THE GEOMETRY OF PERMUTATION MODULES

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Abstract. We consider the derived category of permutation modules over a finite group, in positive characteristic. We stratify this tensor triangulated category using Brauer quotients. We describe the spectrum of its compact objects, by reducing the problem to elementary abelian groups and then by using a twisted form of cohomology to express the spectrum locally in terms of the graded endomorphism ring of the unit. Together, these results yield a classification of thick and of localizing ideals.

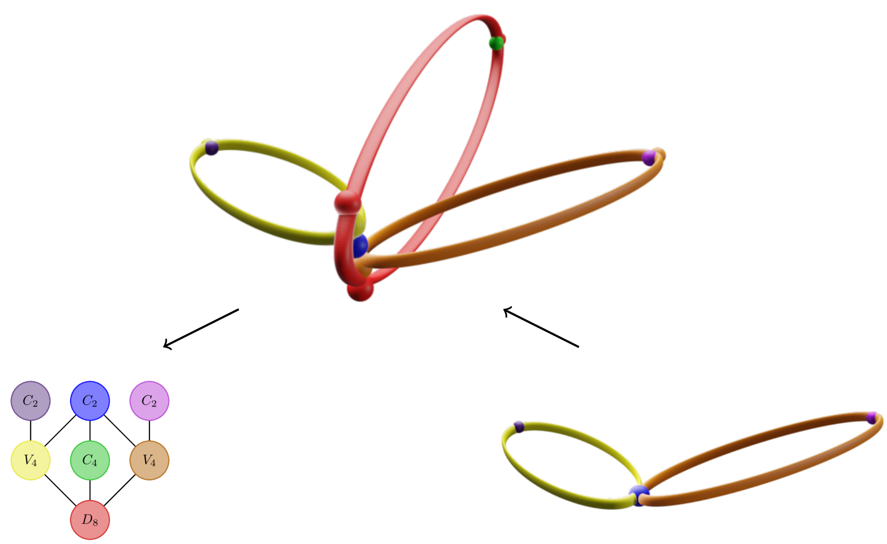

Figure 1. The geometry for the dihedral group D_8.

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Contents

1. Preamble

Fix a field k of positive characteristic p. Let G be a finite group. We often write 'tt' to abbreviate 'tensor triangulated' or 'tensor-triangular'.

Topic. Among k-linear representations of G, *permutation modules* are perhaps the easiest to grasp: They are simply the k-linearizations $k(X)$ of G-sets X. They play an important role throughout equivariant mathematics, in subjects as varied as derived equivalences [Ric96], Mackey functors [Yos83], or equivariant homotopy theory [MNN17], to name a few. The authors' original interest stems from yet another connection, with Voevodsky's theory of motives [Voe00] and specifically Artin motives. For elaboration on this theme, we refer the reader to [BG21].

We consider a 'small' tt-category, the homotopy category

$$\mathcal{K}(G) = \mathcal{K}(G; k) := \mathrm{K_b}\big(\,\mathrm{perm}(G; k)^\natural\big) \tag{1.1}$$

of bounded complexes of finitely generated permutation kG-modules, idempotent-completed. It sits as the compact part of the 'big' tt-category

$$\mathcal{T}(G) = \mathrm{DPerm}(G; k) \tag{1.2}$$

obtained, for instance, by closing $\mathcal{K}(G)$ under coproducts and triangles in the homotopy category $\mathrm{K}(\mathrm{Mod}(kG))$ of all kG-modules. We call $\mathrm{DPerm}(G; k)$ the *derived category of permutation kG-modules*. See Recollection 3.2 for details.

As we shall discuss in this preamble, these tt-categories of permutation modules are interesting and important for a variety of reasons. To begin with, they stand at the crossroad of several subjects, as alluded to above. Concretely, the tt-category $\mathcal{T}(G)$ is equivalent to:

(i) the derived category of cohomological k-linear Mackey functors over G,

(ii) the homotopy category of modules over the constant Green functor $\mathrm{H}\underline{k}$ in genuine G-spectra,

(iii) the triangulated category of k-linear Artin motives generated by motives of intermediate fields in any Galois extension with Galois group G.

