Proceedings of the International Congress of Mathematicians
Hyderabad, India, 2010

Tensor triangular geometry

Paul Balmer*

Abstract. We survey tensor triangular geometry: Its examples, early theory and first applications. We also discuss perspectives and suggest some problems.

Mathematics Subject Classification (2000). Primary 18E30; Secondary 14F05, 19G12, 19K35, 20C20, 53D37, 55P42.

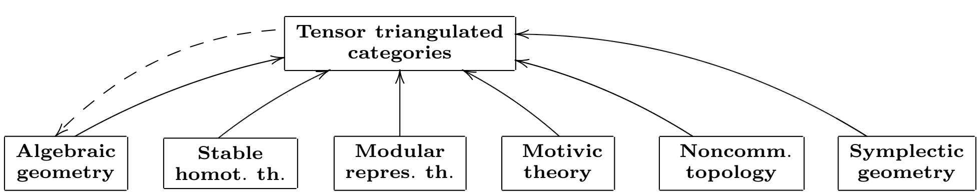
Keywords. Tensor triangulated categories, spectra.

Contents

Introduction

Tensor triangular geometry is the study of tensor triangulated categories by algebro-geometric methods. We invite the reader to discover this relatively new subject.

A great charm of this theory is the profusion of examples to be found throughout pure mathematics, be it in algebraic geometry, stable homotopy theory, modular representation theory, motivic theory, noncommutative topology, or symplectic geometry, to mention some of the most popular. We review them in Section 1. Here is an early photograph of tensor triangular geometry, in the crib:



*Research supported by NSF grant 0654397.

Before climbing into vertiginous abstraction, it is legitimate to enquire about the presence of oxygen in the higher spheres. For instance, some readers might wonder whether tensor triangulated categories do not lose too much information about the more concrete mathematical objects to which they are associated. Our first answer is Theorem 54 below, which asserts that a scheme can be reconstructed from the associated tensor triangulated category, whereas a well-known result of Mukai excludes such reconstruction from the triangular structure alone. Informally speaking, *algebraic geometry embeds into tensor triangular geometry.*

The main tool for this result is the construction of a locally ringed space $\mathrm{Spec}(\mathcal{K}) = \big(\mathrm{Spc}(\mathcal{K})\,,\,\mathcal{O}_{\mathcal{K}}\big)$ for any tensor triangulated category $\mathcal{K}$, which gives back the scheme in the above geometric example. Interestingly, this construction also gives the projective support variety, $\mathcal{V}_G(k)$, in modular representation theory. This unification is one of the first achievements of tensor triangular geometry.

The most interesting part of our $\mathrm{Spec}(\mathcal{K})$ is the underlying space $\mathrm{Spc}(\mathcal{K})$, called the *spectrum* of $\mathcal{K}$. We shall see that determining $\mathrm{Spc}(\mathcal{K})$ is equivalent to the classification of thick triangulated tensor-ideals of $\mathcal{K}$. Indeed, in almost all examples, the classification of all objects of $\mathcal{K}$ is a wild problem. Nevertheless, using subsets of $\mathrm{Spc}(\mathcal{K})$, one can *always* classify objects of $\mathcal{K}$ modulo the basic operations available in $\mathcal{K}$: cones, direct summands and tensor products (Theorem 14). This marks the beginning of tensor triangular geometry, *per se.* See Section 2.

A general goal of this theory is to transpose ideas and techniques between the various areas of the above picture, via the abstract platform of tensor triangulated categories. For instance, from algebraic geometry, we shall abstract the technique of gluing and the concept of being *local.* From modular representation theory, we shall abstract Carlson's Theorem [18] and Rickard's idempotents. And of course many techniques used in triangulated categories have been borrowed from homotopy theory, not the least being the above idea of classifying thick tensor-ideals.

Finally, we also want applications, especially *strict* applications, i. e. results without tensor triangulated categories in the statement but only in the proof. Such applications already exist in algebraic geometry (for K-theory and Witt groups) and in modular representation theory (for endotrivial modules). And applications start to emerge in other areas as well. We discuss this in Section 3.

Let us illustrate our philosophy with a concrete abstraction. Take the notion of $\otimes$-*invertible* object $u \in \mathcal{K}$ (i. e. $u \otimes v \simeq \mathbb{1}$ for some $v \in \mathcal{K}$). This perfectly $\otimes$-triangular concept covers line bundles in algebraic geometry and endotrivial modules in modular representation theory. Now, in algebraic geometry, a line bundle is locally isomorphic to $\mathbb{1}$. Hence, the $\otimes$-triangular geometer asks:

(a) Can one make sense of "locally" in any $\otimes$-triangulated category?

(b) Are all $\otimes$-invertible objects "locally" isomorphic to $\mathbb{1}$, say, up to suspension?

(c) Can one use these ideas to relate line bundles and endotrivial modules?

We shall see that the respective answers are: yes, no (!) and, nonetheless, yes.

Acknowledgements: I'm indebted to many friends and colleagues, that I would like to thank, collectively but very sincerely, for their help and support.

1. Tensor triangulated categories in nature

1.1. Basic definitions.

Let us remind the reader of the notion of triangulated category, introduced by Grothendieck-Verdier [50] forty years ago. See Neeman [41] for a modern reference.

Definition 1. A *triangulated category* is an additive category $\mathcal{K}$ (we can add objects $a \oplus b$ and morphisms $f+g$) with a *suspension* $\Sigma : \mathcal{K} \xrightarrow{\sim} \mathcal{K}$ (treated here as an isomorphism of categories) and a class of so-called *distinguished triangles*

$$\Delta = \Big(\, a \xrightarrow{f} b \xrightarrow{g} c \xrightarrow{h} \Sigma a \,\Big)$$

which are like exact sequences in spirit and are subject to a list of simple axioms:

(TC 1) *Bookkeeping axiom*: Isomorphic triangles are simultaneously distinguished; Δ as above is distinguished if and only if its *rotated* $b \xrightarrow{g} c \xrightarrow{h} \Sigma a \xrightarrow{-\Sigma f} \Sigma b$ is distinguished; $a \xrightarrow{1} a \to 0 \to \Sigma a$ is distinguished for every object a.

(TC 2) *Existence axiom*: Every morphism $f : a \to b$ fits in some distinguished Δ.

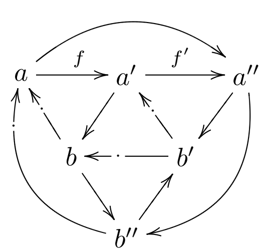
(TC 3) *Morphism axiom*: For every pair of distinguished triangles Δ and Δ'

$$\begin{tikzcd} \Delta = \Big(& a \arrow[r, "f"] \arrow[d, "k"'] & b \arrow[r, "g"] \arrow[d, "\ell"'] & c \arrow[r, "h"] \arrow[d, dotted, "\exists\, m"'] & \Sigma(a) \arrow[d, "\Sigma k"] & \Big) \\ \Delta' = \Big(& a' \arrow[r, "f'"'] & b' \arrow[r, "g'"'] & c' \arrow[r, "h'"'] & \Sigma(a') & \Big), \end{tikzcd}$$

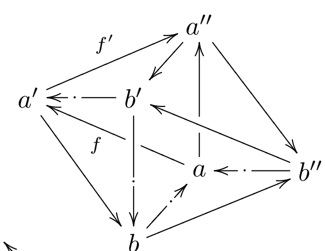
every commutative square (on the left) fits in a morphism of triangles.

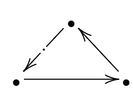
This was also proposed by Puppe in topology but Verdier's notorious addition is:

(TC 4) *Octahedron axiom*: Any two composable morphisms $a \xrightarrow{f} a' \xrightarrow{f'} a''$ fit in a commutative diagram (marked arrows $c \dashrightarrow c'$ mean $c \longrightarrow \Sigma(c')$)



or equivalently



in which the four triangles of the form  are distinguished.

A functor between triangulated categories is *exact* if it commutes with suspension (up to isomorphism) and preserves distinguished triangles in the obvious way.

Remark 2. Assuming (TC 1)-(TC 3), the third object c in a distinguished triangle Δ over a given $f : a \to b$ is unique up to (non-unique) isomorphism and is called the *cone* of f, denoted $\mathrm{cone}(f)$. The octahedron axiom simply says that there is a nice distinguished triangle relating $\mathrm{cone}(f)$, $\mathrm{cone}(f')$ and $\mathrm{cone}(f' \circ f)$.

The power of this axiomatic comes from its remarkable flexibility, compared for instance to the concepts of abelian or exact categories, which are somewhat too "algebraic". As we shall recall below, triangulated categories appear in a priori non-additive frameworks. In fact, the homotopy category of any stable Quillen model category is triangulated, see Hovey [27, Chap. 7].

Definition 3. A *tensor triangulated category* $(\mathcal{K}, \otimes, \mathbb{1})$ is a triangulated category $\mathcal{K}$ equipped with a monoidal structure (see Mac Lane [33, Chap. VII])

$$\mathcal{K} \times \mathcal{K} \xrightarrow{\otimes} \mathcal{K}$$

with unit object $\mathbb{1} \in \mathcal{K}$. We assume $-\otimes-$ exact in each variable, i. e. both functors $a \otimes - : \mathcal{K} \to \mathcal{K}$ and $- \otimes a : \mathcal{K} \to \mathcal{K}$ are exact, for every $a \in \mathcal{K}$. This involves natural isomorphisms $(\Sigma a) \otimes b \simeq \Sigma(a \otimes b)$ and $a \otimes (\Sigma b) \simeq \Sigma(a \otimes b)$ that we assume compatible, in that the two ways from $(\Sigma a) \otimes (\Sigma b)$ to $\Sigma^2(a \otimes b)$ only differ by a sign. Although some of the theory holds without further assumption, we are going to assume moreover that $\otimes$ is *symmetric monoidal*: $a \otimes b \cong b \otimes a$, see [33, § VII.7].

An exact functor F between tensor triangulated categories is $\otimes$*-exact* if it preserves the tensor structure, including the $\mathbb{1}$, up to isomorphisms which are compatible with the isomorphism $F\Sigma \simeq \Sigma F$, in the hopefully obvious way.

Remark 4. This is the most elementary axiomatic for "tensor triangulated"; see details in Hovey-Palmieri-Strickland [28, App. A]. May [34] proposed further compatibility axioms between tensor and octahedra, later extended by Keller-Neeman [30]. However, the elementary Definition 3 suffices for our purpose.

Such structures abound throughout pure mathematics, as we now review. See also [28, 1.2.3] for examples. We cannot provide background, motivation and explanations on all the following subjects and we assume some familiarity with at least some of the examples below, depending on the reader's own interests.

1.2. Examples from algebraic geometry.

Let X be a scheme, here always assumed quasi-compact and quasi-separated (i. e. X admits a basis of quasi-compact open subsets); e.g. X affine, or X noetherian, like a variety over a field. Then $\mathcal{K} = \mathrm{D}^{\mathrm{perf}}(X)$, the derived category of perfect complexes over X, is a tensor triangulated category. See SGA 6 [14] or Thomason [49]. It sits $\mathcal{K} \subset \mathcal{T}$ inside the tensor triangulated category $\mathcal{T} = \mathrm{D}_{\mathrm{Qcoh}(X)}(X)$ of complexes of $\mathcal{O}_X$-modules with quasi-coherent homology. Such a complex is *perfect* if it is locally quasi-isomorphic to a bounded complex of finitely generated projective modules. When X is a quasi-projective variety over a field, $\mathrm{D}^{\mathrm{perf}}(X)$ is simply $\mathrm{D}^{\mathrm{b}}(\mathrm{VB}_X)$ the bounded derived category of vector bundles. The conceptual way of thinking of perfect complexes is as the compact objects in $\mathcal{T}$ (Def. 44). See

Neeman [40] or Bondal-van den Bergh [15, Thm. 3.1.1]. The tensor $\otimes = \otimes^{\mathrm{L}}_{\mathcal{O}_X}$ is the left derived tensor product and the unit $\mathbb{1}$ is $\mathcal{O}_X$ (as a complex concentrated in degree 0).

When $X = \mathrm{Spec}(A)$ is affine, these categories are $\mathcal{T} = \mathrm{D}(A\text{–Mod})$, the derived category of A-modules, and $\mathcal{K} = \mathrm{D}^{\mathrm{perf}}(A) \cong \mathrm{K}^{\mathrm{b}}(A\text{–proj})$, the homotopy category of bounded complexes of finitely generated projective A-modules.

1.3. Examples from stable homotopy theory.

Let $\mathcal{K} = \mathrm{SH}^{\mathrm{fin}}$ be the Spanier-Whitehead stable homotopy category of finite pointed CW-complexes. It sits $\mathcal{K} \subset \mathcal{T}$ as a tensor triangulated subcategory inside $\mathcal{T} = \mathrm{SH}$, the stable homotopy category of topological spectra. The tensor $\otimes = \wedge$ is the smash product and the unit $\mathbb{1} = S^0$ is the sphere spectrum. See Vogt [53]. One can also replace these by equivariant versions, use modules over a ring spectrum, or treat everything over a fixed base space.

