 = \eta_0 \nu_0.$$

Of course μ_0 being very good is also good. By Lemma 3.15, one has:

$$\mathrm{Cone}(\eta_0) \oplus \mathrm{Cone}(\nu_0) \simeq \mathrm{Cone}(\eta_0 \nu_0) = \mathrm{Cone}(\mu_0 \nu_0) \simeq \mathrm{Cone}(\mu_0) \oplus \mathrm{Cone}(\nu_0).$$

Since $\mathrm{Cone}(\nu_0) = P$, one has the result.

The last assertion comes from the fact that there always exists good morphisms $\eta_0 : L \to L^{\perp}$ as we saw in Proposition 3.4. ♯

3.19. Remark. At this point, we know that very good morphisms exist and actually are sufficiently numerous to be found not too far from any given good morphism (recall Remark 3.10). The following result is true:

Theorem – Let $(K, \#)$ be a triangulated category with duality containing $\frac{1}{2}$. Let (P, φ) be a symmetric space and (L, ν_1) be a sub-lagrangian of this space. Adopt the notations of 3.12. Let $\eta_0 : L \to L^{\perp}$ be a very good morphism. Then one can answer positively Question C of 3.5 for this morphism, namely: for any choice of an exact triangle

$$L \xrightarrow{\eta_0} M \xrightarrow{\eta_1} Q \xrightarrow{\eta_2} T(L)$$

there exists a symmetric form $\psi : Q \to Q^{\#}$ such that diagram $\bar{\Omega}$ commutes and such that

$$\eta_1^{\#} \psi \, \eta_1 = -\nu_2 \, \varphi^{-1} \nu_2^{\#}.$$

The proof of this can be found in [2, End of the proof of *théorème* 4.2, p. 52-54]. It was also given in the preprint electronic version of the present article. It is not included here because we are going to give another proof in the case of K satisfying (TR4$^+$). In this later case, we shall prove moreover that this symmetric space (Q, ψ) is Witt-equivalent to $(P, -\varphi)$ and this is the point for which the author has no proof without (TR4$^+$). See Theorem 3.20.

In other words, the following theorem says that we have a positive answer to Questions C and B (and therefore A) of 3.5 if we assume the morphism $L \to L^{\perp}$ to be very good.

3.20. Theorem. *Let $(K,\#)$ be a triangulated category with duality containing $\frac{1}{2}$. Suppose moreover that K satisfies (TR4$^+$).*

$$(\bar{\bar{\Omega}})\qquad \begin{tikzcd}
T^{-1}(M^{\#}) \arrow[r, "\nu_0"] \arrow[d, "T^{-1}(\mu_0^{\#})"'] & L \arrow[r, "\nu_1"] \arrow[d, "\mu_0"] & P \arrow[r, "\nu_2"] \arrow[d, "\varphi", "\simeq"'] & M^{\#} \arrow[d, "\mu_0^{\#}"] \\
T^{-1}(L^{\#}) \arrow[r, "T^{-1}(\nu_0^{\#})"'] \arrow[d, "\mu_2^{\#}"'] & M \arrow[r, "\nu_2^{\#}"'] \arrow[d, "\mu_1"] & P^{\#} \arrow[r, "\nu_1^{\#}"'] & L^{\#} \\
R^{\#} \arrow[d, "\mu_1^{\#}"'] & R \arrow[l, dashed, "\psi=\psi^{\#}"', "\simeq"] \arrow[d, "\mu_2"] & & \\
M^{\#} \arrow[r, "T(\nu_0)"'] & T(L) & &
\end{tikzcd}$$

Let (P,φ) be a symmetric space and (L,ν_1) be a sub-lagrangian of this space. Choose any exact triangle over ν_1, choose any very good morphism $\mu_0 : L \to L^{\perp}$ and choose any exact triangle over μ_0 (as presented in $\bar{\bar{\Omega}}$). Then there exists a symmetric form $\psi : R \to R^{\#}$ such that

1. *The above diagram $\bar{\bar{\Omega}}$ commutes*
2. $\mu_1^{\#}\,\psi\,\mu_1 = -\nu_2\,\varphi^{-1}\nu_2^{\#}$
3. *The space (R,ψ) is Witt-equivalent to $(P,-\varphi)$.*

Moreover, any symmetric form ψ on R satisfying (1) and (2) also satisfies (3).

3.21. Proof. Let us see the "moreover part" first. This is an immediate consequence of Theorem 3.8, since another such form would be isometric to the one whose existence is claimed. Therefore it suffices to prove that there exists a symmetric form satisfying (1), (2) and (3).

We recall the notations: $s := \nu_2\,\varphi^{-1}\nu_2^{\#} : M \to M^{\#}$, $s^{\#} = s$; the previous duality is $* := T^{-1} \circ \#$.

The upper part of $\bar{\bar{\Omega}}$ commutes because μ_0 is good (see definition 3.3). Moreover, from the definition of μ_0 being very good (see definition 3.11), the following triangle is exact:

$$(4)\qquad M \xrightarrow{\nu_2\,\varphi^{-1}\nu_2^{\#}} M^{\#} \xrightarrow{\begin{pmatrix} T(\nu_0) \\ -\mu_0^{\#} \end{pmatrix}} T(L) \oplus L^{\#} \xrightarrow{\begin{pmatrix} T(\mu_0) & \nu_0^{\#} \end{pmatrix}} T(M).$$

From $\bar{\bar{\Omega}}$, we know that $\nu_2^{\#}\mu_0 = \varphi\,\nu_1$. Apply (TR4$^+$) to this relation to find the following octahedron:

$$\begin{tikzcd}
 & & & P^{\#} \arrow[dl, "\nu_1^{\#}"'] \arrow[ddrr, "\nu_2\varphi^{-1}"] & & \\
M \arrow[urrr, "\nu_2^{\#}"] \arrow[ddrr, "\mu_1"'] & & L^{\#} \arrow[ll, "-\nu_0^{\#}"'] \arrow[dd, "-T(\mu_1)\,\nu_0^{\#}"] & & & \\
 & & & L \arrow[uu, "\varphi\,\nu_1"] \arrow[ulll, "\mu_0"'] & & M^{\#} \arrow[ll, "-T\nu_0"'] \arrow[ulll, "\exists\, b"'] \\
 & & R \arrow[ur, "\mu_2"] \arrow[urrr, "\exists\, -a"'] & & &
\end{tikzcd}$$

that is, the existence of two morphisms $a : R \to M^{\#}$ et $b : M^{\#} \to L^{\#}$ such that

(5) $a\,\mu_1 = -\nu_2\,\varphi^{-1}\nu_2^{\#} = -s$
(6) $T(\nu_0)\,a = \mu_2$
(7) $\nu_0^{\#}b = T(\mu_0\,\nu_0)$
(8) $b\,\nu_2\,\varphi^{-1} = \nu_1^{\#}$
(9) the triangle $L^* \xrightarrow{\mu_1\,\nu_0^*} R \xrightarrow{-a} M^{\#} \xrightarrow{b} L^{\#}$ is exact
(10) the triangle $M \xrightarrow{s} M^{\#} \xrightarrow{\begin{pmatrix} T(\nu_0) \\ -b \end{pmatrix}} T(L) \oplus L^{\#} \xrightarrow{\begin{pmatrix} T(\mu_0) & \nu_0^{\#} \end{pmatrix}} T(M)$ is exact
(11) the triangle $M^* \xrightarrow{w} M \xrightarrow{\begin{pmatrix} \nu_2^{\#} \\ \mu_1 \end{pmatrix}} P^{\#} \oplus R \xrightarrow{\begin{pmatrix} \nu_2\varphi^{-1} & a \end{pmatrix}} M^{\#}$ is exact where $w := \mu_0\,\nu_0 \overset{(\bar{\bar{\Omega}})}{=} \nu_0^*\,\mu_0^* = w^*$.