Consequently, while we adopt here the language of permutation modules, our results admit translations into each of the contexts (i), (ii) and (iii).

Main goals. We want to understand the tensor-triangular geometry of these permutation tt-categories. Tensor-triangular geometry [Bal10b] is a way to bring organization to sometimes bewildering tt-categories, be it in topology, algebraic geometry or representation theory. Its fundamental device is the tt-spectrum $\mathrm{Spc}(\mathcal{K})$ of a small tt-category $\mathcal{K}$. Computing $\mathrm{Spc}(\mathcal{K}(G))$ will provide a classification of all thick triangulated $\otimes$-ideals in $\mathcal{K}(G)$. We also want to show that $\mathcal{K}(G)$ strongly controls the big tt-category $\mathcal{T}(G)$, namely the Telescope Conjecture holds for $\mathcal{T}(G)$ and the localizing $\otimes$-ideals of $\mathcal{T}(G)$ are classified by subsets of $\mathrm{Spc}(\mathcal{K}(G))$.

Landscape. Let us place $\mathcal{K}(G)$ among some standard G-equivariant tt-categories:

(a) The equivariant stable homotopy category $\mathrm{SH}(G)^c$ of finite genuine G-spectra.

(b) Kaledin's category of derived Mackey functors $\mathrm{DMack}(G;k)^c$.

(c) The bounded derived category $\mathrm{D_b}(kG)$ of finitely generated kG-modules.

(d) The stable module category $\mathrm{stab}(kG) = \mathrm{mod}(kG)/\,\mathrm{proj}(kG)$.

These categories all fit in a natural sequence of tt-functors, from equivariant homotopy theory to modular representation theory, with our $\mathcal{K}(G)$ at center stage:

$$\mathrm{SH}(G)^c \longrightarrow \mathrm{DMack}(G;k)^c \longrightarrow \mathcal{K}(G;k) \twoheadrightarrow \mathrm{D_b}(kG) \twoheadrightarrow \mathrm{stab}(kG). \tag{1.3}$$

The initial one, $\mathrm{SH}(G)^c$, is topological in nature and its tt-geometry relies heavily on chromatic theory, à la Devinatz-Hopkins-Smith [DHS88, HS98]. The first functor $\mathrm{SH}(G)^c \to \mathrm{DMack}(G;k)^c$ moves to the k-linear world and thus the chromatic refinements disappear from $\mathrm{DMack}(G;k)^c$ onwards. A central feature of the tt-categories in (1.3) is their variance in the group G. Restriction, induction and conjugation turn them into so-called *Mackey 2-functors*. In the language of [BD22], the three Mackey 2-functors $\mathcal{K}(G;k)$, $\mathrm{D_b}(kG)$ and $\mathrm{stab}(kG)$ are moreover *cohomological*. (This categorifies the fact that an ordinary Mackey functor is cohomological if $I_H^K \circ R_H^K$ is multiplication by $[K\,{:}\,H]$.) In other words, the second functor $\mathrm{DMack}(G;k)^c \to \mathcal{K}(G)$ in (1.3) moves us to the cohomological world. The subsequent functors in (1.3) are simply localizations. (For $\mathcal{K}(G) \twoheadrightarrow \mathrm{D_b}(kG)$ this follows from [BG23a]. For $\mathrm{D_b}(kG) \twoheadrightarrow \mathrm{stab}(kG)$ it is a well-known theorem due to Rickard [Ric89], or Buchweitz [Buc21].)

Classical methods. The four categories surrounding our $\mathcal{K}(G)$ in (1.3) have a fairly well-understood tt-geometry, thanks to a series of powerful and widely used techniques that we shall now briefly review with application to $\mathcal{K}(G)$ in mind.