1.4. Examples from modular representation theory.

Let k be a field of positive characteristic and let G be a finite group, or a finite group scheme over k. (The adjective *modular* refers to kG not being semi-simple, i. e. to the existence of non-projective kG-modules.) Then $\mathcal{K} = \mathrm{stab}(kG)$, the stable module category of finitely generated kG-modules, modulo the projectives, is a tensor triangulated category. It sits $\mathcal{K} \subset \mathcal{T}$ inside the bigger tensor triangulated category $\mathcal{T} = \mathrm{Stab}(kG)$, the stable category of arbitrary kG-modules. Objects of $\mathrm{Stab}(kG)$ are k-representations of G and morphisms are equivalence classes of kG-linear maps under the relation $f \sim 0$ when f factors via a projective (which is the same as an injective). The tensor is $\otimes_k$ with diagonal G-action and the unit is the trivial representation $\mathbb{1} = k$. See Happel [23], Carlson [19] or Benson [11]. One can alternatively consider $\mathrm{D}^{\mathrm{b}}(kG\text{–mod})$, inside $\mathrm{D}(kG\text{–Mod})$, with tensor product as above. Rickard [45] proved that the obvious functor $kG\text{–mod} \to \mathrm{D}^{\mathrm{b}}(kG\text{–mod})$ induces an equivalence of $\otimes$-triangulated categories

$$\mathrm{stab}(kG) \cong \mathrm{D}^{\mathrm{b}}(kG\text{–mod}) / \mathrm{K}^{\mathrm{b}}(kG\text{–proj})\,. \tag{5}$$

1.5. Examples from motivic theory.

Let S be the spectrum of a perfect field (or some general base scheme). Then $\mathcal{K} = \mathrm{DM}_{\mathrm{gm}}(S)$, Voevodsky's derived category of geometric motives over S, is a tensor triangulated category. It sits $\mathcal{K} \subset \mathcal{T} = \mathrm{DM}(S)$ inside the derived category of motives over S. The tensor product extends the fiber product $X \times_S Y$. See [52]. The unit $\mathbb{1}$ is simply the motive of the base S (in degree zero).

1.6. Examples from $\mathbb{A}^1$-homotopy theory.

Denote by $\mathcal{K} = \mathrm{SH}^{\mathbb{A}^1}_{\mathrm{gm}}(S)$ the triangulated subcategory generated by smooth S-schemes in the stable $\mathbb{A}^1$-homotopy category $\mathcal{T} = \mathrm{SH}^{\mathbb{A}^1}(S)$ of Morel-Voevodsky; see [51] or [36]. Again, the tensor $\otimes$ is essentially characterized as extending the fiber product $\times_S$ of S-schemes; and again $\mathbb{1}$ is the base S. In some sense, § 1.6 is to § 1.5 what § 1.3 is to § 1.2.

1.7. Examples from noncommutative topology.

It is customary to think of C^*-algebras as noncommutative topological spaces. Let G be a second countable locally compact Hausdorff group – even G trivial is interesting. Then KK^G, the G-equivariant Kasparov category of separable G-C^*-algebras, is a tensor triangulated category, with $\otimes$ given by the minimal tensor product with diagonal G-action. See Meyer [35, § 4] for instance.

As the full category KK^G might be a little too overwhelming at first, we can follow Dell'Ambrogio [21] and consider the triangulated subcategory $\mathcal{K} = \mathcal{K}^G$ generated by the unit $\mathbb{1} = \mathbb{C}$. It actually sits inside the *Bootstrap category* $\mathcal{T} = \mathcal{T}^G$, which is the localizing subcategory of KK^G generated by the unit.

1.8. Further examples.

There are examples in other areas of mathematics. For instance, triangulated categories famously appear in symplectic geometry, where Kontsevich's homological mirror symmetry conjecture [31] predicts an equivalence between the homotopy category of the Fukaya category of Calabi-Yau manifolds and the derived category of their mirror variety. Here, the tensor is a very interesting problem, which has seen recent progress in the work of Subotic [47].

As yet another example, Bühler recently proposed a triangulated category approach to bounded cohomology in [17]. Actually, examples of triangulated categories flourish in many directions, be it in connection to cluster algebras, knot theory, or theoretical physics, to mention a few less traditional examples. In this luxuriant production of triangulated categories, we focus on *tensor* triangulated ones. And even if we "only" have the examples presented so far, the theory already calls for a unified treatment. Well, precisely, here comes one.

2. Abstract tensor triangular geometry

2.1. The spectrum.

The basic idea of tensor triangular geometry, formulated in [1], is the construction of a topological space for every $\otimes$-triangulated category $\mathcal{K}$, called the *spectrum* of $\mathcal{K}$, in which every object b of $\mathcal{K}$ would have a *support*. This support should be understood as the non-zero locus of b. Since this idea admits no obvious formalization a priori, we follow the Grothendieckian philosophy of looking for the *best* such space, in a universal sense. To do this, we have to decide which properties this support should satisfy.

Theorem 6 ([1, Thm. 3.2]). *Let $\mathcal{K}$ be an essentially small $\otimes$-triangulated category. There exists a topological space* $\mathrm{Spc}(\mathcal{K})$ *and closed subsets* $\mathrm{supp}(a) \subset \mathrm{Spc}(\mathcal{K})$ *for all objects $a \in \mathcal{K}$, which form a* support datum *on $\mathcal{K}$, i. e. such that*

(SD 1) $\mathrm{supp}(0) = \varnothing$ *and* $\mathrm{supp}(\mathbb{1}) = \mathrm{Spc}(\mathcal{K})$*,*

(SD 2) $\mathrm{supp}(a \oplus b) = \mathrm{supp}(a) \cup \mathrm{supp}(b)$ *for every $a, b \in \mathcal{K}$,*

(SD 3) $\operatorname{supp}(\Sigma a) = \operatorname{supp}(a)$ *for every* $a \in \mathcal{K}$*,*

(SD 4) $\operatorname{supp}(c) \subset \operatorname{supp}(a) \cup \operatorname{supp}(b)$ *for every distinguished* $a \to b \to c \to \Sigma a$*,*

(SD 5) $\operatorname{supp}(a \otimes b) = \operatorname{supp}(a) \cap \operatorname{supp}(b)$ *for every* $a, b \in \mathcal{K}$

and such that $(\operatorname{Spc}(\mathcal{K}), \operatorname{supp})$ *is the final support datum on* $\mathcal{K}$ *in the sense that for every support datum* (X, σ) *on* $\mathcal{K}$ *(i. e.* X *a space with closed subsets* $\sigma(a) \subset X$ *for all* $a \in \mathcal{K}$ *satisfying (SD 1-5) above), there exists a unique continuous map* $\varphi : X \to \operatorname{Spc}(\mathcal{K})$ *such that* $\sigma(a) = \varphi^{-1}(\operatorname{supp}(a))$ *for every object* $a \in \mathcal{K}$*.*

Before explicitly constructing $\operatorname{Spc}(\mathcal{K})$, let us recall some standard terminology :

Definition 7. A non-empty full subcategory $\mathcal{J} \subset \mathcal{K}$ is a *triangulated subcategory* if for every distinguished triangle $a \to b \to c \to \Sigma a$ in $\mathcal{K}$, when two out of a, b, c belong to $\mathcal{J}$, so does the third; here, we call $\mathcal{J}$ *thick* if it is stable by direct summands: $a \oplus b \in \mathcal{J} \Rightarrow a, b \in \mathcal{J}$ (usual definition of thick) *and* triangulated; we say that $\mathcal{J}$ is $\otimes$-*ideal* if $\mathcal{K} \otimes \mathcal{J} \subset \mathcal{J}$; it is *radical* if $\sqrt[\otimes]{\mathcal{J}} = \mathcal{J}$, that is, $a^{\otimes n} \in \mathcal{J} \Rightarrow a \in \mathcal{J}$.

Construction 8. We baptize the universal support datum $(\operatorname{Spc}(\mathcal{K}), \operatorname{supp})$ of Theorem 6 the *spectrum* of $\mathcal{K}$. The content of the proof is the explicit construction of $\operatorname{Spc}(\mathcal{K})$. A thick $\otimes$-ideal $\mathcal{P} \subsetneq \mathcal{K}$ is called *prime* if it is proper ($\mathbb{1} \notin \mathcal{P}$) and if $a \otimes b \in \mathcal{P}$ implies $a \in \mathcal{P}$ or $b \in \mathcal{P}$. The spectrum of $\mathcal{K}$ is the set of primes :

$$\operatorname{Spc}(\mathcal{K}) := \left\{ \mathcal{P} \subsetneq \mathcal{K} \,\middle|\, \mathcal{P} \text{ is prime} \right\}.$$

(This is where we use $\mathcal{K}$ essentially small.) The *support* of an object $a \in \mathcal{K}$ is

$$\operatorname{supp}(a) := \left\{ \mathcal{P} \in \operatorname{Spc}(\mathcal{K}) \,\middle|\, a \notin \mathcal{P} \right\}.$$

The complements $U(a) := \left\{ \mathcal{P} \in \operatorname{Spc}(\mathcal{K}) \,\middle|\, a \in \mathcal{P} \right\}$, for all $a \in \mathcal{K}$, define an open basis of the topology of $\operatorname{Spc}(\mathcal{K})$. Examples of $\operatorname{Spc}(\mathcal{K})$ are given in § 3.1 below.

Remark 9. Of course, the above notion of *prime* reminds us of commutative algebra. Yet, this analogy is not a good reason for considering primes $\mathcal{P} \subset \mathcal{K}$. On the contrary, $\otimes$-triangular geometers should refrain from believing that everything works in all areas covered by $\otimes$-triangular geometry as simply as in their favorite toy area. The justification for the definition of $\operatorname{Spc}(\mathcal{K})$ is given by the universal property of Theorem 6 and by the Classification Theorem 14 below.

Remark 10. An important question is : Why do we ask $\operatorname{supp}(a)$ to be *closed*? After all, several notions of support involve non-closed subsets, if we deal with "big" objects. For instance, in $\mathrm{D}(\mathbb{Z}\text{–Mod})$, the object $\mathbb{Q}$ should certainly be supported only at (0), which is not closed in $\operatorname{Spec}(\mathbb{Z})$. This is a first indication that our theory is actually well suited for so-called *compact* objects (Def. 44). In fact, the assumption that $\mathcal{K}$ is essentially small points in the same direction : For instance, $\mathrm{D}(\mathbb{Z}\text{–Mod})$ is not essentially small but $\mathrm{D}^{\mathrm{perf}}(\mathbb{Z})$ is. We shall return to this discussion in a few places below, culminating in § 2.6.

Let us now collect some basic facts about the space $\operatorname{Spc}(\mathcal{K})$, all proven in [1].

Proposition 11. *Let $\mathcal{K}$ be an essentially small $\otimes$-triangulated category.*

(a) If $\mathcal{K}$ is non-zero then $\operatorname{Spc}(\mathcal{K})$ *is non-empty.*

(b) The space $\operatorname{Spc}(\mathcal{K})$ *is* spectral *in the sense of Hochster [24], that is, it is quasi-compact and quasi-separated (has a basis of quasi-compact open subsets) and every non-empty closed irreducible subset has a unique generic point (hence* $\operatorname{Spc}(\mathcal{K})$ *is T_0).*

(c) For every $\otimes$-exact functor $F : \mathcal{K} \to \mathcal{L}$, the assignment $\mathcal{Q} \mapsto F^{-1}(\mathcal{Q})$ defines a map $\varphi = \operatorname{Spc}(F) : \operatorname{Spc}(\mathcal{L}) \to \operatorname{Spc}(\mathcal{K})$ which is continuous and spectral (the preimage of a quasi-compact open subset is quasi-compact). So, $\operatorname{Spc}(-)$ *is a contravariant functor. For every $a \in \mathcal{K}$, we have* $\operatorname{supp}(F(a)) = \varphi^{-1}(\operatorname{supp}(a))$.

Remark 12. Hochster [24] observed that a spectral space X has a dual topology with dual-open subsets $Y \subset X$ being the arbitrary unions

(13) $\qquad Y = \cup_{i\in I} Y_i \quad$ with each complement $X \smallsetminus Y_i$ open and quasi-compact.

We call such a dual-open Y a *Thomason subset* of X, in honor of Thomason's insightful result [48, Thm. 4.1], which transposes remarkably well beyond algebraic geometry. When the space X is noetherian (every open is quasi-compact), a subset Y is Thomason if and only if it is specialization closed ($y \in Y \Rightarrow \overline{\{y\}} \subset Y$).

The next two results show that the computation of $\operatorname{Spc}(\mathcal{K})$ is equivalent to the classification of thick $\otimes$-ideals (see Definition 7 for terminology about ideals).