Comparing the two exact triangles over s at our disposal, namely (10) and (4), we get the existence of a morphism $l : M^{\#} \to M^{\#}$ such that the following diagram commutes:

$$\begin{tikzcd}
M \arrow[r, "s"] \arrow[d, equal] & M^{\#} \arrow[r, "\begin{pmatrix} T(\nu_0) \\ -b \end{pmatrix}"] \arrow[d, dashed, "\exists\, 1+l"] & T(L) \oplus L^{\#} \arrow[r, "\begin{pmatrix} T(\mu_0) & \nu_0^{\#} \end{pmatrix}"] \arrow[d, equal] & T(M) \arrow[d, equal] \\
M \arrow[r, "s"'] & M^{\#} \arrow[r, "\begin{pmatrix} T(\nu_0) \\ -\mu_0^{\#} \end{pmatrix}"'] & T(L) \oplus L^{\#} \arrow[r, "\begin{pmatrix} T(\mu_0) & \nu_0^{\#} \end{pmatrix}"'] & T(M)
\end{tikzcd}$$

We immediately get

(12) $l\,s = 0$
(13) $T(\nu_0)\,l = 0$
(14) $\mu_0^{\#}\,(1 + l) = b.$

Composing (13) on the left with $T(\mu_0)$, we get

(15) $T(w)\,l = 0.$

Composing (14) on the right with ν_2, we have

$$\mu_0^{\#}\nu_2 \;+\; \mu_0^{\#}\,l\,\nu_2 = b\,\nu_2 \overset{(8)}{=} \nu_1^{\#}\varphi$$

and taking $\mu_0^{\#}\nu_2 = \nu_1^{\#}\varphi$ from the upper part of $\bar{\bar{\Omega}}$, we end up with:

(16) $\mu_0^{\#}\,l\,\nu_2 = 0.$

From (12) and exact triangle (4), there exists $\bar{l} : T(L) \oplus L^{\#} \to M^{\#}$ such that $l = \bar{l} \cdot \begin{pmatrix} T(\nu_0) \\ -\mu_0^{\#} \end{pmatrix}$. From (13) and the exact triangle over ν_0, there exists a morphism $\tilde{l} : M^{\#} \to P$ such that $l = \nu_2\,\tilde{l}$. Now,

$$l^3 = \bar{l} \cdot \begin{pmatrix} T(\nu_0) \\ -\mu_0^{\#} \end{pmatrix} \cdot l \cdot \nu_2\,\tilde{l} \overset{(13)}{=} \bar{l} \cdot \begin{pmatrix} 0 \\ -\mu_0^{\#} \cdot l \cdot \nu_2 \end{pmatrix} \cdot \tilde{l} \overset{(16)}{=} 0. \tag{17}$$

Observe that from (14), we can construct a isomorphism $\xi : R \to R^{\#}$ such that the following diagram commutes:

$$\begin{tikzcd}
L^* \arrow[r, "\mu_1\,\nu_0^*"] \arrow[d, equal] & R \arrow[r, "-a"] \arrow[d, dashed, "\exists\,\xi"] & M^{\#} \arrow[r, "b"] \arrow[d, "1+l"] & L^{\#} \arrow[d, equal] \\
L^* \arrow[r, "\mu_2^{\#}"'] & R^{\#} \arrow[r, "-\mu_1^{\#}"'] & M^{\#} \arrow[r, "\mu_0^{\#}"'] & L^{\#}
\end{tikzcd} \tag{18}$$

where the two lines are exact triangles by (9) and $(\bar{\bar{\Omega}})$ respectively. Commutativity of (18) gives:

(19) $\xi\,\mu_1\,\nu_0^* = \mu_2^{\#}$
(20) $\mu_1^{\#}\xi = (1+l)\,a$

Composing (20) on the left with $T(\nu_0)$, using (6) and (13) gives the first of the following relations:

(21) $T(\nu_0)\,\mu_1^{\#}\,\xi = \mu_2$
(22) $\mu_1^{\#}\xi\,\mu_1 = -s$

where the second relation is obtained from composing (20) on the right with μ_1, using (5) and (12).

From (16), we have $\mu_0^{\#} l\,\nu_2\,\varphi^{-1} = 0$. Therefore, by the exact triangle over $\mu_0^{\#}$ displayed in the second line of (18), we deduce the existence of a morphism $r : P^{\#} \to R^{\#}$ such that

$$l\,\nu_2\,\varphi^{-1} = \mu_1^{\#}\,r. \tag{23}$$

Now compute:

$$\begin{aligned}\begin{pmatrix}\nu_2 & \mu_1^{\#}\end{pmatrix}\cdot\begin{pmatrix}\varphi^{-1} & 0\\ r & \xi\end{pmatrix} &= \begin{pmatrix}\nu_2\,\varphi^{-1}+\mu_1^{\#}\,r & \mu_1^{\#}\xi\end{pmatrix} \overset{(23)}{=} \begin{pmatrix}\nu_2\,\varphi^{-1}+l\,\nu_2\,\varphi^{-1} & \mu_1^{\#}\xi\end{pmatrix} \overset{(20)}{=}\\ &= \begin{pmatrix}(1+l)\,\nu_2\,\varphi^{-1} & (1+l)\,a\end{pmatrix} = (1+l)\cdot\begin{pmatrix}\nu_2\,\varphi^{-1} & a\end{pmatrix}.\end{aligned}$$

Put this in the following diagram with exact lines (the first being just (11) and the second its dual, using also $w^* = w$):

$$\begin{tikzcd}[ampersand replacement=\&] M^* \arrow[r, "w"] \arrow[d, "1+T^{-1}(l)"'] \& M \arrow[r, "\begin{pmatrix}\nu_2^{\#}\\ \mu_1\end{pmatrix}"] \arrow[d, dashed, "\exists\, 1+p"] \& P^{\#}\oplus R \arrow[r, "\begin{pmatrix}\nu_2\,\varphi^{-1} & a\end{pmatrix}"] \arrow[d, "\begin{pmatrix}\varphi^{-1} & 0\\ r & \xi\end{pmatrix}"] \& M^{\#} \arrow[d, "1+l"] \\ M^* \arrow[r, "w"'] \& M \arrow[r, "\begin{pmatrix}\varphi^{-1}\nu_2^{\#}\\ a^{\#}\end{pmatrix}"'] \& P\oplus R^{\#} \arrow[r, "\begin{pmatrix}\nu_2 & \mu_1^{\#}\end{pmatrix}"'] \& M^{\#} \end{tikzcd} \tag{24}$$

which we complete by $1+p : M \to M$ for some morphism $p : M \to M$ that we shall soon prove to be nilpotent (and a little bit more actually). Immediately from (15), we have $w\,T^{-1}(l) = 0$ and therefore we deduce from (24) the following relations:

(25) $p\,w = 0$
(26) $\nu_2^{\#}\,p = 0$
(27) $a^{\#}(1+p) = r\,\nu_2^{\#} + \xi\,\mu_1$.

Composing the latter with ν_0^* on the right, we find:

$$\underbrace{a^{\#}\nu_0^*}_{\overset{(6)}{=}\ \mu_2^{\#}} + a^{\#}p\,\nu_0^* = r\,\underbrace{\nu_2^{\#}\,\nu_0^*}_{=0} + \underbrace{\xi\,\mu_1\,\nu_0^*}_{\overset{(19)}{=}\ \mu_2^{\#}}$$

which forces $a^{\#}\,p\,\nu_0^* = 0$. Now, as usual, we get from (25) and from the exact triangle in the second line of (24), we get the existence of $\bar{p} : P\oplus R^{\#} \to M$ such that $p = \bar{p}\begin{pmatrix}\varphi^{-1}\nu_2^{\#}\\ a^{\#}\end{pmatrix}$. From (26) and the exact triangle over $\nu_2^{\#}$, we get the existence of $\tilde{p} : M \to L^*$ such that $p = \nu_0^*\,\tilde{p}$. And finally, using $a^{\#}p\,\nu_0^* = 0$:

$$p^3 = \bar{p}\begin{pmatrix}\varphi^{-1}\nu_2^{\#}\\ a^{\#}\end{pmatrix}\cdot p\cdot\nu_0^*\,\tilde{p} \overset{(26)}{=} \bar{p}\begin{pmatrix}0\\ a^{\#}\cdot p\cdot\nu_0^*\end{pmatrix}\cdot\tilde{p} = 0.$$

We are almost done. We have $p^3 = 0$ and $(l^{\#})^3 = 0$ from (17). Moreover from the above factorization $p = \nu_0^*\,\tilde{p}$, we get:

$$l^{\#}\circ p = l^{\#}\,\nu_0^*\,\tilde{p} = \big(T(\nu_0)\,l\big)^{\#}\,\tilde{p} \overset{(13)}{=} 0.$$

Define

$$m := \frac{1}{2}(p + l^{\#})$$

a direct computation using $l^{\#}p = 0$, $p^3 = 0$ and $(l^{\#})^3 = 0$ gives:

$$m^5 = 0$$

which we shall only use to state that $1 + m$ is an isomorphism.