The first obvious idea is to try some induction on the order of G. For each of the tt-categories in (1.3) we can define a so-called

geometric open

inside their tt-spectrum. It is the open complement of all the images of the closed maps induced by restriction to proper subgroups. This geometric open captures what is intrinsically new over G, beyond what is detected by proper subgroups. The name comes from stable homotopy theory (a), as the localization of $\mathrm{SH}(G)$ over the geometric open recovers the classical *geometric G-fixed-points functor*. In fact, a miracle occurs here: That localization of $\mathrm{SH}(G)$ is simply the non-equivariant SH. This fact has allowed [BS17] to describe all points of the spectrum of $\mathrm{SH}(G)^c$: All points come from the non-equivariant chromatic spectrum $\mathrm{Spc}(\mathrm{SH}^c)$ via geometric H-fixed-points, for all subgroups $H \leqslant G$ up to conjugation. The same strategy has been applied by Patchkoria-Sanders-Wimmer [PSW22] to derived Mackey functors over G, where the same miracle occurs: The geometric open boils down to the non-equivariant version, independently of G. One deduces that the spectrum of $\mathrm{DMack}(G;k)^c$ is the set of conjugacy classes of subgroups of G with a certain Alexandrov topology (if G is a p-group, $K \in \overline{\{H\}}$ iff $K \leq_G H$).

This geometric fixed-points method has been formalized by Barthel-Castellana-Heard-Naumann-Pol [BCH+23] for arbitrary equivariant tt-categories. However, the induction process breaks down because the above 'miracle' can evaporate in general: There is no simple description of the geometric open *a priori* and it can heavily depend on the group G. For example, for the stable category of an elementary abelian p-group $G = (C_p)^r$ the geometric open is dense in the spectrum, a projective space of dimension $r-1$, and thus it grows with G. In that respect, $\mathcal{K}(G)$ unfortunately behaves like $\mathrm{stab}(kG)$, and $\mathrm{D_b}(kG)$; the miracle breaks down. Beyond groups with very small p-Sylow the inductive approach of [BS17] hits a wall because the geometric fixed-points of $\mathcal{K}(G)$ are too complicated.

The second important method goes back to Serre's 1965 Theorem [Ser65]. In modern lingo, it says that the geometric open of $\mathrm{D_b}(kG)$ is empty unless G is an elementary abelian p-group. As a consequence, and through further work of Quillen [Qui71], the tt-geometry of $\mathrm{D_b}(kG)$ and $\mathrm{stab}(kG)$ reduces to elementary abelian subgroups of G. Unfortunately again, Serre's result does not hold for $\mathcal{K}(G)$: The geometric open is non-empty for *every* p-group G. Ipso facto, one cannot reduce the tt-geometry of $\mathcal{K}(G)$ to the elementary abelian subgroups of G.

Here is a third classical method. Work of Benson-Carlson-Rickard [BCR97] determines the tt-geometry of the derived and stable categories by using the cohomology $\mathrm{H}^\bullet(G,k)$, that we can view as the graded endomorphism ring $\mathrm{End}^\bullet_{\mathrm{D_b}(kG)}(\mathbb{1})$ of the $\otimes$-unit object $\mathbb{1}$ in $\mathrm{D_b}(kG)$. Reformulated in the language of [Bal10a], their result implies that the *comparison map*, which exists for every tt-category $\mathcal{K}$,

$$\mathrm{comp}_{\mathcal{K}}\colon \mathrm{Spc}(\mathcal{K}) \to \mathrm{Spec^h}(\mathrm{End}^\bullet_{\mathcal{K}}(\mathbb{1})), \tag{1.4}$$

is a homeomorphism in the case of $\mathcal{K} = \mathrm{D_b}(kG)$. The case of $\mathrm{stab}(kG)$ only differs from the above by removing the closed point, *i.e.* the 'irrelevant' ideal $\mathrm{H}^+(G;k)$. Again, these ideas have been pushed and generalized, most famously in a corpus of work affectionately known as 'BIK', after Benson-Iyengar-Krause [BIK11]. So we could hope that the BIK methods might apply to our tt-category of permutation modules $\mathcal{K}(G)$. Alas, the graded ring $\mathrm{End}^\bullet_{\mathcal{K}(G)}(\mathbb{1})$ is just the field k and its spectrum, a meagre singleton, refuses to entertain any idea of geometry.