Theorem 14 (Classification of thick tensor-ideals [1, Thm. 4.10]). *Let $\mathcal{K}$ be an essentially small $\otimes$-triangulated category. Then the assignment*

$$Y \;\longmapsto\; \mathcal{K}_Y := \big\{\, a \in \mathcal{K} \,\big|\, \operatorname{supp}(a) \subset Y \,\big\}\,, \tag{15}$$

induces a bijection between Thomason subsets Y of the spectrum, see (13)*, and radical thick $\otimes$-ideals $\mathcal{J}$ of $\mathcal{K}$. Its inverse is $\mathcal{J} \mapsto \operatorname{supp}(\mathcal{J}) := \bigcup_{a\in\mathcal{J}} \operatorname{supp}(a)$.*

Being radical is a mild condition, as we shall see in Remark 23. Theorem 14 admits the following converse:

Theorem 16. *If the radical thick $\otimes$-ideals of $\mathcal{K}$ are classified as in* (15)*, by the Thomason subsets of a support datum (X,σ) with X spectral in the sense of Hochster, then the map $\varphi : X \to \operatorname{Spc}(\mathcal{K})$ of Theorem 6 is a homeomorphism.*

Theorem 16 was originally proven in [1, Thm. 5.2] under the assumption that X be a noetherian space. The ideal proof is due to Buan-Krause-Solberg [16, Cor. 5.2], who also extended our spectrum to lattices of ideals.

Remark 17. In categories like $\mathcal{K} = \mathrm{SH}^{\mathrm{fin}}$ or $\mathcal{K} = \mathrm{D}^{\mathrm{perf}}(A)$, which are generated by the unit $\mathbb{1}$, every thick subcategory is automatically $\otimes$-ideal. Similarly, $\mathcal{K} = \operatorname{stab}(kG)$ is generated by the unit $\mathbb{1} = k$ for G a p-group. However, the global study requires the tensor, see Remark 53.

We now indicate what happens to the spectrum under the few general constructions which are available for arbitrary $\otimes$-triangulated categories.

Theorem 18. *Let $\mathcal{K}$ be an essentially small $\otimes$-triangulated category.*

(a) Let $\mathcal{J} \subset \mathcal{K}$ be a thick $\otimes$-ideal. Then Verdier localization $\mathcal{K} \xrightarrow{q} \mathcal{K}/\mathcal{J}$ (Remark 19) induces a homeomorphism from $\mathrm{Spc}(\mathcal{K}/\mathcal{J})$ *onto the subspace* $\{\mathcal{P} \,|\, \mathcal{P} \supset \mathcal{J}\}$ *of* $\mathrm{Spc}(\mathcal{K})$*. For instance, if $\mathcal{J} = \langle a \rangle = \mathcal{K}_{\mathrm{supp}(a)}$ is the thick $\otimes$-ideal generated by one object $a \in \mathcal{K}$, then* $\mathrm{Spc}(\mathcal{K}/\langle a \rangle) \simeq U(a)$ *is open in* $\mathrm{Spc}(\mathcal{K})$.

(b) Idempotent completion $\iota : \mathcal{K} \to \mathcal{K}^\natural$ (see [10] or Remark 22 below) induces a homeomorphism $\mathrm{Spc}(\iota) : \mathrm{Spc}(\mathcal{K}^\natural) \xrightarrow{\sim} \mathrm{Spc}(\mathcal{K})$.

(c) Let $u \in \mathcal{K}$ be an object such that the cyclic permutation $(123) : u^{\otimes 3} \xrightarrow{\sim} u^{\otimes 3}$ *is the identity and consider $F : \mathcal{K} \to \mathcal{K}[u^{\otimes -1}]$. Then* $\mathrm{Spc}(F)$ *yields a homeomorphism from* $\mathrm{Spc}(\mathcal{K}[u^{\otimes -1}])$ *onto the closed subspace* $\mathrm{supp}(u)$ *of* $\mathrm{Spc}(\mathcal{K})$.

Proof. (a) and (b) are [1, Prop. 3.11 and Cor. 3.14]. For (c), recall that $\mathcal{K}[u^{\otimes -1}]$ has objects (a, m) with $a \in \mathcal{K}$ and $m \in \mathbb{Z}$ (the formal $a \otimes u^{\otimes m}$) and morphisms $\mathrm{Hom}\big((a, m), (b, n)\big) = \mathrm{colim}_{k \to +\infty} \mathrm{Hom}_{\mathcal{K}}(a \otimes u^{\otimes m+k}, b \otimes u^{\otimes n+k})$. This category inherits from $\mathcal{K}$ a unique $\otimes$-triangulation and the functor $F : a \mapsto (a, 0)$ is $\otimes$-exact. The assumption on (123) ensures that the tensor structure on $\mathcal{K}[u^{\otimes -1}]$ is well-defined on morphisms. Then, the inverse of $\mathrm{Spc}(F)$ is defined by $\mathcal{P} \mapsto \mathcal{P}[u^{\otimes -1}]$ for every prime $\mathcal{P} \subset \mathcal{K}$ such that $\mathcal{P} \in \mathrm{supp}(u)$, that is, $u \notin \mathcal{P}$. Indeed, the latter condition implies that $\mathcal{P}[u^{\otimes -1}]$ is both proper and prime in $\mathcal{K}[u^{\otimes -1}]$. $\square$

Remark 19. Recall that the Verdier quotient $q : \mathcal{K} \longrightarrow \mathcal{K}/\mathcal{J}$ is the universal functor out of $\mathcal{K}$ such that $q(\mathcal{J}) = 0$. It is the localization of $\mathcal{K}$ with respect to the morphisms s in $\mathcal{K}$ such that $\mathrm{cone}(s) \in \mathcal{J}$. It can be constructed by keeping the same objects as $\mathcal{K}$ and defining morphisms as equivalence classes of fractions $\cdot \xleftarrow{s} \cdot \to \cdot$ with $\mathrm{cone}(s) \in \mathcal{J}$, under amplification.

We now introduce a very useful condition on $\mathcal{K}$:

Definition 20. A $\otimes$-triangulated category $\mathcal{K}$ is *rigid* if there exists an exact functor $D : \mathcal{K}^{\mathrm{op}} \to \mathcal{K}$ and a natural isomorphism $\mathrm{Hom}_{\mathcal{K}}(a \otimes b, c) \cong \mathrm{Hom}_{\mathcal{K}}(b, Da \otimes c)$ for every $a, b, c \in \mathcal{K}$. One calls Da the *dual* of a. In the terminology of [33] and [28], $(\mathcal{K}, \otimes)$ is *closed* symmetric monoidal and every object is *strongly dualizable*.

Hypothesis 21. From now on, we assume our $\otimes$-triangulated category $\mathcal{K}$ to be essentially small, rigid and idempotent complete.

Remark 22. Following up on Remark 10, the assumption that $\mathcal{K}$ is rigid is another indication that our input category $\mathcal{K}$ cannot be chosen too big. Much milder is the assumption that $\mathcal{K}$ is *idempotent-complete*, i. e. every idempotent $e = e^2 : a \to a$ in $\mathcal{K}$ yields a decomposition $a = \mathrm{im}(e) \oplus \ker(e)$, since $\mathcal{K}$ can always be idempotent completed $\mathcal{K} \overset{\iota}{\hookrightarrow} \mathcal{K}^\natural$ (see [10]) without changing the spectrum (Thm. 18 (b)).

Remark 23. Under Hypothesis 21, some natural properties become true in $\mathcal{K}$. For instance, $\mathrm{supp}(a) = \varnothing$ forces $a = 0$ (not only $\otimes$-nilpotent) by [3, Cor. 2.5]. Moreover, if $\mathrm{supp}(a) \cap \mathrm{supp}(b) = \varnothing$ then $\mathrm{Hom}_{\mathcal{K}}(a, b) = 0$, see [3, Cor. 2.8]. Finally, every thick $\otimes$-ideal $\mathcal{J} \subset \mathcal{K}$ is automatically radical $\sqrt[\otimes]{\mathcal{J}} = \mathcal{J}$ by [3, Prop. 2.4].

2.2. Localization.

Let us introduce the most important basic construction of $\otimes$-triangular geometry, which gives a meaning to "the category $\mathcal{K}$ *over some open* U of its spectrum".

Construction 24. For every quasi-compact open $U \subset \operatorname{Spc}(\mathcal{K})$, with closed complement $Z := \operatorname{Spc}(\mathcal{K}) \smallsetminus U$, we define the tensor triangulated category $\mathcal{K}(U)$ as

$$\mathcal{K}(U) := \big(\mathcal{K}/\mathcal{K}_Z\big)^{\natural}.$$

It is the idempotent completion of the Verdier quotient (Rem. 19) $\mathcal{K}/\mathcal{K}_Z$ of $\mathcal{K}$ by the thick $\otimes$-ideal $\mathcal{K}_Z = \big\{\, a \in \mathcal{K} \,\big|\, \operatorname{supp}(a) \subset Z \,\big\}$ of those objects supported outside U. We have a natural functor $\operatorname{res}_U : \mathcal{K} \to \mathcal{K}(U)$. One can prove that $\operatorname{Spc}(\operatorname{res}_U)$ induces a conceptually pleasant homeomorphism, see [9, Prop. 1.11],

$$\operatorname{Spc}(\mathcal{K}(U)) \cong U\,.$$

Hence quasi-compactness of U is necessary since $\operatorname{Spc}(\mathcal{K})$ is always quasi-compact, see Prop. 11 (b). Informally, the category $\mathcal{K}(U)$ is the piece of $\mathcal{K}$ living above the open U. For every $a, b \in \mathcal{K}$, we abbreviate $\operatorname{Hom}_{\mathcal{K}(U)}(\operatorname{res}_U(a), \operatorname{res}_U(b))$ by $\operatorname{Hom}_U(a, b)$. In the same spirit, we say that something about $\mathcal{K}$ happens "over U", when it happens in the category $\mathcal{K}(U)$ after applying the restriction functor res_U.

Theorem 25 ([5, § 4]). *Let $\mathcal{K}$ be a $\otimes$-triangulated category as in Hypothesis 21.*

(a) The topological space $\operatorname{Spc}(\mathcal{K})$ is local (i. e. every open cover contains the whole space) if and only if $a \otimes b = 0$ implies $a = 0$ or $b = 0$. Then $\{0\}$ is the unique closed point of $\operatorname{Spc}(\mathcal{K})$ and we call $\mathcal{K}$ a local *$\otimes$-triangulated category.*

(b) For every $\mathcal{P} \in \operatorname{Spc}(\mathcal{K})$, the category $\mathcal{K}/\mathcal{P}$ is local in the above sense. Its idempotent completion $(\mathcal{K}/\mathcal{P})^{\natural}$ is the colimit of the $\mathcal{K}(U)$ over those quasi-compact open $U \subset \operatorname{Spc}(\mathcal{K})$ containing the point $\mathcal{P} \in \operatorname{Spc}(\mathcal{K})$.

Remark 26. Roughly speaking, $\mathcal{K}/\mathcal{P}$ (or rather $(\mathcal{K}/\mathcal{P})^{\natural}$) is the *stalk* of $\mathcal{K}$ at the point $\mathcal{P} \in \operatorname{Spc}(\mathcal{K})$. The support $\operatorname{supp}(a) = \big\{\, \mathcal{P} \,\big|\, a \notin \mathcal{P} \,\big\} = \big\{\, \mathcal{P} \,\big|\, a \neq 0 \text{ in } \mathcal{K}/\mathcal{P} \,\big\}$ of an object $a \in \mathcal{K}$ can now be understood as the points of $\operatorname{Spc}(\mathcal{K})$ where a does not vanish in the stalk. This expresses the non-zero locus of a, as initially wanted.

Remark 27. Amusingly, a *local* $\otimes$-triangulated category $\mathcal{K}$ (i. e. $a \otimes b = 0 \;\Rightarrow\; a$ or $b = 0$) could hastily be baptized "integral" if one was to follow algebraic gut feeling. Extending standard terminology to $\otimes$-triangular geometry requires some care. Indeed, "local" is correct because of the conceptual characterization of Theorem 25 (a). And comfortingly, for X a scheme, the $\otimes$-triangulated category $\mathcal{K} = \mathrm{D}^{\mathrm{perf}}(X)$ is local if and only if $X \cong \operatorname{Spec}(A)$ with A a local ring.

Remark 28. When $\mathcal{K}$ is local, $\operatorname{Spc}(\mathcal{K})$ has a unique closed point by Thm. 25 (a). Then, the smallest possible support for a non-zero object is exactly that closed point $*$. We define $\operatorname{FL}(\mathcal{K}) := \big\{\, a \in \mathcal{K} \,\big|\, \operatorname{supp}(a) \subset * \,\big\}$ and call such objects the *finite length* objects, by analogy with commutative algebra. (This somewhat improper terminology might need improvement; see the comments in Remark 27.)