Taking the mean of the morphism displayed in (24) and its dual we get the following one:

$$\text{(28)} \qquad \begin{array}{ccccccc}
M^* & \xrightarrow{\ w\ } & M & \xrightarrow{\begin{pmatrix}\nu_2^{\#}\\ \mu_1\end{pmatrix}} & P^{\#}\oplus R & \xrightarrow{(\nu_2\,\varphi^{-1}\quad a)} & M^{\#} \\
{\scriptstyle 1+m^*}\downarrow\simeq & & \simeq\downarrow{\scriptstyle 1+m} & & \downarrow{\scriptstyle\begin{pmatrix}\varphi^{-1} & \frac12 r^{\#}\\ \frac12 r & \frac{\xi+\xi^{\#}}{2}\end{pmatrix}} & & \simeq\downarrow{\scriptstyle 1+m^{\#}} \\
M^* & \xrightarrow[\ w\ ]{} & M & \xrightarrow[\begin{pmatrix}\varphi^{-1}\nu_2^{\#}\\ a^{\#}\end{pmatrix}]{} & P\oplus R^{\#} & \xrightarrow[(\nu_2\quad \mu_1^{\#})]{} & M^{\#}.
\end{array}$$

Put $\chi := \dfrac{\xi + \xi^{\#}}{2}$. Using (19) and the dual of (21), we immediately observe that

(29) $\chi : R \to R^{\#}$ fits in diagram $\bar{\bar{\Omega}}$

(30) $\mu_1^{\#}\chi\,\mu_1 = -s$

where the second property is immediately obtained from (22) and its dual.

Diagram (28) and Lemma 2.1 tells us that the symmetric space

$$\Big(\, P^{\#} \oplus R\ ,\ \begin{pmatrix}\varphi^{-1} & \frac12 r^{\#}\\ \frac12 r & \chi\end{pmatrix}\Big)$$

is neutral. Therefore, so is any isometric space like:

$$\begin{pmatrix}1 & 0\\ -\frac12 r\,\varphi & 1\end{pmatrix}\cdot\begin{pmatrix}\varphi^{-1} & \frac12 r^{\#}\\ \frac12 r & \chi\end{pmatrix}\cdot\begin{pmatrix}1 & 0\\ -\frac12 r\,\varphi & 1\end{pmatrix}^{\#} = \begin{pmatrix}\varphi^{-1} & 0\\ 0 & \psi\end{pmatrix}$$

where $\psi := \chi - \frac14 r\,\varphi\,r^{\#}$. Therefore the space (R,ψ) is Witt-equivalent to $(P^{\#}, -\varphi^{-1})$ which in turn is isometric to $(P, -\varphi)$. So, the symmetric space (R,ψ) satisfies condition (3) of the theorem. Let us check conditions (1) and (2).

Direct computation gives both:

$$\begin{aligned}
T(\nu_0)\,\mu_1^{\#}\,\psi &= T(\nu_0)\,\mu_1^{\#}\,\chi - \frac14\,T(\nu_0)\,\mu_1^{\#}\,r\,\varphi\,r^{\#} \overset{(29)}{=} \mu_2 - \frac14\,T(\nu_0)\cdot(\mu_1^{\#}\,r\,\varphi)\cdot r^{\#}\\
&\overset{(23)}{=} \mu_2 - \frac14\,T(\nu_0)\,l\,\nu_2\,r^{\#} \overset{(13)}{=} \mu_2
\end{aligned}$$

which insures that ψ fits into diagram $\bar{\bar{\Omega}}$, and

$$\begin{aligned}
\mu_1^{\#}\,\psi\,\mu_1 &= \mu_1^{\#}\,\chi\,\mu_1 - \frac14\,\mu_1^{\#}\,r\,\varphi\,r^{\#}\,\mu_1 \overset{(30)}{=} -s - \frac14\,\mu_1^{\#}\,r\,\varphi\,r^{\#}\,\mu_1\\
&\overset{(23)}{=} -s - \frac14\,l\,\nu_2\,\varphi^{-1}\varphi\,\varphi^{-1}\nu_2^{\#}\,l^{\#} = -s - \frac14\,l\,s\,l^{\#} \overset{(12)}{=} -s.
\end{aligned}$$

♯

3.22. Corollary – Sub-lagrangian Theorem. *Let $(K, \#)$ be a triangulated category with duality containing $\frac{1}{2}$. Suppose moreover that K satisfies (TR4$^+$).*

Let (P, φ) be a symmetric space and (L, ν_1) be a sub-lagrangian of this space. Choose any exact triangle over ν_1 and any morphism $\eta_0 : L \to M$ such that the following diagram commutes:

$$\begin{array}{ccccccc} T^{-1}(M^\#) & \xrightarrow{\nu_0} & L & \xrightarrow{\nu_1} & P & \xrightarrow{\nu_2} & M^\# \\ {\scriptstyle T^{-1}(\eta_0^\#)}\downarrow & & {\scriptstyle \eta_0}\downarrow & & {\scriptstyle \simeq}\downarrow{\scriptstyle \varphi} & & \downarrow{\scriptstyle \eta_0^\#} \\ T^{-1}(L^\#) & \xrightarrow[T^{-1}(\nu_0^\#)]{} & M & \xrightarrow[\nu_2^\#]{} & P^\# & \xrightarrow[\nu_1^\#]{} & L^\#. \end{array}$$

Then, there exists another morphism $\mu_0 : L \to M$ such that

(1) Cone(η_0) *and* Cone(μ_0) *are stably isomorphic,*
(2) Cone(μ_0) *carries a symmetric form Witt-equivalent to (P, φ).*

In particular, for any lagrangian L, there exists a morphism $\mu_0 : L \to L^\perp = M$ and a symmetric form on Cone(μ_0) *which is Witt-equivalent to (P, φ).*

3.23. Proof. This is just juxtaposition of Theorem 3.17 and Theorem 3.20. ♯

3.24. Remark. The final assertion of the above corollary is the answer to the initial naive questions of 3.1. This assertion, although it sounds very nice, is hard to use in general. In the classical case, it was the ideal formulation because there was only one way to send a sub-lagrangian into its orthogonal. Here, there are several (η_0, μ_0, ...), some better than others, as I hope the reader noticed.

Assertions (1) and (2) of the corollary will be more useful.

3.25. Remark. The results of this section might be used to establish some link between the Witt group proposed here and the quotient of it defined by Youssin in [15].

4. Localization context and residue homomorphism.

4.1. Localization of triangulated categories with duality.

Let us consider what we can call an *exact sequence of triangulated categories*:

$$0 \longrightarrow J \longrightarrow K \longrightarrow L \longrightarrow 0.$$

By that, we understand that L is a localization of K, that is $L = S^{-1}K$ for a class S of morphisms. We can always suppose that S is saturated (*i.e.* S is formed by all the morphisms of K which become isomorphisms in L) and that L is obtained by calculus of fraction (nice exercise). Then J is supposed to be the *kernel* of this localization, *i.e.* the full subcategory of K over the objects which become isomorphic to zero in L.