The challenge. In summary, the classical methods that worked so well for $\mathrm{SH}(G)^c$ and $\mathrm{DMack}(G;k)^c$ on the one hand, and those that worked for $\mathrm{D_b}(kG)$ and $\mathrm{stab}(kG)$ on the other, all fall short in the case of $\mathcal{K}(G)$:

(✓=works, ✗=fails)	$\mathrm{SH}(G)^c$ & $\mathrm{DMack}(G;k)^c$	$\mathcal{K}(G;k)$	$\mathrm{D_b}(kG)$ & $\mathrm{stab}(kG)$
Geom. fixed-pts	✓	✗	✗
Elem. ab. subgps	✗	✗	✓
Comp. map & BIK	✗	✗	✓

This turn of events might seem surprising considering that $\mathcal{K}(G)$ ought to be the most accessible one among the five tt-categories in our list. Indeed, the mere construction of $\mathrm{SH}(G)$ and $\mathrm{DMack}(G;k)$ is highly involved and the modular representations that make up $\mathrm{D_b}(kG)$ and $\mathrm{stab}(kG)$ are notoriously wild, whereas $\mathcal{K}(G)$ is simply the bounded homotopy category of an additive category with finitely many isomorphism classes of indecomposables.

As we shall demonstrate in this article, the tt-geometry of $\mathcal{K}(G)$ just *is* very complex. It combines the complexity of its neighbors in (1.3), $\mathrm{DMack}(G;k)^c$ and $\mathrm{D_b}(kG)$, in a way reminiscent of how $\mathrm{SH}(G)^c$ combines the complexity of $\mathrm{DMack}(G;k)^c$ and SH^c. More precisely, just as the underlying set

$$\mathrm{Spc}(\mathrm{SH}(G)^c) = \amalg_H \mathrm{Spc}(\mathrm{SH}^c) \tag{1.5}$$

decomposes, over conjugacy classes of subgroups $H \le G$, into chromatic strata, so it will be shown that

$$\mathrm{Spc}(\mathcal{K}(G)) = \amalg_H \mathrm{Spc}(\mathrm{D_b}(k(G/\!\!/H))) \tag{1.6}$$

decomposes, over conjugacy classes of p-subgroups $H \le G$, into cohomological support varieties for the associated Weyl groups $G/\!\!/H$. Figure 2 may help the reader visualize the various phenomena at play. Each of them can be thought of as contributing a 'dimension' to the spectrum.

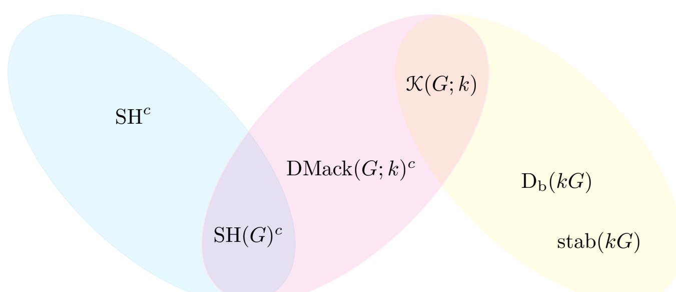

FIGURE 2. Related tt-categories and tt-geometric phenomena: chromatic (cyan), group-combinatorial (magenta), and Mackey-cohomological (yellow).

So, how do we approach the tt-geometry of $\mathcal{K}(G)$ given that the classical methods fail us? Let us discuss in broad strokes how the ideas behind those methods can still guide us to the solution, with suitably reinvented tools.