We now use $\mathcal{K}(U)$ to create a structure sheaf on $\mathrm{Spc}(\mathcal{K})$.

Construction 29. For every quasi-compact open $U \subset \mathrm{Spc}(\mathcal{K})$, we can consider the commutative ring $\mathrm{End}_{\mathcal{K}(U)}(\mathbb{1})$. Since the unit $\mathbb{1}$ of $\mathcal{K}(U)$ is simply the restriction of the unit $\mathbb{1}$ of $\mathcal{K}$, and since $(\mathcal{K}(U))(V) \cong \mathcal{K}(V)$ for every $V \subset U \cong \mathrm{Spc}(\mathcal{K}(U))$, we obtain a presheaf of commutative rings $\mathrm{p}\mathcal{O}_{\mathcal{K}}$, at least on the open basis consisting of quasi-compact open subsets. This presheaf $\mathrm{p}\mathcal{O}_{\mathcal{K}}(U) = \mathrm{End}_U(\mathbb{1})$ is already useful in itself but can also be sheafified into a sheaf $\mathcal{O}_{\mathcal{K}}$ of commutative rings on $\mathrm{Spc}(\mathcal{K})$. We denote by

$$\mathrm{Spec}(\mathcal{K}) := \big(\,\mathrm{Spc}(\mathcal{K})\,,\,\mathcal{O}_{\mathcal{K}}\big)$$

the corresponding ringed space. It is a locally ringed space by [5, Cor. 6.6].

Remark 30. The above construction has an obvious algebro-geometric bias and one should not expect too much from this sheaf of rings $\mathcal{O}_{\mathcal{K}}$ in general. Still, it will be important in Theorems 54 and 57 below. Our preferred presheaf on $\mathrm{Spc}(\mathcal{K})$ is not $\mathcal{O}_{\mathcal{K}}$ but the more fundamental “presheaf” of $\otimes$-triangulated categories: $U \mapsto \mathcal{K}(U)$ of Construction 24.

2.3. Support and decomposition.

Here comes the first $\otimes$-triangular result which really opens the door to geometry. It extends a famous result of Carlson [18] in representation theory.

Theorem 31 ([3, Thm. 2.11]). *Let $\mathcal{K}$ be a $\otimes$-triangulated category as in Hypothesis 21 and let $a \in \mathcal{K}$ be an object. Suppose that its support is disconnected, i. e. $\mathrm{supp}(a) = Y_1 \sqcup Y_2$ with each Y_i closed and $Y_1 \cap Y_2 = \varnothing$. Then the object decomposes accordingly, that is, $a \simeq a_1 \oplus a_2$ with $\mathrm{supp}(a_1) = Y_1$ and $\mathrm{supp}(a_2) = Y_2$.*

It is easy to build counter-examples to the above statement if we remove the assumption that $\mathcal{K}$ is idempotent complete, see [3, Ex. 2.13]. This explains why we insist on idempotent-completion, for instance in the construction of $\mathcal{K}(U)$ above. Theorem 31 has the following application.

Theorem 32 ([3, Thm. 3.24]). *Let $\mathcal{K}$ be a $\otimes$-triangulated category as in Hypothesis 21 and assume that $\mathrm{Spc}(\mathcal{K})$ is a noetherian topological space (every open is quasi-compact). Let $\dim : \mathrm{Spc}(\mathcal{K}) \to \mathbb{Z} \cup \{\pm\infty\}$ be a dimension function, i. e. $\mathcal{Q} \subsetneq \mathcal{P} \Rightarrow \dim(\mathcal{Q}) + 1 \le \dim(\mathcal{P})$. Consider the filtration of $\mathcal{K}$ by the $\otimes$-ideals $\mathcal{K}_{(d)} := \big\{\, a \in \mathcal{K} \,\big|\, \dim(\mathcal{P}) \le d \text{ for all } \mathcal{P} \in \mathrm{supp}(a)\,\big\}$. Then for every finite $d \in \mathbb{Z}$, the corresponding subquotient $\mathcal{K}_{(d)}/\mathcal{K}_{(d-1)}$ decomposes into a coproduct of local parts. More precisely, after idempotent completion, we have an equivalence*

$$\big(\mathcal{K}_{(d)}/\mathcal{K}_{(d-1)}\big)^{\natural} \xrightarrow{\sim} \coprod_{\mathcal{P}\in\mathrm{Spc}(\mathcal{K}),\ \dim(\mathcal{P})=d} \big(\,\mathrm{FL}(\mathcal{K}/\mathcal{P})\big)^{\natural}$$

where the subcategories of finite-length objects $\mathrm{FL}(\mathcal{K}/\mathcal{P})$ are the ones of Remark 28.

Examples of dimension functions, $\dim(\mathcal{P})$, include the Krull dimension of the irreducible closed $\overline{\{\mathcal{P}\}}$, or the opposite of its Krull codimension, in $\mathrm{Spc}(\mathcal{K})$.

2.4. Gluing and Picard groups.

The true power of Theorem 31 appears in the following gluing method.

Theorem 33 (B.-Favi [9, Cor. 5.8 and 5.10]). *Let $\mathcal{K}$ be a $\otimes$-triangulated category as in Hypothesis 21 and let* $\mathrm{Spc}(\mathcal{K}) = U_1 \cup U_2$ *be a cover with both U_i quasi-compact open. Set $U_{12} := U_1 \cap U_2$ and consider the commutative square of $\otimes$-triangulated categories and restriction functors*

$$\begin{tikzcd} \mathcal{K} \arrow[r] \arrow[d] & \mathcal{K}(U_1) \arrow[d] \\ \mathcal{K}(U_2) \arrow[r] & \mathcal{K}(U_{12})\,. \end{tikzcd}$$

(a) *Gluing of morphisms: For every pair of objects $a, b \in \mathcal{K}$, we have a Mayer-Vietoris long exact sequence of abelian groups*

$$\cdots \xrightarrow{\partial} \mathrm{Hom}_{\mathcal{K}}(a,b) \to \begin{array}{c} \mathrm{Hom}_{U_1}(a,b) \\ \oplus \\ \mathrm{Hom}_{U_2}(a,b) \end{array} \to \mathrm{Hom}_{U_{12}}(a,b) \xrightarrow{\partial} \mathrm{Hom}_{\mathcal{K}}(a,\Sigma b) \to \cdots$$

(b) *Gluing of objects: Given two objects $a_i \in \mathcal{K}(U_i)$, $i = 1, 2$, and an isomorphism $\sigma : a_1 \overset{\sim}{\to} a_2$ over U_{12}, there exists a triple (a, f_1, f_2) where a is an object of $\mathcal{K}$ and $f_i : a \overset{\sim}{\to} a_i$ is an isomorphism over U_i such that $\sigma \circ f_1 = f_2$ over U_{12}. This gluing is unique up to possibly non-unique isomorphism of triples in $\mathcal{K}$.*

Remark 34. The apparently anodyne non-uniqueness of the isomorphism in (b) has a cost. Namely, gluing of three objects over three open subsets is still possible but without uniqueness [9, Cor. 5.11]. And gluing of more than three pieces might simply not exist unless some connectivity conditions are imposed [9, Thm. 5.13].

Here is an application of the gluing technique to Picard groups.

Definition 35. The *Picard group*, $\mathrm{Pic}(\mathcal{K})$, is the group of isomorphism classes of $\otimes$-invertible objects of $\mathcal{K}$, that is, those $u \in \mathcal{K}$ for which there exists $v \in \mathcal{K}$ with $u \otimes v \simeq \mathbb{1}$. (As $\mathcal{K}$ is rigid, $v \simeq Du$.) This does not use the triangulation.

We can now construct $\otimes$-invertible objects by gluing copies of the $\otimes$-unit $\mathbb{1}$.

Definition 36. For every quasi-compact open $U \subset \mathrm{Spc}(\mathcal{K})$, denote by $\mathbb{G}_{\mathrm{m}}(U) := \mathrm{Aut}_U(\mathbb{1})$ the group of automorphisms of $\mathbb{1}$ in $\mathcal{K}(U)$.

Theorem 37 (B.-Favi [9, Thm. 6.7]). *Under Hypothesis 21, if* $\mathrm{Spc}(\mathcal{K}) = U_1 \cup U_2$ *with each U_i quasi-compact, then gluing induces a well-defined group homomorphism $\delta : \mathbb{G}_{\mathrm{m}}(U_{12}) \to \mathrm{Pic}(\mathcal{K})$, where $U_{12} := U_1 \cap U_2$. We have an exact sequence*

$$\begin{tikzcd} \cdots \mathrm{Hom}_{U_{12}}(\Sigma\mathbb{1},\mathbb{1}) \arrow[r, "1+\partial"] & \mathbb{G}_{\mathrm{m}}(\mathrm{Spc}(\mathcal{K})) \arrow[r] & \mathbb{G}_{\mathrm{m}}(U_1) \oplus \mathbb{G}_{\mathrm{m}}(U_2) \arrow[r] & \mathbb{G}_{\mathrm{m}}(U_{12}) \arrow[dll, "\delta"'] \\ & \mathrm{Pic}(\mathcal{K}) \arrow[r] & \mathrm{Pic}(\mathcal{K}(U_1)) \oplus \mathrm{Pic}(\mathcal{K}(U_2)) \arrow[r] & \mathrm{Pic}(\mathcal{K}(U_{12}))\,, \end{tikzcd}$$

which continues on the left as in Theorem 33 (where ∂ also comes from).

It remains an open problem how to extend this sequence on the right, say, with Brauer groups. The other natural thing one might want to do is to glue any $\mathbb{G}_{\mathrm{m}}$-cocycle on $\mathrm{Spc}(\mathcal{K})$ into an invertible object of $\mathcal{K}$. Then the difficulty of gluing more than three pieces (Remark 34) becomes an obstacle. It can be circumvented in positive characteristic p, at the price of inverting p on the Picard group :

Theorem 38 ([6, Thm. 3.9]). *Let p be a prime and $\mathcal{K}$ a $\otimes$-triangulated $\mathbb{Z}/p$-category satisfying Hypothesis 21. Let $\check{H}^1(\mathrm{Spc}(\mathcal{K}),\mathbb{G}_{\mathrm{m}})$ be the first Čech cohomology group with coefficients in the above presheaf of units $\mathbb{G}_{\mathrm{m}}$. Let $\mathrm{Pic}_{loc.tr.}(\mathcal{K}) := \big\{ [u] \,\big|\, u \simeq \mathbb{1}$ in $\mathcal{K}/\mathcal{P}$ for all $\mathcal{P} \in \mathrm{Spc}(\mathcal{K}) \big\} \subset \mathrm{Pic}(\mathcal{K})$ be the subgroup of locally (very) trivial invertibles. Then, gluing induces a well-defined isomorphism β*

$$\check{H}^1(\mathrm{Spc}(\mathcal{K}),\mathbb{G}_{\mathrm{m}}) \underset{\mathbb{Z}}{\otimes} \mathbb{Z}[1/p] \xrightarrow[\simeq]{\beta} \mathrm{Pic}_{loc.tr.}(\mathcal{K}) \underset{\mathbb{Z}}{\otimes} \mathbb{Z}[1/p] \ \subset\ \mathrm{Pic}(\mathcal{K}) \underset{\mathbb{Z}}{\otimes} \mathbb{Z}[1/p]\,.$$

We call $\mathbb{1}$ the *very* trivial $\otimes$-invertible because the right notion of a *trivial* $\otimes$-invertible is probably one of the form $\Sigma^n\mathbb{1}$ for some $n \in \mathbb{Z}$. See more in § 4.5.

Remark 39. In algebraic geometry, invertible objects are (shifted) line bundles. Hence they are locally trivial for the Zariski topology, which explains why the Picard group, $\mathrm{Pic}(X)$, is the first Zariski cohomology group of $\mathbb{G}_{\mathrm{m}}$. However, there are local $\otimes$-triangulated categories with non-trivial Picard group. See Remark 71 for an example in modular representation theory. The following result shows that the Picard group can be as large as one wants with given (even local) spectrum.

Proposition 40 (B. - Rahbar Virk). *Let $\mathcal{K}$ be a local $\otimes$-triangulated category (*$\mathrm{Spc}(\mathcal{K})$ *connected is enough). Let G be an abelian group. Define a tensor on the triangulated category $\mathcal{L} := \coprod_G \mathcal{K}$ by $a_g \otimes b_h := (a\otimes b)_{g+h}$, where $a_g \in \mathcal{L}$ is the object corresponding to $a \in \mathcal{K}$ in the copy indexed by $g \in G$. Then* $\mathrm{Spc}(\mathcal{L}) \cong \mathrm{Spc}(\mathcal{K})$ *whereas* $\mathrm{Pic}(\mathcal{L}) \cong \mathrm{Pic}(\mathcal{K}) \times G$.