Suppose moreover that K is endowed with a duality (or a skew-duality) $\#$ such that $\#(S) = S$. Then $S^{-1}K$ inherits a unique duality that can be called the localization of $\#$ and that we shall lazily also denote by $\#$. It is then clear that $\#$ restricts to a duality on J. In other words, we have an *exact sequence of triangulated categories with duality*:

$$\text{(LOC)} \qquad \begin{array}{ccccccccc} 0 & \longrightarrow & J & \xrightarrow{j} & K & \xrightarrow{q} & S^{-1}K & \longrightarrow & 0 \\ & & {\scriptstyle \#}\downarrow & & {\scriptstyle \#}\downarrow & & {\scriptstyle \#}\downarrow & & \\ 0 & \longrightarrow & J & \xrightarrow[j]{} & K & \xrightarrow[q]{} & S^{-1}K & \longrightarrow & 0. \end{array}$$

Additional details on localization of triangulated categories are to be found in [14] or in [1].

4.2. Example. Let X be a noetherian regular separated scheme and U an open subscheme. Consider $\mathrm{D}^{\mathrm{b}}_{\mathrm{lf}}(X)$ the derived category of bounded complexes of locally free $\mathcal{O}_X$-modules of finite rank. Then $\mathrm{D}^{\mathrm{b}}_{\mathrm{lf}}(U)$ is a localization of $\mathrm{D}^{\mathrm{b}}_{\mathrm{lf}}(X)$. It is moreover a localization of triangulated categories with duality (*confer* [1, *théorème* 4.17, p. 28]).

The kernel category J is here the (derived) category of complexes whose homology is concentrated on Y, the closed complement of U in X. It is a good question to wonder how the Witt groups of J might be related to the Witt groups of Y.

This very application of the present paper is my main motivation and it will hopefully appear in forthcoming works. However, this and the 12-term localization sequence seem to me independent questions. In other words, first, there is a localization sequence which is easy to formulate in the triangulated framework and second, we shall try to interpret J in terms of Y. This touches another point, namely the difficulty usually encountered to define a good residue of a form on U with values in some Witt groups of Y. When it exists, this residue will actually be decomposable as the general residue presented hereafter followed by some homomorphism between the Witt groups of J and those of Y.

4.3. Remark. If we have an exact sequence (LOC), functoriality of Witt groups naturally induces short complexes:

$$\mathrm{W}^n(J) \xrightarrow{\mathrm{W}^n(j)} \mathrm{W}^n(K) \xrightarrow{\mathrm{W}^n(q)} \mathrm{W}^n(S^{-1}K)$$

for all $n \in \mathbb{Z}$. To get a long (periodic) exact sequence, we are going to construct connecting homomorphisms between $\mathrm{W}^n(S^{-1}K)$ and $\mathrm{W}^{n+1}(J)$. This will use the cone construction of 1.6 - 1.10. First of all we shall recall from [2] the construction of $\mathrm{W}(S^{-1}K)$ using S-spaces.

4.4. Definition. Let $(K, \#, \varpi)$ be a triangulated category with δ-duality ($\delta = \pm 1$). Let S be a system of morphisms in K. An *S-space* is a pair (A, s) where A is an object in K and s is a morphism $s : A \to A^\#$ such that:

(1) $s = s^\#\,\varpi_A$
(2) $s \in S$.

We say that two S-spaces (A, s) and (B, t) are *S-isometric* if there exists an object C and morphisms u, v in S such that $u : C \to A$, $v : C \to B$ and

$$u^\# s\, u = v^\# t\, v.$$

Instead of S-spaces, the term *S-lattice* or *symmetric S-lattice* might be more conceptually accurate.

4.5. Proposition. *Let $J \overset{j}{\hookrightarrow} K \overset{q}{\twoheadrightarrow} S^{-1}K$ be a localization of triangulated categories with δ-duality in the sense of 4.1. The application $(A, s) \mapsto \big(q(A), q(s)\big)$ induces a well defined monoid homomorphism between the set of S-isometry classes of S-spaces and the Witt monoid* $\mathrm{MW}(S^{-1}K)$ *of* $S^{-1}K$ *(see definition 1.13). This homomorphism is an isomorphism.*

Through this application, $\mathrm{NW}(K)$ *maps surjectively onto* $\mathrm{NW}(S^{-1}K)$.

4.6. Proof. Exercise. Solution in [2, *lemme* 2.11, p. 98]. $\sharp$

4.7. Remark. Of course, a symmetric space is an S-space. The above proposition describes the Witt monoid of $S^{-1}K$ in terms of S-spaces in K. Neutral spaces in $S^{-1}K$ are obtained by the neutral spaces of K. In other words, the Witt group of $S^{-1}K$ can be computed as follows. Take S-isometry classes of S-spaces and divide out by neutral spaces of K. As a consequence of that, to define an group homomorphism

$$\mathrm{W}(S^{-1}K) \longrightarrow G$$

to a group G, it suffices to define it additively on S-spaces, to check that it is invariant under S-isometries and that it sends neutral spaces of K to zero in G.

We are going to check that such an homomorphism is induced by the cone construction of 1.6 - 1.10.

4.8. Theorem. *Let $J \overset{j}{\hookrightarrow} K \overset{q}{\twoheadrightarrow} S^{-1}K$ be a localization of triangulated categories with δ-duality in the sense of 4.1. Let (A,s) and (B,t) be S-isometric S-spaces in K. Then*

$$[\mathrm{Cone}(A,s)] = [\mathrm{Cone}(B,t)] \in \mathrm{W}^1(J).$$

4.9. Proof. The proof goes in several steps. First of all, observe that the cone construction is additive:

$$\mathrm{Cone}\big((A,s)\perp(B,t)\big) \simeq \mathrm{Cone}(A,s)\perp\mathrm{Cone}(B,t)$$

and is well defined up to strong isometry:

$$\mathrm{Cone}(B, h^{\#}\, s\, h) \simeq \mathrm{Cone}(A,s)$$

if $h : B \overset{\sim}{\to} A$ is an isomorphism in K.

We shall only prove the case where $\#$ is a duality, that is $\delta = +1$. The skew-duality goes as well. It is convenient to abbreviate

$$\flat := T \circ \#$$

the shifted skew-duality. If (A,s) is an S-space for $\#$, then $\mathrm{Cone}(A,s)$ is a symmetric space for $(K, \flat, -\varpi)$. See, if necessary, (Γ) in Theorem 1.6 and definition 1.10.

4.10. Lemma. *Let (P,φ) be a symmetric space in K and let (L,ν_1) be a sub-lagrangian of this space such that L belongs to J, the kernel category. Choose an exact triangle*

$$T^{-1}(M^{\#}) \xrightarrow{\nu_0} L \xrightarrow{\nu_1} P \xrightarrow{\nu_2} M^{\#}.$$

Then $\nu_2 \in S$ and $L^{\perp} = M$ inherits from φ a structure of S-space, namely $(M, \nu_2\,\varphi^{-1}\nu_2^{\#})$, such that

$$\mathrm{Cone}(M, \nu_2\,\varphi^{-1}\nu_2^{\#}) = \big(T(L)\oplus L^{\#}, \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}\big).$$

In particular, $[\mathrm{Cone}(M, \nu_2^{\#}\varphi^{-1}\nu_2)] = 0$ in $\mathrm{W}^1(J)$.

4.11. Proof. Define $s := \nu_2\,\varphi^{-1}\nu_2^{\#} : M \to M^{\#}$. By Theorem 3.17, there exists a very good morphism $\mu_0 : L \to M$ (*confer* definition 3.11). This means in particular that the following triangle is exact:

$$M \xrightarrow{s} M^{\#} \xrightarrow{\begin{pmatrix} T(\nu_0) \\ -\mu_0^{\#} \end{pmatrix}} T(L)\oplus L^{\#} \xrightarrow{\begin{pmatrix} T(\mu_0) & \nu_0^{\#} \end{pmatrix}} T(M).$$

Since one can use any exact triangle over s to construct $\mathrm{Cone}(M,s)$, let us use the above one.