New methods. While geometric fixed-points definitely remain insufficient, a different type of fixed-points functors, the *modular fixed-points*, will prove very useful. They are the correct analogue at the level of $\mathcal{K}(G)$ of Brauer quotients, a well-known tool for the study of permutation modules. Firstly, we will use them to describe all points of the spectrum of $\mathcal{K}(G)$, arriving at (1.6) above. In addition, they allow us to circumvent the failure of Serre's theorem for $\mathcal{K}(G)$ and instead of a reduction to elementary abelian *subgroups*, obtain a reduction to elementary abelian *subquotients* of G. Finally, for G elementary abelian, although the comparison with cohomology and the BIK method still cannot be used globally on $\mathcal{K}(G)$, we will produce an open cover of $\mathrm{Spc}(\mathcal{K}(G))$ over which the comparison map is indeed a homeomorphism. In other words, BIK will work on small enough pieces of the category $\mathcal{K}(G)$. Their determination will involve a 'twisted' version of cohomology which epimorphically maps to the group cohomology for each Weyl group.

We explain these ideas in more detail and state precise theorems in the introduction to Part I (Section 2), where we discuss modular fixed-points and the 'stratification' results about the big tt-category $\mathcal{T}(G)$, and in the introduction to Part II (Section 10) where we focus on the topology of $\mathrm{Spc}(\mathcal{K}(G))$ and produce the announced local analysis for G elementary abelian.

Illustration. A geometric paper should include pictures and there will be many of those below. The title page shows what happens for $G = D_8$, the dihedral group of order 8, at the prime $p = 2$. Hopefully, the beauty of Figure 1 will entice the reader to proceed beyond this preamble.

At the bottom right of Figure 1, we recognize the projective support variety of D_8 consisting of two copies of $\mathbb{P}^1_k$ glued together at an $\mathbb{F}_2$-rational point. It is the spectrum of the stable module category $\mathrm{stab}(kD_8)$ and also the spectrum of $\mathrm{D_b}(kD_8)$ with its 'irrelevant' closed point punctured out. This 'puncturing' process produces more geometric pictures, displaying classical projective varieties associated to graded rings instead of their full homogeneous spectra. At the bottom left, we recognize the lattice of conjugacy classes of subgroups of D_8, with the Alexandrov topology, which is the spectrum of $\mathrm{DMack}(D_8; k)^c$ with the closed point (the trivial subgroup) punctured out for coherence. In the center of this triptych sits the spectrum of $\mathcal{K}(D_8)$ in majesty, with its closed points removed. It has three irreducible components, each of which is a $\mathbb{P}^1_k$ with multiple $\mathbb{F}_2$-rational points doubled. The components meet in some of these doubled points. This spectrum is presented in detail at the end of the paper, in Example 18.17.

The two maps in Figure 1 are the images under the contravariant functor $\mathrm{Spc}(-)$ of the tt-functors in (1.3), ignoring $\mathrm{SH}(D_8)$. The colors are chosen to indicate where each point goes in a hopefully self-explanatory way. We see that the right-hand projective support variety $\mathrm{Spc}(\mathrm{stab}(kD_8))$ embeds as an open subset of $\mathrm{Spc}(\mathcal{K}(D_8))$, meeting two of the three irreducible components. These two components are detected by the two Klein-four subgroups of D_8. The third component is detected by the announced modular fixed-points and relies on the elementary abelian Klein-four $D_8/Z(D_8)$ that appears as a *quotient*.

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1.7. *Terminology.* A 'tensor category' is an additive category with a symmetric-monoidal product additive in each variable. We say 'tt-category' for 'tensor triangulated category' and 'tt-ideal' for 'thick (triangulated) $\otimes$-ideal'. We say 'big' tt-category for a rigidly-compactly generated tt-category, as in [BF11]. We write $\mathrm{Spc}(\mathcal{K})$ for the tt-spectrum of a tt-category $\mathcal{K}$. For an object $x \in \mathcal{K}$, we write $\mathrm{open}(x) = \{\, \mathcal{P} \in \mathrm{Spc}(\mathcal{K}) \,|\, x \in \mathcal{P} \,\}$ to denote the open complement of $\mathrm{supp}(x)$.