Proof. Easy exercise using the $\otimes$-invertible objects $\mathbb{1}_g \in \mathcal{L}$ for all $g \in G$ and the fact that every object of $\mathcal{L}$ is a finite direct sum $\underset{g\in G}{\oplus} a(g)_0 \otimes \mathbb{1}_g$ with $a(g) \in \mathcal{K}$. ☐

2.5. Comparing triangular spectra and algebraic spectra.

Remark 41. It should be clear by now that the main key to the geometry of a given $\otimes$-triangulated category $\mathcal{K}$, is the determination of its spectrum, $\mathrm{Spc}(\mathcal{K})$. We have seen in Theorem 16 that this problem amounts to the classification of thick $\otimes$-ideals of $\mathcal{K}$. This is very nice when the latter classification has been kindly performed by our predecessors but in most new areas such a classification is not yet under roof and actually constitutes a very interesting challenge. See § 4.1 below. To study $\mathrm{Spc}(\mathcal{K})$ without classification, we need some comparison with other spaces that might appear in examples. This is the purpose of [5], where we relate $\mathrm{Spc}(\mathcal{K})$ to the spectrum of the endomorphism ring $\mathrm{R}_{\mathcal{K}} = \mathrm{End}_{\mathcal{K}}(\mathbb{1})$ of the $\otimes$-unit $\mathbb{1}$, and to the homogeneous spectrum of the graded ring $\mathrm{R}^{\bullet}_{\mathcal{K}} = \mathrm{Hom}_{\mathcal{K}}(\mathbb{1}, \Sigma^{\bullet}\mathbb{1})$.

Theorem 42 ([5, Thm. 5.3]). *There exist two natural continuous maps*

$$\rho^{\bullet}_{\mathcal{K}} : \mathrm{Spc}(\mathcal{K}) \longrightarrow \mathrm{Spec}^{\mathrm{h}}(\mathrm{R}^{\bullet}_{\mathcal{K}}) \qquad \textit{and} \qquad \rho_{\mathcal{K}} : \mathrm{Spc}(\mathcal{K}) \longrightarrow \mathrm{Spec}(\mathrm{R}_{\mathcal{K}})$$

defined by $\rho^{\bullet}_{\mathcal{K}}(\mathcal{P}) = \bigoplus_{d\in\mathbb{Z}} \left\{ f \in \mathrm{R}^{d}_{\mathcal{K}} \;\middle|\; \mathrm{cone}(f) \notin \mathcal{P} \right\}$ *and* $\rho_{\mathcal{K}}(\mathcal{P}) = \rho^{0}_{\mathcal{K}}(\mathcal{P})$.

In fact, these maps are often surjective (yet, not always, see [5, Ex. 8.3]):

Theorem 43 ([5, § 7]). *With the notation of Theorem 42, we have:*

(a) Suppose that $\mathcal{K}$ *is connective, i. e. that* $\mathrm{Hom}(\Sigma^i \mathbb{1}, \mathbb{1}) = 0$ *for* $i < 0$ *(which reads* $\mathrm{R}^{d}_{\mathcal{K}} = 0$ *for* $d > 0$*). Then* $\rho_{\mathcal{K}} : \mathrm{Spc}(\mathcal{K}) \to \mathrm{Spec}(\mathrm{R}_{\mathcal{K}})$ *is a surjective map.*

(b) Suppose that $\mathrm{R}^{\bullet}_{\mathcal{K}}$ *is coherent (e.g. noetherian) in the graded sense. Then both* $\rho^{\bullet}_{\mathcal{K}} : \mathrm{Spc}(\mathcal{K}) \to \mathrm{Spec}^{\mathrm{h}}(\mathrm{R}^{\bullet}_{\mathcal{K}})$ *and* $\rho_{\mathcal{K}}$ *are surjective maps.*

Injectivity is more delicate, see Theorem 51. However, in "algebraic" examples, these maps are (local) homeomorphisms, see Remark 56 and Theorem 57.

2.6. Non-compact objects.

As indicated a couple of times above, the natural input $\mathcal{K}$ to our $\otimes$-triangular geometry machine consists of small enough categories. Let us now be more precise.

Definition 44. Let $\mathcal{T}$ be a triangulated category admitting arbitrary small coproducts $\coprod_{i\in I} t_i$. An object $c \in \mathcal{T}$ is called *compact* if for every set of objects $\{t_i\}_{i\in I}$ in $\mathcal{T}$, the natural map $\coprod_{i\in I} \mathrm{Hom}_{\mathcal{T}}(c, t_i) \to \mathrm{Hom}_{\mathcal{T}}(c, \coprod_{i\in I} t_i)$ is an isomorphism. The subcategory $\mathcal{T}^c$ of compact objects is triangulated but not closed under coproducts. We say that $\mathcal{T}$ is a *compactly generated tensor triangulated category* if

(i) $\mathcal{T}^c$ generates $\mathcal{T}$, that is, $\mathcal{T} = \mathrm{Loc}(\mathcal{T}^c)$ is the smallest localizing (i. e. closed under small coproducts) triangulated subcategory of $\mathcal{T}$ which contains $\mathcal{T}^c$.

(ii) $\mathcal{T}^c$ is essentially small, $\mathcal{T}^c$ is rigid and $\mathbb{1}$ is compact.

In that case, an object is compact if and only if it is rigid (i. e. strongly dualizable) and the $\otimes$-triangulated category $\mathcal{K} := \mathcal{T}^c$ of rigid-compact objects satisfies our Hypothesis 21. We can then apply the above $\otimes$-triangular geometry to $\mathcal{K} = \mathcal{T}^c$.

Examples 45. Examples § 1.2-1.4 fit in this picture with the $\mathcal{T}$ provided each time. (Examples § 1.5–1.7 require some care.) In [28], $\otimes$-triangulated categories $\mathcal{T}$ as above are studied under the name *unital algebraic stable homotopy categories.*

Remark 46. Our spectrum $\mathrm{Spc}(\mathcal{K})$ is the right space for the compact part but $\mathrm{Spc}(\mathcal{T})$ is not an appropriate invariant of $\mathcal{T}$ for it might not even be a set. Moreover, we do not need supports of non-compact objects to be closed and we would like $\mathrm{supp}(\coprod_{i\in I} t_i) = \cup_{i\in I} \mathrm{supp}(t_i)$. The question of $\mathrm{supp}(s \otimes t)$ is not entirely clear. One expects $\mathrm{supp}(s \otimes t) \subset \mathrm{supp}(s) \cap \mathrm{supp}(t)$ with equality when s is compact. Putting all this together, one can actually define a "big spectrum" of $\mathcal{T}$ as the universal space with supports, satisfying (SD' 1)-(SD' 7) below. Since it is not clear yet how useful this big spectrum can be, we do not make a theory out of this. The following result, due independently to Pevtsova-Smith [43] and Dell'Ambrogio, indicates that such a big spectrum might often coincide with $\mathrm{Spc}(\mathcal{K})$ anyway.

Theorem 47 ([21, Thm. 3.1]). *Let $\mathcal{T}$ be a compactly generated $\otimes$-triangulated category as in Definition 44. Let X be a topological space with a choice of a subset $\sigma(t) \subset X$ for every object $t \in \mathcal{T}$ satisfying the following conditions:*

(SD'1) $\sigma(0) = \varnothing$ *and* $\sigma(\mathbb{1}) = X$,

(SD'2) $\sigma(s \oplus t) = \sigma(s) \cup \sigma(t)$ *for every* $s, t \in \mathcal{T}$,

(SD'3) $\sigma(\Sigma t) = \sigma(t)$ *for every* $t \in \mathcal{T}$,

(SD'4) $\sigma(u) \subset \sigma(s) \cup \sigma(t)$ *for every distinguished triangle* $s \to t \to u \to \Sigma s$,

(SD'5) $\sigma(s \otimes t) \subset \sigma(s) \cap \sigma(t)$ *for every* $s, t \in \mathcal{T}$*, with equality if* s *or* t *is compact,*

(SD'6) $\sigma(\coprod_{i \in I} t_i) = \cup_{i \in I} \sigma(t_i)$ *for every set* $\{t_i\}_{i \in I}$ *of objects of* $\mathcal{T}$,

(SD'7) $\sigma(c)$ *is closed for every compact object* $c \in \mathcal{T}^c$.

In particular (X, σ) is a support datum on $\mathcal{K} = \mathcal{T}^c$. Suppose moreover:

(i) X is spectral in the sense of Hochster [24], see Proposition 11 (b).

(ii) An open $U \subset X$ is quasi-compact if and only if $U = X \smallsetminus \sigma(c)$ for $c \in \mathcal{T}^c$.

(iii) For $t \in \mathcal{T}$, if $\sigma(t) = \varnothing$ then $t = 0$.

Then the canonical map $X \to \mathrm{Spc}(\mathcal{T}^c)$ of Theorem 6 is a homeomorphism.

In examples where $\mathcal{T}$ is given with such supports, Theorem 47 might be used to compute $\mathrm{Spc}(\mathcal{K})$. Conversely, $\mathrm{Spc}(\mathcal{K})$, for $\mathcal{K} = \mathcal{T}^c$, yields information about the big category $\mathcal{T}$, via the following inflating technique, see [41, Chap. 4]:

Remark 48. For $U \subset \mathrm{Spc}(\mathcal{K})$ quasi-compact open with closed complement Z, set $\mathcal{T}_Z = \mathrm{Loc}(\mathcal{K}_Z)$ the localizing subcategory of $\mathcal{T}$ generated by $\mathcal{K}_Z \subset \mathcal{K}$. In [8], we define the category "$\mathcal{T}$ over U" as the localization $\mathcal{T}(U) := \mathcal{T}/\mathcal{T}_Z$. The $\otimes$-triangulated category $\mathcal{T}(U)$ remains compactly generated and Neeman's generalization [41, Thm. 4.4.9] of Thomason's result (Rem. 55) reads: $\big(\mathcal{T}(U)\big)^c = \mathcal{K}(U)$. This also justifies the idempotent completion in the definition of $\mathcal{K}(U)$.

Transposing Rickard's idempotents [46] to $\otimes$-triangular geometry gives:

Theorem 49 (B.-Favi [8]). *Let $\mathcal{T}$ be a compactly generated $\otimes$-triangulated category (Def. 44) and $\mathcal{K} = \mathcal{T}^c$ its compact objects. For every Thomason subset $Y \subset \mathrm{Spc}(\mathcal{K})$, there exists a distinguished triangle $e(Y) \to \mathbb{1} \to f(Y) \to \Sigma(e(Y))$ in $\mathcal{T}$ such that $e(Y) \otimes f(Y) = 0$ (hence $e(Y)^{\otimes 2} \simeq e(Y)$ and $f(Y)^{\otimes 2} \simeq f(Y)$ are $\otimes$-idempotents) and such that $f(Y) \otimes - : \mathcal{T} \longrightarrow \mathcal{T}$ realizes Bousfield localization with respect to $\mathcal{T}_Y := \mathrm{Loc}(\mathcal{K}_Y) = e(Y) \otimes \mathcal{T}$, the localizing subcategory of $\mathcal{T}$ generated by the compact objects $\mathcal{K}_Y = \big\{ a \in \mathcal{K} \,\big|\, \mathrm{supp}(a) \subset Y \big\}$. Moreover, for every pair of Thomason subsets $Y_1, Y_2 \subset \mathrm{Spc}(\mathcal{K})$, we have isomorphisms $e(Y_1 \cap Y_2) \cong e(Y_1) \otimes e(Y_2)$ and $f(Y_1 \cup Y_2) \cong f(Y_1) \otimes f(Y_2)$ and two Mayer-Vietoris distinguished triangles in $\mathcal{T}$:*

$$e(Y_1 \cap Y_2) \longrightarrow e(Y_1) \oplus e(Y_2) \longrightarrow e(Y_1 \cup Y_2) \longrightarrow \Sigma e(Y_1 \cap Y_2)$$

$$f(Y_1 \cap Y_2) \longrightarrow f(Y_1) \oplus f(Y_2) \longrightarrow f(Y_1 \cup Y_2) \longrightarrow \Sigma f(Y_1 \cap Y_2)\,.$$

Using these $\otimes$-idempotents, we get the announced definition of a support inside $\mathrm{Spc}(\mathcal{K})$, for all objects of $\mathcal{T}$ (compare Benson-Iyengar-Krause [13]):

Theorem 50 (B.-Favi [8, § 7]). *Let $\mathcal{T}$ and $\mathcal{K} = \mathcal{T}^c$ be as above and suppose that $\mathrm{Spc}(\mathcal{K})$ is noetherian. Define $\kappa(\mathcal{P}) = e(\overline{\{\mathcal{P}\}}) \otimes f(\mathrm{supp}(\mathcal{P})) \in \mathcal{T}$, for all $\mathcal{P} \in \mathrm{Spc}(\mathcal{K})$ (here $\mathrm{supp}(\mathcal{P})$ is the Thomason subset corresponding to $\mathcal{P}$ in the Classification Theorem 14). Then, the support admits the following extension to all objects $t \in \mathcal{T}$:*

$$\mathrm{supp}(t) := \left\{ \mathcal{P} \in \mathrm{Spc}(\mathcal{K}) \,\middle|\, t \otimes \kappa(\mathcal{P}) \neq 0 \right\}.$$

This support satisfies all properties (SD'1)-(SD'7) of Theorem 47.