By definition, $\mathrm{Cone}(M,s) = \big(T(L)\oplus L^{\#}, \psi\big)$ for some form $\psi : T(L)\oplus L^{\#} \longrightarrow \big(T(L)\oplus L^{\#}\big)^{\flat} = L^{\#}\oplus T(L)$ such that $\psi^{\flat} = -\psi$ and such that the following diagram commutes:

$$\begin{tikzcd}
M \arrow[r, "s"] \arrow[d, equal] & M^{\#} \arrow[rr, "s_1 := \begin{pmatrix} T(\nu_0) \\ -\mu_0^{\#} \end{pmatrix}"] \arrow[d, equal] & & T(L)\oplus L^{\#} \arrow[rr, "s_2 := \begin{pmatrix} T(\mu_0) & \nu_0^{\#} \end{pmatrix}"] \arrow[d, "\psi"] & & T(M) \arrow[d, equal] \\
M \arrow[r, "s"'] & M^{\#} \arrow[rr, "-s_2^{\flat} = \begin{pmatrix} -\mu_0^{\#} \\ -T(\nu_0) \end{pmatrix}"'] & & L^{\#}\oplus T(L) \arrow[rr, "s_1^{\flat} = \begin{pmatrix} \nu_0^{\#} & -T(\mu_0) \end{pmatrix}"'] & & T(M).
\end{tikzcd}$$

Clearly, it suffices to take $\psi = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}$. $\sharp$

4.12. Lemma. *Let (P,φ) be a symmetric space in K and let $s : C \to P$ be a morphism in S. Then*

$$[\mathrm{Cone}(C, s^{\#}\varphi\, s)] = 0 \quad \textit{in} \quad \mathrm{W}^1(J).$$

4.13. Proof. Choose an exact triangle over s :

$$(1) \qquad C \xrightarrow{s} P \xrightarrow{s_1} D \xrightarrow{s_2} T(C).$$

Observe that $s \in S$ forces

$$(2) \qquad D \in J.$$

Define

$$t := s^{\#}\varphi\, s : C \to C^{\#}.$$

We want to prove that $\mathrm{Cone}(C,t)$ is Witt-trivial in $(J,\flat)$. That is, we have to find some object N in J and a morphism $w : N \to N^{\#}$ (observe that $T^{-1}\circ\flat = \#$) such that $w = w^{\#}$ and $\mathrm{Cone}(C,t)\perp\mathrm{Cone}(N,w)$ is neutral in J. Choose $N := D^{\#}$ and $w := -s_1\,\varphi^{-1}s_1^{\#} : D^{\#} \to D$. Observe immediately that $w^{\#} = w$.

Choose an exact triangle over t of the form :

$$C \xrightarrow{t} C^{\#} \xrightarrow{t_1} T(L) \xrightarrow{t_2} T(C)$$

and observe that $t \in S$ implies

$$(3) \qquad L \in J.$$

Write down the composition axiom for the relation

$$t = (s^{\#}\varphi)\, s$$

where the exact triangle over $s^{\#}$ (and then over $s^{\#}\varphi$) is constructed from exact triangle (1). This way, one finds an exact triangle

$$D^{\#} \xrightarrow{w} D \xrightarrow{w_1} T(L) \xrightarrow{w_2} T(D^{\#})$$

and a collection of relations among which we shall only use :

$$(4) \qquad T(\varphi^{-1})\, s_1^{\flat} w_2 = T(s)\, t_2.$$

Let us baptize $\nu_1 : L \to P$ the morphism $\nu_1 := s \circ T^{-1}t_2$. We are going to use Lemma 4.10. To do that, observe that (L,ν_1) is a sub-lagrangian of (P,φ) :

$$\nu_1^{\#}\varphi\,\nu_1 \overset{(4)}{=} \big(\varphi^{-1}s_1^{\#}T^{-1}(w_2)\big)^{\#}\varphi\,\big(s\,T^{-1}(t_2)\big) = \big(T(w_2^{\#})\,s_1\,\varphi^{-1}\big)\,\varphi\,\big(s\,T^{-1}(t_2)\big) = 0.$$

To apply the previous lemma, we also need an exact triangle over ν_1. To find it, note that

$$t\circ(-T^{-1}s_2) = -s^{\#}\varphi\, s\, T^{-1}(s_2) = 0.$$

The octahedron applied to this relation gives us an exact triangle :

$$T^{-1}(C^{\#})\oplus T^{-1}(D) \xrightarrow{\nu_0} L \xrightarrow{\nu_1} P \xrightarrow{\nu_2} C^{\#}\oplus D$$

where $\nu_2 = \begin{pmatrix} x \\ s_1 \end{pmatrix}$ for some morphism $x : P \to C^{\#}$ such that $x\,s = t$. But $t = s^{\#}\varphi\, s$. This means that $(s^{\#}\varphi - x)\,s = 0$. We proceed like in 3.14. Using exact triangle (1), we have the existence of some morphism $a : D \to C^{\#}$ such that $s^{\#}\varphi - x = a\, s_1$, which, in turn, can be seen as

$$\begin{pmatrix} 1 & a \\ 0 & 1 \end{pmatrix}\begin{pmatrix} x \\ s_1 \end{pmatrix} = \begin{pmatrix} s^{\#}\varphi \\ s_1 \end{pmatrix}.$$

Using this endomorphism of $C^{\#}\oplus D$, one can then suppose that $\nu_2 = \begin{pmatrix} s^{\#}\varphi \\ s_1 \end{pmatrix}$, changing ν_0 as well, but keeping the $\nu_1 : L \to P$ we liked. Now, having Lemma 4.10 in mind, compute

$$\nu_2\,\varphi^{-1}\nu_2^{\#} = \begin{pmatrix} s^{\#}\varphi \\ s_1 \end{pmatrix}\varphi^{-1}\begin{pmatrix} \varphi s & s_1^{\#} \end{pmatrix} = \begin{pmatrix} t & 0 \\ 0 & -w \end{pmatrix}.$$

Since $L \in J$ by (3), Lemma 4.10 insures us that the cone of the S-space

$$(M, \nu_2\,\varphi^{-1}\nu_2^{\#}) = \left(C\oplus D^{\#}, \begin{pmatrix} t & 0 \\ 0 & -w \end{pmatrix}\right)$$

is neutral in J. But it is obviously the orthogonal sum of the cone of t and of the cone of $-w : D^{\#} \to D$, the latter being neutral in J because $D^{\#} \in J$ by (2). Therefore, $[\mathrm{Cone}(t)] = 0$ in $\mathrm{W}^1(J)$ as claimed. ♯

4.14. Proof of Theorem 4.8. Let (A,s) and (B,t) be S-isometric S-spaces. It suffices to do the case when there exists a morphism $u : B \to A$, $u \in S$, such that

$$t = u^{\#} s\, u. \tag{1}$$

Recall that the cone construction is additive and invariant under strong isometries in K (see 4.9). Now consider

$$\begin{pmatrix} 1 & 0 \\ u^{\#} & 1 \end{pmatrix} \cdot \begin{pmatrix} s & 0 \\ 0 & -t \end{pmatrix} \cdot \begin{pmatrix} 1 & u \\ 0 & 1 \end{pmatrix} \overset{(1)}{=} \begin{pmatrix} s & s\,u \\ u^{\#}\, s & 0 \end{pmatrix}.$$

It suffices to prove that $[\mathrm{Cone}\big(A \oplus B, \begin{pmatrix} s & s\,u \\ u^{\#}\, s & 0 \end{pmatrix}\big)] = 0$ in $\mathrm{W}^1(J)$. Now, consider the morphism

$$\begin{pmatrix} 1 & 0 \\ 0 & s\,u \end{pmatrix} : A \oplus B \longrightarrow A \oplus A^{\#}$$

which obviously belongs to S and consider the symmetric form

$$\varphi := \begin{pmatrix} s & 1 \\ 1 & 0 \end{pmatrix}$$

over $P := A \oplus A^{\#}$. The result is now a direct consequence of Lemma 4.12 since

$$\begin{pmatrix} 1 & 0 \\ 0 & s\,u \end{pmatrix}^{\#} \cdot \varphi \cdot \begin{pmatrix} 1 & 0 \\ 0 & s\,u \end{pmatrix} = \begin{pmatrix} s & s\,u \\ u^{\#}\, s & 0 \end{pmatrix}.$$

$\natural$

4.15. Mea culpa. The careful reader of [1] will object that Theorem 5.15 there already states that this connecting homomorphism is well defined. The proof of Lemma 5.11 *loc. cit.* on which this theorem relies, was left as a (difficult) exercise. Solution to this problem is the above Theorem 4.8.