For subgroups $H, K \leqslant G$, we write $H \leqslant_G K$ to say that H is G-conjugate to a subgroup of K, that is, $H^g \leqslant K$ for some $g \in G$. We write $\sim_G$ for G-conjugation. As always $H^g = g^{-1} H\, g$ and ${}^g H = g\, H\, g^{-1}$. We write $\mathrm{Sub}_p(G)$ for the set of p-subgroups of G and $\mathrm{Sub}_p(G)/_G$ for its G-orbits under G-conjugation. We write $N_G(H, K)$ for $\{\, g \in G \,|\, H^g \leqslant K \,\}$ and $N_G H = N_G(H, H)$ for the normalizer. For each subgroup $H \leqslant G$, its Weyl group is $G /\!\!/ H = (N_G H)/H$.

1.8. *Convention.* When a notation involves a subgroup H of an ambient group G, we drop the mention of G if no ambiguity can occur, for instance Res_H for Res^G_H. The mention of the field k is sometimes dropped, for readability.

Part I. Modular fixed-points and stratification

2. Introduction to Part I

Having sketched the broad context and the aims of the article, let us turn to the content of Part I in more detail.

Stratification. In colloquial terms, one of our main results says that the big derived category $\mathcal{T}(G)$ of permutation modules given in (1.2) is strongly controlled by its compact part $\mathcal{K}(G)$ described in (1.1):

2.1. **Theorem** (Theorem 9.11)**.** *The derived category of permutation modules* $\mathcal{T}(G)$ *is* stratified *by* $\mathrm{Spc}(\mathcal{K}(G))$ *in the sense of Barthel-Heard-Sanders* [BHS21].

Let us remind the reader of BHS-stratification. What we establish in Theorem 9.11 is an inclusion-preserving bijection between the localizing $\otimes$-ideals of $\mathcal{T}(G)$ and the subsets of the spectrum $\mathrm{Spc}(\mathcal{K}(G))$. This bijection is defined via a canonical support theory on $\mathcal{T}(G)$ that exists once we know that $\mathrm{Spc}(\mathcal{K}(G))$ is a noetherian space (Proposition 9.1). Note that Theorem 2.1 cannot be obtained via 'BIK-stratification' as in Benson-Iyengar-Krause [BIK11], since the endomorphism ring of the unit $\mathrm{End}^{\bullet}_{\mathcal{K}(G)}(\mathbb{1}) = k$ is too small. However, we shall see that [BIK11] plays an important role in our proof, albeit indirectly. An immediate consequence of stratification is the Telescope Property (Corollary 9.12):

2.2. **Corollary.** *Every smashing $\otimes$-ideal of $\mathcal{T}(G)$ is generated by its compact part.*

The key question is now to understand the spectrum $\mathrm{Spc}(\mathcal{K}(G))$. Recall from [BG23a, Theorem 5.13] that the innocent-looking category $\mathcal{K}(G)$ actually captures much of the wilderness of modular representation theory. It admits as Verdier quotient the derived category $\mathrm{D_b}(kG)$ of *all* finitely generated kG-modules. By Benson-Carlson-Rickard [BCR97], the spectrum of $\mathrm{D_b}(kG)$ is the homogeneous spectrum of the cohomology ring $\mathrm{H}^{\bullet}(G, k)$. We deduce in Proposition 3.22 that $\mathrm{Spc}(\mathcal{K}(G))$ contains an open piece V_G

$$\mathrm{Spec^h}(\mathrm{H}^{\bullet}(G, k)) \cong \mathrm{Spc}(\mathrm{D_b}(kG)) =: V_G \hookrightarrow \mathrm{Spc}(\mathcal{K}(G)) \tag{2.3}$$

that we call the *cohomological open* of G.