Note that (i) and (ii) of Theorem 47 are trivial here. It is not clear when this support detects vanishing, i. e. when $t \otimes \kappa(\mathcal{P}) = 0$ for all $\mathcal{P} \in \mathrm{Spc}(\mathcal{K})$ implies $t = 0$.

3. Examples and applications

We now apply the theory of Part 2 to the examples of Part 1.

3.1. Classification of thick $\otimes$-ideals, after Hopkins.

Such classifications began in stable homotopy theory, see § 1.3, long before the start of $\otimes$-triangular geometry. Via Theorem 16, this becomes:

Theorem 51 (Hopkins-Smith [26], see [5, Cor. 9.5]). *The spectrum of $\mathrm{SH}^{\mathrm{fin}}$ is*

$$\begin{array}{ccccc}
\mathcal{P}_{2,\infty} & \mathcal{P}_{3,\infty} & \cdots & \mathcal{P}_{p,\infty} & \cdots \\
| & | & & | & \\
\vdots & \vdots & & \vdots & \\
| & | & & | & \\
\mathcal{P}_{2,n+1} & \mathcal{P}_{3,n+1} & \cdots & \mathcal{P}_{p,n+1} & \cdots \\
| & | & & | & \\
\mathcal{P}_{2,n} & \mathcal{P}_{3,n} & \cdots & \mathcal{P}_{p,n} & \cdots \\
| & | & & | & \\
\vdots & \vdots & & \vdots & \\
| & | & & | & \\
\mathcal{P}_{2,1} & \mathcal{P}_{3,1} & \cdots & \mathcal{P}_{p,1} & \cdots \\
 & & \mathrm{SH}^{\mathrm{fin}}_{\mathrm{tor}} & &
\end{array}$$

The lines $\mathcal{P}{-}\mathcal{P}'$ indicate that the higher prime is in the closure of the lower one. For every prime number p and every $n \geq 1$, the prime $\mathcal{P}_{p,n}$ of $\mathrm{SH}^{\mathrm{fin}}$ is the kernel of the n-th Morava K-theory (composed with localization at p) and $\mathcal{P}_{p,\infty} = \cap_{n\geq 1}\mathcal{P}_{p,n}$ is the kernel of localization at p. Finally, $\mathrm{SH}^{\mathrm{fin}}_{\mathrm{tor}} := \mathrm{Ker}(\mathrm{H}(-,\mathbb{Q}))$ is the subcategory of torsion spectra. The surjective continuous map $\rho = \rho_{\mathrm{SH}^{\mathrm{fin}}} : \mathrm{Spc}(\mathrm{SH}^{\mathrm{fin}}) \longrightarrow \mathrm{Spec}(\mathbb{Z})$ of Theorem 42 is given by $\rho(\mathrm{SH}^{\mathrm{fin}}_{\mathrm{tor}}) = (0)$ and $\rho(\mathcal{P}_{p,n}) = p\mathbb{Z}$ for all $1 \leq n \leq \infty$.

Remark 52. This example yields many observations. First, $\mathrm{Spc}(\mathrm{SH}^{\mathrm{fin}})$ is not noetherian and the closed subsets $\{\mathcal{P}_{n,\infty}\}$ are not the support of any object. In particular, in the local category $\mathrm{SH}^{\mathrm{fin}}_p$ at p, we have $\mathrm{FL}(\mathrm{SH}^{\mathrm{fin}}_p) = 0$. Finally, $\mathrm{Spec}(\mathrm{SH}^{\mathrm{fin}})$ is a locally ringed space but is not a scheme. See more in [5].

Remark 53. Hopkins [25] also understood that this classification could be transposed to algebra and indicated that (15) should provide the classification for $\mathcal{K} = \mathrm{D}^{\mathrm{perf}}(A)$, with the subsets $Y \subset \mathrm{Spec}(A)$ being all specialization closed subsets. The actual proof of this statement requires A to be noetherian and was given by Neeman [39]. But it is Thomason who nailed down the dual-open subsets (our Thomason subsets) in [48, Thm. 3.15]. His result settles the non-noetherian affine case and, most interestingly, works for any quasi-separated scheme if one insists on $\otimes$-*ideal* thick subcategories. Via Theorem 16 and Construction 29, this yields:

Theorem 54 (Reconstruction [1, Thm. 6.3]). *Let X be a quasi-compact and quasi-separated scheme. We have an isomorphism* $\mathrm{Spec}(\mathrm{D}^{\mathrm{perf}}(X)) \simeq X$ *of ringed spaces.*

Remark 55. Under the underlying homemorphism $\mathrm{Spc}(\mathrm{D}^{\mathrm{perf}}(X)) \simeq X$, we can reformulate another famous result of Thomason's [49, § 5]: *For every quasi-compact $U \subset X$, we have* $\mathcal{K}(U) \cong \mathrm{D}^{\mathrm{perf}}(U)$, where $\mathcal{K}(U)$ is as in Construction 24.

Remark 56. The map $\varphi : X \to \mathrm{Spc}(\mathrm{D}^{\mathrm{perf}}(X))$ of Theorem 16 sends $x \in X$ to $\mathrm{Ker}\big(\mathrm{D}^{\mathrm{perf}}(X) \to \mathrm{D}^{\mathrm{perf}}(\mathcal{O}_{X,x})\big)$. For $X = \mathrm{Spec}(A)$ affine and $\mathfrak{p} \in \mathrm{Spec}(A)$, the quotient $\mathcal{K}/\varphi(\mathfrak{p}) \cong \mathrm{D}^{\mathrm{perf}}(A_{\mathfrak{p}})$ is indeed the expected local category. Let us make two further observations. First, φ reverses inclusions, i. e. if $\mathfrak{p} \subset \mathfrak{q}$ in A then $\varphi(\mathfrak{p}) \supset \varphi(\mathfrak{q})$ in $\mathcal{K}$. This phenomenon is in line with other mildly surprising facts, to an algebraist's eye, like $\overline{\{\mathcal{P}\}} = \big\{\, \mathcal{Q} \,\big|\, \mathcal{Q} \subset \mathcal{P} \,\big\}$ for every $\mathcal{P} \in \mathrm{Spc}(\mathcal{K})$.

Secondly, an inverse to φ is given by the map $\rho_{\mathcal{K}} : \mathrm{Spc}(\mathcal{K}) \to \mathrm{Spec}(\mathrm{R}_{\mathcal{K}}) = \mathrm{Spec}(A)$ of Theorem 42. Hence $\mathcal{K} = \mathrm{D}^{\mathrm{perf}}(A)$ provides an example where $\rho_{\mathcal{K}}$ is not only surjective, as follows from Theorem 43, but also injective. Interestingly, one can actually give a direct proof of the injectivity of $\rho_{\mathcal{K}}$ in this case and obtain the Hopkins-Neeman-Thomason classification for $\mathrm{D}^{\mathrm{perf}}(A)$ by Theorem 14. See details in [5, Rem. 8.4].

Walking in Hopkins's steps, Benson-Carlson-Rickard [12] and later Friedlander-Pevtsova [22] performed the classification in modular representation theory for finite groups and finite group schemes. Combined with Theorem 16, this reads:

Theorem 57 ([1, Thm. 6.3] and [5, Cor. 9.5]). *Let k be a field of positive characteristic and G be a finite group (scheme over k). See Section 1.4. Consider the graded-commutative cohomology ring* $\mathrm{H}^\bullet(G,k)$. *Then, for the derived category* $\mathcal{K} = \mathrm{D}^{\mathrm{b}}(kG\text{–mod})$, *the map $\rho^\bullet_{\mathcal{K}}$ of Theorem 42 induces an isomorphism*

$$\mathrm{Spec}(\mathrm{D}^{\mathrm{b}}(kG\text{–mod})) \simeq \mathrm{Spec}^{\mathrm{h}}(\mathrm{H}^\bullet(G,k))$$

between the triangular spectrum of $\mathcal{K}$ and the homogeneous spectrum of the cohomology. Via (5), *it restricts to an isomorphism* $\mathrm{Spec}(\mathrm{stab}(kG)) \simeq \mathrm{Proj}(\mathrm{H}^\bullet(G,k))$, *where the latter is the so-called* projective support variety $\mathcal{V}_G(k)$.

Indeed, Friedlander and Pevtsova were able to reconstruct the structure sheaf of $\mathcal{V}_G$ by computing the triangular structure sheaf $\mathcal{O}_{\mathcal{K}}$ of our Construction 29.

Recently, Krishna [32, Thm. 7.10] proved that the spectrum of the category of perfect complexes over a (reasonable) stack is the associated moduli space.

3.2. Further computations.

It is now natural to turn to other, newer areas, where the classification of thick $\otimes$-ideals is not yet known, to see whether the spectrum can be computed by some other means. Here are some first results in motivic theory and noncommutative topology. In both cases, the spectrum is only known in the simplest $\otimes$-triangulated category that one can produce. But these should be considered as bridgeheads in two unknown (but friendly) territories.

Let us start with motivic theory, see § 1.5-§ 1.6. Here, the simplest category is probably that of mixed Tate motives with rational coefficients, i. e. the triangulated subcategory of $\mathrm{DM}(k)_{\mathbb{Q}}$ generated by the Tate objects $\mathbb{Q}(i)$, for all $i \in \mathbb{Z}$.

Theorem 58 (Peter [42]). *Let k be a number field and $\mathrm{DMT}(k)_{\mathbb{Q}}$ be the triangulated category of mixed Tate motives. Then $\mathrm{Spc}(\mathrm{DMT}(k)_{\mathbb{Q}})$ is just a point.*

At the other end of the motivic game, the computation of the spectrum of $\mathrm{SH}^{\mathbb{A}^1}_{\mathrm{gm}}(S)$ as in § 1.6 is probably a difficult long-term challenge. Using Theorem 43 and Morel's computation [37] of $\mathrm{End}_{\mathrm{SH}^{\mathbb{A}^1}}(\mathbb{1})$, we can still get :

Theorem 59 ([5, Cor. 10.1]). *Let $\mathcal{K} = \mathrm{SH}^{\mathbb{A}^1}_{\mathrm{gm}}(k)$ for a perfect field k of characteristic different from 2 as in Section 1.6. Then the continuous map $\rho_{\mathcal{K}}$ of Theorem 42 defines a surjection from the triangular spectrum $\mathrm{Spc}(\mathcal{K})$ onto the Zariski spectrum $\mathrm{Spec}(\mathrm{GW}(k))$ of the Grothendieck-Witt ring of quadratic forms over k.*

The second area we want to discuss is noncommutative topology, see § 1.7. In that case, the baby $\otimes$-triangulated category is the thick subcategory $\mathcal{K}^G$ of KK^G generated by the unit. The ring of endomorphisms of the unit $R(G) = \mathrm{End}_{KK^G}(\mathbb{1})$ is the Grothendieck group of continuous complex representations of G.

Theorem 60 (Dell'Ambrogio [21]). *Let G be a finite group. Then the map $\rho_{\mathcal{K}^G}$ of Theorem 42 is split surjective. It is a homeomorphism for G trivial, i. e. $\mathrm{Spc}(\mathcal{K}) \simeq \mathrm{Spec}(\mathbb{Z})$ where $\mathcal{K} \subset KK$ is the triangulated subcategory generated by $\mathbb{1} = \mathbb{C}$.*

Dell'Ambrogio also conjectured [21, Conj. 1.3] that $\rho_{\mathcal{K}^G}$ is injective for every finite group G. Again, our surjectivity Theorem 43 applies in big generality :

Theorem 61 ([5, Cor. 8.8]). *Let G be a compact Lie group. Then the continuous map $\rho_{\mathcal{K}^G} : \mathrm{Spc}(\mathcal{K}^G) \to \mathrm{Spec}(R(G))$ of Theorem 42 is surjective.*

Remark 62. A famous result of Quillen in modular representation theory of a finite group G asserts that $\mathcal{V}_G$ is covered by the images of the $\mathcal{V}_E$ under the maps $\mathrm{Spc}(\mathrm{res}^G_E) : \mathcal{V}_E \to \mathcal{V}_G$, where $E < G$ runs through the elementary abelian p-subgroups. Dell'Ambrogio explains in [21] how the celebrated Baum-Connes conjecture with coefficients would follow from an analogous property in KK-theory, namely that the spectrum of KK^G (G as in §1.7) be covered by the images of the various spectra of KK^H, where $H < G$ runs through compact subgroups.