4.16. Corollary and definition. *Let $J \overset{j}{\hookrightarrow} K \overset{q}{\twoheadrightarrow} S^{-1}K$ be a localization of triangulated categories with δ-duality in the sense of 4.1. The cone construction of 1.6 and 1.10 induce well defined homomorphisms*

$$\begin{aligned} \partial^n : \mathrm{W}^n(S^{-1}K) &\longrightarrow \mathrm{W}^{n+1}(J) \\ x &\longmapsto [\mathrm{Cone}(A,s)] \\ &\qquad \textit{where } (A,s) \textit{ is any } S\textit{-space such that } x = [q(A), q(s)] \end{aligned}$$

for all $n \in \mathbb{Z}$. They will be called the residue homomorphisms.

4.17. Proof. This is immediate from Theorem 4.8 and Remark 4.7. In fact, the cone of any (neutral) form in K is of course trivial since the cone of an isomorphism is zero.

The theorem was stated for $n = 0$ but this is of course enough: apply it to $T^n(K, \#, \varpi)$. $\natural$

4.18. Exercise. Check that the $\partial^n : \mathrm{W}^n(S^{-1}K) \longrightarrow \mathrm{W}^{n+1}(J)$ commute with the periodicity isomorphisms $\mathrm{W}^n \overset{\sim}{\to} \mathrm{W}^{n+4}$ of Proposition 1.14.

4.19. Exercise. Check that the sequence obtained by using the natural homomorphisms of 4.3 and the residue homomorphisms of 4.16 is a complex (very easy).

4.20. Exercise. Understand the analogy between the residue homomorphisms presented here and the classical second residue, in the case of a valuation ring for instance. Understand first why an S-space is a good generalization of a lattice.

5. The main result.

5.1. Definition. We shall say, for short, that an additive category in which $A \oplus B \simeq B$ implies $A \simeq 0$ is *weakly cancellative.* Recall that Nakayama's lemma forces the category of finitely generated modules over a local ring to be weakly cancellative. Therefore, so is the category of coherent $\mathcal{O}_X$-modules over a scheme. Moreover if $\mathcal{E}$ is an exact subcategory of a weakly cancellative abelian category, then $\mathcal{E}$ and $\mathrm{D}^{\mathrm{b}}(\mathcal{E})$ are weakly cancellative as well.

5.2. Theorem. *Let $J \overset{j}{\hookrightarrow} K \overset{q}{\twoheadrightarrow} S^{-1}K$ be a localization of triangulated categories with duality in the sense of 4.1. Suppose that $\frac{1}{2} \in K$, that K satisfies (TR4$^+$) of section 0 and that $S^{-1}K$ is weakly cancellative. Then, using the natural homomorphisms $\mathrm{W}^n(j)$ and $\mathrm{W}^n(q)$ and the residue homomorphisms ∂^n of definition 4.16, we obtain the following long exact exact sequence:*

$$\cdots \longrightarrow \mathrm{W}^{n-1}(S^{-1}K) \xrightarrow{\partial^{n-1}} \mathrm{W}^n(J) \xrightarrow{\mathrm{W}^n(j)} \mathrm{W}^n(K) \xrightarrow{\mathrm{W}^n(q)} \mathrm{W}^n(S^{-1}K) \xrightarrow{\partial^n} \mathrm{W}^{n+1}(J) \longrightarrow \cdots$$

of localization.

5.3. Proof. We are now well prepared to prove this theorem. We shall only establish exactness of the sequence at $\mathrm{W}(J)$, $\mathrm{W}(K)$ and $\mathrm{W}(S^{-1}K)$; the last one is already done in [1] and in [2]. The same results for the shifted groups, using the translated dualities go as well. The reader can establish the analogous to the results of sections 2 and 3 for skew-dualities. Details are left as an exercise.

Exactness at $\mathrm{W}(J)$.

Let $x \in \mathrm{Ker}\big(\mathrm{W}(J) \to \mathrm{W}(K)\big)$. Choose a symmetric space such that $x = [P, \varphi]$. Then (P, φ) is Witt-trivial in K. By Theorem 2.5, (P, φ) is neutral in K. This means by definition 1.12 that

$$(P, \varphi) = \mathrm{Cone}(A, u)$$

for some object A in K and some morphism $u : A \to T^{-1}(A^{\#})$ such that

$$T^{-1}(u^{\#}) = u.$$

The reader puzzled down by the sign should re-read Remark 1.9 and its neighbors.

Then $\mathrm{Cone}(u) = P \in J$ and therefore u becomes an isomorphism in $S^{-1}K$ (where P would be zero). Then, since S is saturated, $u \in S$. In other words, (A, u) is an S-space for the skew-duality $T^{-1} \circ \#$ and

$$(P, \varphi) = \mathrm{Cone}(A, u).$$

By definition 4.16,

$$(P, \varphi) = \partial^{-1}\big(q(A), q(u)\big).$$

This is the announced result. Observe that we did not use that K satisfies (TR4$^+$). The key point is of course the very strong Theorem 2.5.

Exactness at $\mathrm{W}(K)$.

The first step is to give a criterion for symmetric spaces over K to become neutral in $S^{-1}K$. This should be compared with Lemma 2.1 condition (3). In $S^{-1}K$ the new isomorphisms are essentially the morphisms of S and therefore we obtain new neutral forms using the criterion of the lemma. We prove hereafter that they are the only one.

5.4. Lemma. *Let $J \overset{j}{\hookrightarrow} K \overset{q}{\twoheadrightarrow} S^{-1}K$ be a localization of triangulated categories with duality in the sense of 4.1. Suppose that $\frac{1}{2} \in K$. Then a symmetric space (P, φ) in K is neutral in $S^{-1}K$ if and only if there exists a morphism of exact triangles:*

$$\begin{tikzcd} T^{-1}(M^{\#}) \arrow[r, "\nu_0"] \arrow[d, "T^{-1}(h^{\#})"'] & L \arrow[r, "\nu_1"] \arrow[d, "h"] & P \arrow[r, "\nu_2"] \arrow[d, "\varphi", "\simeq"'] & M^{\#} \arrow[d, "h^{\#}"] \\ T^{-1}(L^{\#}) \arrow[r, "T^{-1}(\nu_0^{\#})"'] & M \arrow[r, "\nu_2^{\#}"'] & P^{\#} \arrow[r, "\nu_1^{\#}"'] & L^{\#} \end{tikzcd}$$

with $h \in S$.

5.5. Proof. Suppose that there exists such a diagram. Then localize it and apply Lemma 2.1 in $S^{-1}K$. You will obtain neutrality of $\big(q(P), q(\varphi)\big)$.

The converse is the interesting part. The reader should pay attention to distinguish diagrams in $S^{-1}K$ from those in K. Hereafter, $*$ shall denote $T^{-1} \circ \#$ from time to time.

Let (P, φ) be a symmetric space in K neutral in $S^{-1}K$. By definition, there exists an exact triangle in $S^{-1}K$:

$$T^{-1}(A^{\#}) \xrightarrow{a_0} A \xrightarrow{a_1} P \xrightarrow{a_2} A^{\#} \tag{1}$$

such that

(2) $a_2 = a_1^{\#}\varphi$ in $S^{-1}K$;
(3) $a_0 = a_0^*$ in $S^{-1}K$.