3.3. Applications to algebraic geometry.

The following result is an immediate corollary of Theorem 54 :

Corollary 63. *Let X and Y be two quasi-separated (e. g. noetherian) schemes. If their derived categories of perfect complexes are equivalent* $\mathrm{D}^{\mathrm{perf}}(X) \simeq \mathrm{D}^{\mathrm{perf}}(Y)$ *as* tensor *triangulated categories then the schemes $X \simeq Y$ are isomorphic.*

Remark 64. A $\otimes$-triangular equivalence $\mathrm{D}_{\mathrm{Qcoh}(X)}(X) \simeq \mathrm{D}_{\mathrm{Qcoh}(Y)}(Y)$ restricts to a $\otimes$-triangular equivalence on the compact parts, $\mathrm{D}^{\mathrm{perf}}(X) \simeq \mathrm{D}^{\mathrm{perf}}(Y)$, hence implies $X \simeq Y$ as well. This reconstruction result is known to fail without the tensor : There exist non-isomorphic schemes, even abelian varieties, with triangular equivalent derived categories. See Mukai [38].

Remark 65. In homological mirror symmetry, or more generally each time that one expects a given triangulated category $\mathcal{K}$ to be equivalent to $\mathrm{D}^{\mathrm{perf}}(X)$ for some (maybe conjectural) scheme X, it becomes interesting to construct the tensor product on $\mathcal{K}$ which should correspond to that of $\mathrm{D}^{\mathrm{perf}}(X)$. See [47]. In this situation, the scheme X *must* be $\mathrm{Spec}(\mathcal{K})$ by Theorem 54. This does not guarantee that $\mathcal{K} = \mathrm{D}^{\mathrm{perf}}(X)$ but it tells us what X must be.

The abstract results of $\otimes$-triangular geometry apply in particular to $\mathcal{K} = \mathrm{D}^{\mathrm{perf}}(X)$. For instance, the filtration by (co)dimension of support in Theorem 32 yields a spectral sequence in any cohomology theory "defined" on derived categories, like K-theory or Witt theory, for instance. In particular, we get the following generalization of Quillen's famous spectral sequence for regular schemes [44] :

Theorem 66 ([4, Thm. 1]). *Let X be a (topologically) noetherian scheme of finite Krull dimension. Then there is a cohomologically indexed and converging spectral sequence in Thomason non-connective K-theory [49], of local-global nature :*

$$E_1^{p,q} = \bigoplus_{x \in X^{(p)}} K_{-p-q}(\mathcal{O}_{X,x} \ \textit{on}\ \{x\}) \qquad \xRightarrow[p,q,n\in\mathbb{Z}]{p+q=n} \qquad K_{-n}(X)\,.$$

Remark 67. This theorem is a first *strict* application of $\otimes$-triangular geometry, since the statement does not involve $\otimes$-triangulated categories. Yet, the deeper result is Theorem 32 which says that the quotient $\mathrm{D}^{\mathrm{perf}}(X)_{(d)} / \mathrm{D}^{\mathrm{perf}}(X)_{(d-1)}$ decomposes, up to idempotent completion, as the coproduct of the categories $\mathrm{FL}(\mathrm{D}^{\mathrm{perf}}(\mathcal{O}_{X,x})) = \big\{\, a \in \mathrm{D}^{\mathrm{perf}}(\mathcal{O}_{X,x}) \,\big|\, \mathrm{supp}(a) \subset \{x\} \,\big\}$ over all $x \in X_{(d)}$.

This illustrates the "boomerang effect" of abstraction : Inspired by Quillen [44], we started from the well-known fact that for a *regular* scheme, the above quotient is exactly equivalent to $\coprod_{x\in X_{(d)}} \mathrm{FL}(\mathrm{D}^{\mathrm{perf}}(\mathcal{O}_{X,x}))$, without idempotent completion, and we tried to extend it to $\otimes$-triangular geometry. This simply fails ! But it works if one adds the idempotent completion to the picture. Then, Theorem 32 holds in *all* areas of $\otimes$-triangular geometry. Now, this yields a gain even in algebraic geometry where we started, for we understand that the regularity assumption was not that important after all. In K-theory, the idempotent completion explains the presence of negative K-theory in Theorem 66. Of course, all this has its origin in

Thomason's description of $\mathrm{D}^{\mathrm{perf}}(U)$ (Remark 55) and it is fair to say that he had everything in [49] to prove Theorem 66. It is nonetheless remarkable that these ideas extend so far beyond algebraic geometry.

3.4. Applications to modular representation theory.

In modular representation theory, see § 1.4, the filtration Theorem 32 applied to $\mathcal{K} = \mathrm{stab}(kG)$ recovers, and slightly improves, a result of Carlson-Donovan-Wheeler [20, Thm. 3.5]. Let us rather comment on the Picard group, $\mathrm{Pic}(\mathrm{stab}(kG))$, which is a classical invariant, known as the group $T(G) = T_k(G)$ of endotrivial kG-modules up to isomorphism. A kG-module M is *endotrivial* if $\mathrm{End}_k(M) \simeq k \oplus (\mathrm{proj})$ which simply means that $M^* \otimes M \simeq \mathbb{1}$ in $\mathrm{stab}(kG)$. We proved :

Theorem 68 (B.-Benson-Carlson [7]). *The endotrivial modules obtained by the gluing technique of Theorem 37 generate a finite-index subgroup of $T(G)$.*

Remark 69. Recall the $\otimes$-triangulated category $\mathcal{K}(U)$ of Construction 24 for every quasi-compact open $U \subset \mathrm{Spc}(\mathcal{K})$. In algebraic geometry, for X a scheme and $\mathcal{K} = \mathrm{D}^{\mathrm{perf}}(X)$, Thomason proved $\mathcal{K}(U) \simeq \mathrm{D}^{\mathrm{perf}}(U)$, see Remark 55. In other words, the construction $(\mathcal{K}, U) \longmapsto \mathcal{K}(U)$ "stays inside algebraic geometry".

On the other hand, for $\mathcal{K} = \mathrm{stab}(kG)$ and $U \subset \mathcal{V}_G(k)$ non-trivial, $\mathcal{K}(U)$ is never equivalent to a stable category $\mathrm{stab}(kH)$, no matter what finite group H one tries. See [6, Prop. 4.2]. Hence, although Thomason's result *does* work abstractly and transposes to modular representation theory via the $\otimes$-triangular construction $\mathcal{K}(U)$, the resulting construction takes us out of basic modular representation theory. Here is a nice *strict* application of Theorem 38 (without $\otimes$-triangulated categories in the statement) :

Theorem 70 ([6, Thm. 4.7]). *Let G be a finite group and $\mathcal{V}_G = \mathrm{Proj}(\mathrm{H}^\bullet(G,k))$ its projective support variety over a field k of characteristic dividing the order of G. Then gluing induces an injection $\beta : \mathrm{Pic}(\mathcal{V}_G) \underset{\mathbb{Z}}{\otimes} \mathbb{Z}[1/p] \hookrightarrow T(G) \underset{\mathbb{Z}}{\otimes} \mathbb{Z}[1/p]$. Combining with Theorem 68, we obtain a rational isomorphism*

$$\mathrm{Pic}(\mathcal{V}_G) \underset{\mathbb{Z}}{\otimes} \mathbb{Q} \simeq T(G) \underset{\mathbb{Z}}{\otimes} \mathbb{Q}\,.$$

Remark 71. The above result fails integrally. For instance, for $G = Q_8$ the quaternion group and k containing a cubic root of unity, the group of endotrivials is $T(Q_8) = \mathbb{Z}/4 \oplus \mathbb{Z}/2$ although $\mathrm{Spc}(\mathrm{stab}(kQ_8)) = \mathcal{V}_{Q_8}(k) = *$ is just a point, hence $\mathrm{Pic}(\mathcal{V}_{Q_8}) = 0$. Note also that $\mathrm{stab}(kQ_8)$ is a local $\otimes$-triangulated category.

3.5. Intra-utero applications.

While $\otimes$-triangular geometry was still in the making, $\otimes$-triangulated categories showed useful in the theory of Witt groups of quadratic forms over schemes. This abstract theory, of so-called *triangular Witt groups*, has been quite useful. It led to the proof of the Gersten conjecture for Witt groups, among many other (strict) applications, including the computation of several classical Witt groups. For a survey, the interested reader is referred to [2]. In retrospect, many of these triangular Witt groups results fit very well in the language of $\otimes$-triangular geometry.

4. Problems

We have already mentioned a few open questions in the above text. In conclusion, we briefly suggest some additional directions of possible interest. We refrain from insisting on the wildest dreams (as in Remarks 62 and 65 for instance) and favor of a few problems reasonably close to the current stage of the theory.

4.1. Computing the spectrum in more examples.

As discussed in § 2.5, the most basic question is to compute $\mathrm{Spc}(\mathcal{K})$ for more $\otimes$-triangulated categories $\mathcal{K}$, preferably without using the classification of thick $\otimes$-ideals, in order to deduce the latter via Theorem 14 and show off a little. Theorem 47 offers an angle of attack. Still, we need more results telling us how to compare $\mathrm{Spc}(\mathcal{K})$ to other spaces. Such a comparison is provided by the maps $\rho_{\mathcal{K}}$ and $\rho^{\bullet}_{\mathcal{K}}$ of Theorem 42. We have seen that these maps are often surjective (Thm. 43). It then becomes interesting to decide when they are injective and more generally to study their fibers.

In algebraic examples like $\mathcal{K} = \mathrm{D}^{\mathrm{perf}}(A)$ or $\mathcal{K} = \mathrm{D}^{\mathrm{b}}(kG\text{–mod})$, the map $\rho^{\bullet}_{\mathcal{K}}$ is injective (see § 3.1) but we have seen in the very first example (Thm. 51) that injectivity fails completely outside algebra. The tempting guess would be :

Conjecture 72. The map $\rho^{\bullet}_{\mathcal{K}}$ is (locally) injective when $\mathcal{K}$ is "algebraic enough".

Here "algebraic enough" could mean those triangulated categories $\mathcal{K}$ which arise as stable categories of Frobenius exact categories, or, alternatively, those $\mathcal{K}$ which are the derived category of some dg-category, see Keller [29]. It might also be necessary to add some hypothesis like $\mathcal{K}$ being locally generated by $\mathbb{1}$.

Remark 73. By Hochster [24], any spectral space, like our $\mathrm{Spc}(\mathcal{K})$, is the spectrum of *some* commutative ring. It would be pleasant to construct such a ring explicitly in terms of $\mathcal{K}$. The above use of $\mathrm{R}_{\mathcal{K}}$ and $\mathrm{R}^{\bullet}_{\mathcal{K}}$ was a first attempt to do this.

4.2. Image of algebraic geometry in $\otimes$-triangular geometry.

We have seen in Theorem 54 that a scheme X can be reconstructed from the $\otimes$-triangulated category $\mathrm{D}^{\mathrm{perf}}(X)$. An important question is to decide which $\otimes$-triangulated categories $\mathcal{K}$ are $\otimes$-equivalent to $\mathrm{D}^{\mathrm{perf}}(\mathrm{Spec}(\mathcal{K}))$. Actually, it would also be interesting to know when the locally ringed space $\mathrm{Spec}(\mathcal{K})$ is a scheme. As already mentioned in Remark 65, this could have consequences beyond algebraic geometry, as for instance in homological mirror symmetry.

Also interesting would probably be the tensor-triangular characterization of some properties of morphisms of schemes, like being smooth or étale.

4.3. Residue fields.

In examples, triangular primes $\mathcal{P} \subset \mathcal{K}$ are often the kernel of a tensor functor $\mathcal{K} \to \mathcal{F}$ with $\mathcal{F} = \mathrm{VB}_k$ being the category of k-vector spaces over a field k (in algebraic geometry), or $\mathcal{F}$ being the category of graded modules over a graded field

$k[t,t^{-1}]$ (in homotopy theory), or $\mathcal{F} = \mathrm{stab}(kC_p)$ being the stable category of kC_p-modules, for C_p the cyclic group of order $p = \mathrm{char}(k)$ (in modular representation theory, although this case is still unclear). This observation calls for two things:

(a) The definition of $\otimes$-*triangular fields* $\mathcal{F}$, which would imply in particular that $\mathrm{Spc}(\mathcal{F}) = \{*\}$ is reduced to a point.

(b) The construction, for every local category $\mathcal{K}$ (Thm. 25), of a conservative $\otimes$-exact functor $\pi : \mathcal{K} \to \mathcal{F}$ into some $\otimes$-triangular field, that would be a "residue field". Conservative means that $\mathrm{Ker}(\pi) = 0$, i.e. that the image of $\mathrm{Spc}(\pi) : \{*\} = \mathrm{Spc}(\mathcal{F}) \longrightarrow \mathrm{Spc}(\mathcal{K})$ would be the unique closed point of $\mathrm{Spc}(\mathcal{K})$.