Trivial Remark. *Consider an exact triangle in $S^{-1}K$ as in (1) satisfying (2) and (3). Choose any morphism $s : B \to A$ in K such that $s \in S$. Set $b_1 = a_1 \circ s$ and $w = s^{-1}a_0\,(s^{-1})^*$. Then we have an exact triangle in $S^{-1}K$:*

$$T^{-1}(B^{\#}) \xrightarrow{w} B \xrightarrow{b_1} P \xrightarrow{b_1^{\#}\varphi} B^{\#}$$

with $w = w^*$ in $S^{-1}K$.

To prove this, it suffices to observe the commutative diagram in $S^{-1}K$:

$$\begin{tikzcd} T^{-1}(A^{\#}) \arrow[r, "a_0"] \arrow[d, "s^*"', "\simeq"] & A \arrow[r, "a_1"] & P \arrow[r, "a_2"] \arrow[d, equal] & A^{\#} \arrow[d, "s^{\#}"', "\simeq"] \\ T^{-1}(B^{\#}) \arrow[r, "w"'] & B \arrow[r, "b_1"'] \arrow[u, "s"', "\simeq"] & P \arrow[r, "b_1^{\#}\varphi"'] & B^{\#} \end{tikzcd}$$

which commutes because, in $S^{-1}K$, we have $b_1^{\#}\varphi = (a_1\,s)^{\#}\varphi = s^{\#}a_1^{\#}\varphi \overset{(2)}{=} s^{\#}a_2$. Of course (3) implies that $w^* = w$. This finishes the proof of the trivial remark.

Consider the exact triangle (1). Write a_1 as a fraction: $a_1 = b_1\,s^{-1} \,:\, A \xleftarrow{s} B \xrightarrow{b_1} P$ for some object B and some morphism $s : B \to A$ in S. We have $b_1 = a_1\,s$. Then the remark tells us that we can replace (1) by an exact triangle in $S^{-1}K$:

$$T^{-1}(B^{\#}) \xrightarrow{w} B \xrightarrow{b_1} P \xrightarrow{b_1^{\#}\varphi} B^{\#}$$

with $w = w^*$ in $S^{-1}K$. Then it is immediate that $b_1^{\#}\varphi\,b_1 = 0$ in $S^{-1}K$. This implies that there exists a morphism $s' : B' \to B$ such that $b_1^{\#}\varphi\,b_1 \circ s' = 0$ in K. Use again the trivial remark to replace b_1 by $b_1\,s'$. Summarizing up, we have the following situation:

(4) we have an exact triangle in $S^{-1}K$:

$$T^{-1}(B^{\#}) \xrightarrow{w} B \xrightarrow{b_1} P \xrightarrow{b_1^{\#}\varphi} B^{\#},$$

(5) $w = w^*$ in $S^{-1}K$,
(6) b_1 is a morphism in K $\big($meaning $= q(b_1)\big)$
(7) $b_1^{\#}\varphi\, b_1 = 0$ <u>in K</u>.

Since we have (6), we can choose an exact triangle in K containing b_1:

$$T^{-1}(C) \xrightarrow{b_0} B \xrightarrow{b_1} P \xrightarrow{b_2} C. \tag{8}$$

In $S^{-1}K$, we can compare (4) and (8), or if you prefer, (4) and the localization of (8):

$$\begin{tikzcd}
T^{-1}(B^{\#}) \arrow[r, "w"] \arrow[dd, dashed, "\exists\,\simeq"'] & B \arrow[r, "b_1"] \arrow[dd, equal] & P \arrow[r, "b_1^{\#}\varphi"] \arrow[dd, equal] & B^{\#} \arrow[d, dashed, "\exists\, t^{\#}"] \\
 & & & D^{\#} \\
T^{-1}(C) \arrow[r, "b_0"'] & B \arrow[r, "b_1"'] & P \arrow[r, "b_2"'] & C \arrow[u, dashed, "\exists\, u"']
\end{tikzcd} \tag{9}$$

There exists an isomorphism $B^{\#} \xrightarrow{\sim} C$, that can be expressed as a fraction like displayed above. Moreover, the relation $t^{\#}b_1^{\#}\varphi = u\, b_2$ is valid in $S^{-1}K$ and therefore in K up to some composition, say on the left, by a morphism in S. Up to a modification of D, t and u, one can suppose (9) and moreover:

$$t^{\#}b_1^{\#}\varphi = u\, b_2 \qquad \underline{\text{in } K}.$$

Since S is saturated, we have $t, u \in S$. Define $c_1 := b_1\, t : D \to P$ so that the above relation becomes:

$$c_1^{\#}\varphi = u\, b_2 \qquad \underline{\text{in } K}. \tag{10}$$

Choose an exact triangle over c_1 in K and use the definition of c_1, to construct the following morphism of exact triangles <u>in K</u>:

$$\begin{tikzcd}
T^{-1}(E) \arrow[r, "c_0"] \arrow[d, dashed, "T^{-1}(v)"'] & D \arrow[r, "c_1"] \arrow[d, "t"] & P \arrow[r, "c_2"] \arrow[d, equal] & E \arrow[d, dashed, "\exists\, v \in S"] \\
T^{-1}(C) \arrow[r, "b_0"'] & B \arrow[r, "b_1"'] & P \arrow[r, "b_2"'] & C.
\end{tikzcd} \tag{11}$$

There exists a morphism $v : E \to C$ making (11) commute and it is necessarily in S, which is saturated. Now define

$$l := u\, v : E \to D^{\#}.$$

Note that $l \in S$. Compose on the left the third square of (11) with u to obtain $l\, c_2 = u\, b_2$ and therefore, because of (10), we have:

$$l\, c_2 = c_1^{\#}\varphi \qquad \underline{\text{in } K}. \tag{12}$$

Define also in $S^{-1}K$:

$$z := t^{-1} \circ w \circ (t^{-1})^* : T^{-1}(D^{\#}) \to D.$$

Relation (5) forces $z^* = z$ in $S^{-1}K$. Now compute:

$$\begin{aligned} z\, T^{-1}(l) &= z\, T^{-1}(u\, v) = t^{-1}w\, (t^{-1})^*\, T^{-1}(u)\, T^{-1}(v) \\ &\overset{(9)}{=} t^{-1}\, b_0\, T^{-1}(v) \overset{(11)}{=} c_0 \qquad\qquad \text{in } S^{-1}K. \end{aligned}$$

Regrouping this last relation with (12), we have the following commutative diagram in $S^{-1}K$:

$$\begin{tikzcd}
T^{-1}(E) \arrow[r, "c_0"] \arrow[d, "T^{-1}(l)"'] & D \arrow[r, "c_1"] \arrow[d, equal] & P \arrow[r, "c_2"] \arrow[d, equal] & E \arrow[d, "l"] \\
T^{-1}(D^{\#}) \arrow[r, "z"'] & D \arrow[r, "c_1"'] & P \arrow[r, "c_1^{\#}\varphi"'] & D^{\#}.
\end{tikzcd} \tag{13}$$

But relation (12) was true in K and therefore can be expressed as the commutativity of the following right hand square in K:

$$\begin{tikzcd} T^{-1}(E) \arrow[r, "c_0"] \arrow[d, "T^{-1}(l)"'] & D \arrow[r, "c_1"] \arrow[d, dashed, "\exists m"] & P \arrow[r, "c_2"] \arrow[d, "\varphi"] & E \arrow[d, "l"] \\ T^{-1}(D^{\#}) \arrow[r, "c_0^*"'] & E^{\#} \arrow[r, "c_2^{\#}"'] & P^{\#} \arrow[r, "c_1^{\#}"'] & D^{\#}. \end{tikzcd} \tag{14}$$

where the first line is the exact triangle over c_1 we chose in (11) and where the second line is simply the dual of the first. Now, choose in K a morphism $m : D \to E^{\#}$ such that the above diagram commutes. Observe that $m \in S$. Define

$$h := \frac{1}{2}(m + l^{\#}) : D \to E^{\#}.$$

It is obvious from (14) that the following diagram commutes in K (easy exercise):

$$\begin{tikzcd} T^{-1}(E) \arrow[r, "c_0"] \arrow[d, "T^{-1}(h)"'] & D \arrow[r, "c_1"] \arrow[d, "h"] & P \arrow[r, "c_2"] \arrow[d, "\varphi"] & E \arrow[d, "h"] \\ T^{-1}(D^{\#}) \arrow[r, "c_0^*"'] & E^{\#} \arrow[r, "c_2^{\#}"'] & P^{\#} \arrow[r, "c_1^{\#}"'] & D^{\#}. \end{tikzcd} \tag{15}$$

The only non-trivial point is to check that $h \in S$, which is of course not true simply because $l, m \in S$. To prove this last point, we are going to establish that $h^{\#}$ also makes diagram (13) commute (in place of l). This is enough because S is saturated. In view of the definition of h, it is of course sufficient to establish that $m^{\#}$ makes diagram (13) commute.

In other words, we have to establish:

(16) $z\, T^{-1}(m^{\#}) = c_0$ in $S^{-1}K$

(17) $m^{\#}\, c_2 = c_1^{\#} \varphi$ in $S^{-1}K$.

But (17) is already true in view of the middle square of (14), by applying $\#$. Now simply compute in $S^{-1}K$:

$$z\, m^* = z^*\, m^* = (m \underbrace{z}_{\overset{(13)}{=} c_0\, T^{-1}(l^{-1})})^* = \big(m\, c_0\, T^{-1}(l^{-1})\big)^* \overset{(14)}{=} (c_0^*)^* = c_0.$$

Putting everything together, we have the commutative diagram (15) with exact lines in K, for a morphism $h : D \to E^{\#}$ such that $h \in S$. This is exactly the claim of the lemma. The proof of this lemma is octahedron-free! ♯

It is now easy to establish exactness of the sequence at $\mathrm{W}(K)$. It suffices to prove that a form satisfying the condition of the lemma is Witt-equivalent in K to a form (R, ψ) with $R \in J$. In the terminology of § 3, the morphism h would be a *good* morphism $L \to L^{\perp}$ (see definition 3.3). Sub-lagrangian Theorem 3.22 applied to $\eta_0 = h$, precisely asserts that there exists a form ψ on some $R = \mathrm{Cone}(\mu_0)$, Witt-equivalent to (P, φ) and such that R and $\mathrm{Cone}(h)$ are stably isomorphic, say

$$R \oplus B \simeq \mathrm{Cone}(h) \oplus B$$

for some object B in K – from § 3 we know that we could choose $B = P$. Localize the above isomorphism in $S^{-1}K$:

$$q(R) \oplus q(B) \simeq q\big(\mathrm{Cone}(h)\big) \oplus q(B).$$

But $q\big(\mathrm{Cone}(h)\big)$ is isomorphic to 0 in $S^{-1}K$ because $h \in S$. Therefore, since $S^{-1}K$ is weakly cancellative,

$$q(R) \simeq 0 \qquad \text{in } S^{-1}K.$$

This means by definition that $R \in J$. Then, $[P, \varphi] = [R, \psi] \in \mathrm{Im}\big(\mathrm{W}(j)\big)$.

Exactness at $\mathrm{W}(S^{-1}K)$.

This was established in [1, Theorem 5.17, p. 124] and in [2, *théorème* 2.21, p. 103]. At that time, W^1 was written W_1^- to recall the 1 from "$T^1 \circ \#$" and the minus sign from "$-\varpi$". $\sharp$

Theorem 5.2, Proposition 1.14 and Exercise 4.18 immediately give the following corollary.

5.6. Corollary. *Let $J \overset{j}{\hookrightarrow} K \overset{q}{\twoheadrightarrow} S^{-1}K$ be a localization of triangulated categories with duality in the sense of 4.1. Suppose that $\frac{1}{2} \in K$, that K satisfies (TR4$^+$) of section 0 and that $S^{-1}K$ is weakly cancellative. Then, we have the following*

$$
\begin{tikzcd}
 & \mathrm{W}(J) \arrow[r] & \mathrm{W}(K) \arrow[r] & \mathrm{W}(S^{-1}K) \arrow[dr, bend left] & \\
\mathrm{W}^3(S^{-1}K) \arrow[ur, bend left] & & & & \mathrm{W}^1(J) \arrow[d] \\
\mathrm{W}^3(K) \arrow[u] & & \textit{12-term cyclic exact sequence.} & & \mathrm{W}^1(K) \arrow[d] \\
\mathrm{W}^3(J) \arrow[u] & & & & \mathrm{W}^1(S^{-1}K) \arrow[dl, bend left] \\
 & \mathrm{W}^2(S^{-1}K) \arrow[ul, bend left] & \mathrm{W}^2(K) \arrow[l] & \mathrm{W}^2(J) \arrow[l] &
\end{tikzcd}
$$

$\sharp$

5.7. Remark. Of course, when we shall have [3] under roof, this result will turn out to be a generalization of the localization exact sequences of Karoubi [8], Pardon [11] and Ranicki [13]. Since I tried to keep the present article free of derived categories and complexes, I shall not enter into too many details. The reader should be aware that this result specializes to existing ones, that he could find in the references that were given several times in this article.

Bibliography

[1] Paul BALMER, *Derived Witt groups of a scheme*, Journal of pure and applied algebra 141, 2, pp. 101-129 (1999).

[2] Paul BALMER, *Groupes de Witt dérivés d'un schéma*, Thèse de doctorat, Lausanne (1998).[1]

[3] Paul BALMER, *Triangular Witt groups, Part II: From usual to derived*, to appear in *Mathematische Zeitschrift*, now available on the Web at www.mathematik.uni-bielefeld.de/LAG, (1999). [1]

[4] A. BEILINSON, J. BERNSTEIN, P. DELIGNE, *Faisceaux Pervers*, Astérisque 100, Société Mathématique de France (1982).

[5] J.-L. COLLIOT-THÉLÈNE, J.-J. SANSUC, *Fibrés quadratiques et composantes connexes réelles*, Mathematische Annalen, Band 244, pp. 105-134 (1979).

[6] Fernando FERNÁNDEZ-CARMENA, *On the injectivity of the map of the Witt group of a scheme into the Witt group of its function field*, Mathematische Annalen 277, pp. 453-468 (1987).

[1] Available on the author's web page: http://www.math.uwo.ca/~ balmer/

[7] Robin HARTSHORNE, *Residues and duality,* Springer Lecture Notes in Mathematics 20 (1966).

[8] Max KAROUBI, *Localisation de formes quadratiques I & II,* Annales scientifiques de l'École Normale Supérieure de Paris, 4^e série, 7, pp. 359-404, (1974) & 4^e série, 8, pp. 99-155, (1975).

[9] Manfred KNEBUSCH, *Symmetric bilinear forms over algebraic varieties,* In : Conf. on Quadratic Forms, Kingston 1976, Queen's Papers on Pure and Applied Mathematics 46, pp. 103-283, (1977).

[10] M. OJANGUREN, R. PARIMALA, R. SRIDHARAN, V. SURESH, *Witt groups of the punctured spectrum of 3-dimensional regular local ring and a purity theorem,* To appear in : Journal of the London Mathematical Society.

[11] William PARDON, *The exact sequence of a localization for Witt groups,* In : Algebraic K-theory, Evanston 1976, Springer Lecture Notes in Mathematics 551, pp. 336-379, (1977).

[12] Andrew RANICKI, *The algebraic theory of surgery I & II,* Proc. of the London Mathematical Society (3) 40, pp. 87-192 & pp. 193-287 (1980).

[13] Andrew RANICKI, *Exact Sequences in the Algebraic Theory of Surgery,* Mathematical Notes 26, Princeton University Press (1981).

[14] Jean-Louis VERDIER, *Catégories dérivées des catégories abéliennes,* Astérisque 239, Société mathématique de France (1996).

[15] Boris YOUSSIN, *Witt groups of derived categories,* Journal of K-theory, volume 11, No 4, pp. 373-395, (1997).

[16] C.T.C. WALL, *Surgery on compact manifolds*, Academic Press (1970), 2nd ed. A.M.S. (1999).

[17] Charles WEIBEL, *An introduction to homological algebra*, Cambridge University Press (1994).

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