Note that there might be several such residue field functors, as seems to be the case in modular representation theory. It is not at all clear whether such functors can be constructed from the $\otimes$-triangular structure alone but they should certainly be looked for in examples where one tries to determine $\mathrm{Spc}(\mathcal{K})$.

Regarding the definition of $\otimes$-triangular fields, the naive idea of requesting the category $\mathcal{F}$ to be semi-simple does not cover $\mathrm{stab}(kC_p)$ for instance. Indeed, $\mathrm{Spc}(\mathrm{stab}(kC_p))$ is a point but there is no non-zero $\otimes$-exact functor from $\mathrm{stab}(kC_p)$ into a semi-simple $\otimes$-category as soon as $p \geq 3$. (For $p = 2$, $\mathrm{stab}(kC_2) \cong \mathrm{VB}_k$.) Currently, my favorite guess is to define $\mathcal{F}$ to be a triangular field if every non-zero object $x \in \mathcal{F}$ is faithful (i.e. $x \otimes f = 0$ forces $x = 0$ or $f = 0$). This covers all three examples above and still forces $\mathrm{Spc}(\mathcal{F}) = \{*\}$ but there is no solid conceptual motivation for this definition at this stage, beyond unification of examples.

4.4. Nilpotence.

A clear understanding of nilpotence phenomena in triangulated categories still eludes us, even in the presence of a tensor. First, we do not know how to define *reduced* $\otimes$-triangulated categories. Nor do we know how to construct $\mathrm{D}^{\mathrm{perf}}(X_{\mathrm{red}})$ out of the $\otimes$-triangulated category $\mathcal{K} = \mathrm{D}^{\mathrm{perf}}(X)$, except via the odious cheat: $\mathrm{D}^{\mathrm{perf}}\big((\mathrm{Spec}(\mathcal{K}))_{\mathrm{red}}\big)$. For instance, even when $\mathrm{Spc}(\mathcal{K}) = \{*\}$ is a point, that is, when $\mathcal{K}$ is something like an "artinian local" $\otimes$-triangulated category, it is not clear how to obtain a residue field (§ 4.3) by reduction modulo nilpotents.

Also, there seems to be no obvious way to construct a $\otimes$-triangulated category "$\mathcal{K}$ over Z", for a closed subset $Z \subset \mathrm{Spc}(\mathcal{K})$ of the spectrum, say, with what should be the "reduced structure". Neither do I know which closed subsets $Z \subset \mathrm{Spc}(\mathcal{K})$ are the support of an object $u \in \mathcal{K}$ as in Theorem 18 (c). Again, this relates to the residue field of § 4.3 when $\mathcal{K}$ is local and $Z = \{*\}$ is the closed point.

4.5. Torsion in the Picard group.

This is a follow-up on Remarks 39 and 71. First, let us note that the isomorphism $\mathrm{Pic}(\mathcal{V}_G) \otimes \mathbb{Q} \simeq T(G) \otimes \mathbb{Q}$ of Theorem 70 is still unknown for G a finite group *scheme*, because we do not know whether the Picard group is locally torsion in that case. We have seen in Proposition 40 that the Picard group can be locally wild. Yet, the example $\coprod_G \mathcal{K}$ can be ruled out if we further require $\mathcal{K}$ to be generated by $\mathbb{1}$, as a thick triangulated subcategory. Hence the following hope survives:

Conjecture 74. Let $\mathcal{K}$ be a $\otimes$-triangulated category as in Hypothesis 21. Assume that $\mathcal{K}$ is local (Thm. 25) and that $\mathcal{K}$ is generated by $\mathbb{1}$. Let $u \in \mathcal{K}$ be $\otimes$-invertible. Then there exists $m > 0$ such that $u^{\otimes m}$ is trivial in the sense that $u^{\otimes m} \simeq \Sigma^n \mathbb{1}$ for some $n \in \mathbb{Z}$. That is, $\mathrm{Pic}(\mathcal{K})$ is rationally trivial: $\mathrm{Pic}(\mathcal{K}) \otimes_{\mathbb{Z}} \mathbb{Q} = \mathbb{Q} \cdot [\Sigma\mathbb{1}]$.

References

[1] P. Balmer. The spectrum of prime ideals in tensor triangulated categories. *J. Reine Angew. Math.*, 588:149–168, 2005.

[2] P. Balmer. Witt groups. In *Handbook of K-theory. Vol. 2*, pages 539–576. Springer, Berlin, 2005.

[3] P. Balmer. Supports and filtrations in algebraic geometry and modular representation theory. *Amer. J. Math.*, 129(5):1227–1250, 2007.

[4] P. Balmer. Niveau spectral sequences on singular schemes and failure of generalized Gersten conjecture. *Proc. Amer. Math. Soc.*, 137(1):99–106, 2009.

[5] P. Balmer. Spectra, spectra, spectra – Tensor triangular spectra versus Zariski spectra of endomorphism rings. Preprint, 2009.

[6] P. Balmer. Picard groups in triangular geometry and applications to modular representation theory. *Trans. Amer. Math. Soc.*, 362(7), 2010.

[7] P. Balmer, D. J. Benson, and J. F. Carlson. Gluing representations via idempotent modules and constructing endotrivial modules. *J. Pure Appl. Algebra*, 213(2):173–193, 2009.

[8] P. Balmer and G. Favi. Generalized tensor idempotents and the telescope conjecture. Preprint, 24 pages, available at `www.math.ucla.edu/~balmer`, 2009.

[9] P. Balmer and G. Favi. Gluing techniques in triangular geometry. *Q. J. Math.*, 51(4):415–441, 2007.

[10] P. Balmer and M. Schlichting. Idempotent completion of triangulated categories. *J. Algebra*, 236(2):819–834, 2001.

[11] D. J. Benson. *Representations and cohomology I & II*, volume 30 & 31 of *Cambridge Studies in Advanced Mathematics*. Cambridge University Press, 1998.

[12] D. J. Benson, J. F. Carlson, and J. Rickard. Thick subcategories of the stable module category. *Fund. Math.*, 153(1):59–80, 1997.

[13] D. J. Benson, S. B. Iyengar, and H. Krause. Local cohomology and support for triangulated categories. *Ann. Sci. Éc. Norm. Supér. (4)*, 41(4):573–619, 2008.

[14] P. Berthelot, A. Grothendieck, and L. Illusie, editors. *SGA 6 : Théorie des intersections et théorème de Riemann-Roch*. Springer LNM 225. 1971.

[15] A. Bondal and M. van den Bergh. Generators and representability of functors in commutative and noncommutative geometry. *Mosc. Math. J.*, 3(1):1–36, 258, 2003.

[16] A. B. Buan, H. Krause, and Ø. Solberg. Support varieties: an ideal approach. *Homology, Homotopy Appl.*, 9(1):45–74, 2007.

[17] T. Bühler. On the algebraic foundations of bounded cohomology. ETH Thesis 2008. To appear in *Mem. Amer. Math. Soc.*

[18] J. F. Carlson. The variety of an indecomposable module is connected. *Invent. Math.*, 77(2):291–299, 1984.

[19] J. F. Carlson. *Modules and group algebras.* Lectures in Mathematics ETH Zürich. Birkhäuser Verlag, Basel, 1996.

[20] J. F. Carlson, P. W. Donovan, and W. W. Wheeler. Complexity and quotient categories for group algebras. *J. Pure Appl. Algebra*, 93(2):147–167, 1994.

[21] I. Dell'Ambrogio. Tensor triangular geometry and KK-theory. Preprint, 33 pages, available online, 2009.

[22] E. M. Friedlander and J. Pevtsova. Π-supports for modules for finite group schemes. *Duke Math. J.*, 139(2):317–368, 2007.

[23] D. Happel. *Triangulated categories in the representation theory of finite-dimensional algebras*, volume 119 of *LMS Lecture Note.* Cambr. Univ. Press, Cambridge, 1988.

[24] M. Hochster. Prime ideal structure in commutative rings. *Trans. Amer. Math. Soc.*, 142:43–60, 1969.

[25] M. J. Hopkins. Global methods in homotopy theory. In *Homotopy theory (Durham, 1985)*, volume 117 of *LMS Lect. Note*, pages 73–96. Cambridge Univ. Press, 1987.

[26] M. J. Hopkins and J. H. Smith. Nilpotence and stable homotopy theory. II. *Ann. of Math. (2)*, 148(1):1–49, 1998.

[27] M. Hovey. *Model categories*, volume 63 of *Mathematical Surveys and Monographs.* American Mathematical Society, 1999.

[28] M. Hovey, J. H. Palmieri, and N. P. Strickland. Axiomatic stable homotopy theory. *Mem. Amer. Math. Soc.*, 128(610), 1997.

[29] B. Keller. Deriving DG categories. *Ann. Sci. École Norm. Sup. (4)*, 27(1):63–102, 1994.

[30] B. Keller and A. Neeman. The connection between May's axioms for a triangulated tensor product and Happel's description of the derived category of the quiver D_4. *Doc. Math.*, 7:535–560, 2002.

[31] M. Kontsevich. Homological algebra of mirror symmetry. In *Proc. of the Intern. Congress of Mathematicians (Zürich, 1994)*, pages 120–139. Birkhäuser, 1995.

[32] A. Krishna. Perfect complexes on Deligne-Mumford stacks and applications. *J. K-Theory*, 4(3):559–603, 2009.

[33] S. Mac Lane. *Categories for the working mathematician*, volume 5 of *Graduate Texts in Mathematics.* Springer-Verlag, New York, 1998.

[34] J. P. May. The additivity of traces in triangulated categories. *Adv. Math.*, 163(1):34–73, 2001.

[35] R. Meyer. Categorical aspects of bivariant K-theory. In *K-theory and noncommutative geometry*, EMS Ser. Congr. Rep., pages 1–39. Eur. Math. Soc., 2008.

[36] F. Morel. An introduction to $\mathbb{A}^1$-homotopy theory. In *Contemporary developments in algebraic K-theory*, ICTP Lect. Notes, XV, pages 357–441. Trieste, 2004.

[37] F. Morel. On the motivic π_0 of the sphere spectrum. In *Axiomatic, enriched and motivic homotopy theory*, volume 131 of *NATO Sci. Ser. II*, pages 219–260. Kluwer, Dordrecht, 2004.

[38] S. Mukai. Duality between $D(X)$ and $D(\hat{X})$ with its application to Picard sheaves. *Nagoya Math. J.*, 81:153–175, 1981.

[39] A. Neeman. The chromatic tower for $D(R)$. *Topology*, 31(3):519–532, 1992.

[40] A. Neeman. The Grothendieck duality theorem via Bousfield's techniques and Brown representability. *J. Amer. Math. Soc.*, 9(1):205–236, 1996.

[41] A. Neeman. *Triangulated categories*, volume 148 of *Annals of Mathematics Studies*. Princeton University Press, 2001.

[42] T. Peter. Prime ideals of mixed Tate motives. Preprint, 2010.

[43] J. Pevtsova. Spectra of tensor triangulated categories. MSRI talk, 2008 April 4, available online at `www.msri.org`.

[44] D. Quillen. Higher algebraic K-theory. I. In *Algebraic K-theory, I: Higher K-theories (Proc. Conf., Battelle Memorial Inst., Seattle, Wash., 1972)*, pages 85–147. Lecture Notes in Math., Vol. 341. Springer, Berlin, 1973.

[45] J. Rickard. Derived categories and stable equivalence. *J. Pure Appl. Algebra*, 61(3):303–317, 1989.

[46] J. Rickard. Idempotent modules in the stable category. *J. London Math. Soc. (2)*, 56(1):149–170, 1997.

[47] A. Subotic. Monoidal structures on the Fukaya category. PhD, Harvard (2010).

[48] R. W. Thomason. The classification of triangulated subcategories. *Compositio Math.*, 105(1):1–27, 1997.

[49] R. W. Thomason and T. Trobaugh. Higher algebraic K-theory of schemes and of derived categories. In *The Grothendieck Festschrift, Vol. III*, volume 88 of *Progr. Math.*, pages 247–435. Birkhäuser, Boston, MA, 1990.

[50] J.-L. Verdier. Des catégories dérivées des catégories abéliennes. *Astérisque*, 239, 1996.

[51] V. Voevodsky. $\mathbf{A}^1$-homotopy theory. In *Proceedings of the International Congress of Mathematicians, Vol. I (Berlin, 1998)*, pages 579–604, 1998.

[52] V. Voevodsky, A. Suslin, and E. M. Friedlander. *Cycles, transfers, and motivic homology theories*, volume 143 of *Annals of Mathematics Studies*. Princeton University Press, Princeton, NJ, 2000.

[53] R. Vogt. *Boardman's stable homotopy category*. Lecture Notes Series, No. 21. Matematisk Institut, Aarhus Universitet, Aarhus, 1970.

UCLA Mathematics Department, Los Angeles, CA 90095-1555
E-mail: balmer@math.ucla.edu