In good logic, the closed complement of V_G is

$$\mathrm{Spc}(\mathcal{K}(G)) \smallsetminus V_G \quad = \quad \mathrm{Supp}(\mathcal{K}_{\mathrm{ac}}(G)) \tag{2.4}$$

the support of the tt-ideal $\mathcal{K}_{\mathrm{ac}}(G) = \mathrm{Ker}(\mathcal{K}(G) \twoheadrightarrow \mathrm{D_b}(kG))$ of acyclic objects. The problem becomes to understand this closed subset $\mathrm{Supp}(\mathcal{K}_{\mathrm{ac}}(G))$. To appreciate the issue, let us say a word of closed points. Corollary 7.31 gives the complete list: There is one closed point $\mathcal{M}(H)$ of $\mathrm{Spc}(\mathcal{K}(G))$ for every conjugacy class of p-subgroups $H \leqslant G$. The cohomological open V_G only contains one closed point, for the trivial subgroup $H = 1$. All other closed points $\mathcal{M}(H)$ for $H \neq 1$ are to be found in the complement $\mathrm{Supp}(\mathcal{K}_{\mathrm{ac}}(G))$. It will turn out that $\mathrm{Spc}(\mathcal{K}(G))$ is substantially richer than the cohomological open V_G, in a way that involves p-local information about G. To understand this, we need the right notion of fixed-points.

Modular fixed-points. Let $H \leqslant G$ be a subgroup. We abbreviate by

$$G/\!\!/H := W_G(H) = N_G(H)/H \tag{2.5}$$

the Weyl group of H in G. If $H \trianglelefteqslant G$ is normal then of course $G/\!\!/H = G/H$.

For every G-set X, its H-fixed-points X^H is canonically a $(G/\!\!/H)$-set. We also have a naive fixed-points functor $M \mapsto M^H$ on kG-modules but it does not 'linearize' fixed-points of G-sets, that is, $k(X)^H$ differs from $k(X^H)$ in general. And it does not preserve the tensor product. We would prefer a *tensor*-triangular functor

$$\Psi^H\colon \mathcal{T}(G) \to \mathcal{T}(G/\!\!/H) \tag{2.6}$$

such that $\Psi^H(k(X)) = k(X^H)$ for every G-set X.

A related problem was encountered long ago for the G-equivariant stable homotopy category $\mathrm{SH}(G)$, see [LMSM86]: The naive fixed-points functor (a. k. a. the 'genuine' or 'categorical' fixed-points functor) is not compatible with taking suspension spectra, and it does not preserve the smash product. To solve both issues, topologists invented *geometric* fixed-points Φ^H. As we saw in the preamble, those geometric fixed-points functors already played an important role in tensor-triangular geometry [BS17, BGH20, PSW22] and it would be reasonable, if not very original, to try the same strategy for $\mathcal{T}(G)$. Unfortunately they do *not* give us the wanted Ψ^H of (2.6), as we explain in Remark 4.11.

In summary, we need a third notion of fixed-points functor Ψ^H, which is neither the naive one $(-)^H$, nor the 'geometric' one Φ^H imported from topology. It turns out (see Warning 5.1) that it can only exist in characteristic p when H is a p-subgroup. The good news is that this is the only restriction (see Section 5):

2.7. **Proposition.** *For every p-subgroup $H \leqslant G$ there exists a coproduct-preserving tensor-triangular functor on the big derived category of permutation modules* (1.2)

$$\Psi^H\colon \quad \mathcal{T}(G) \longrightarrow \mathcal{T}(G/\!\!/H)$$

such that $\Psi^H(k(X)) \cong k(X^H)$ for every G-set X. In particular, this functor preserves compacts and restricts to a tt-functor $\Psi^H\colon \mathcal{K}(G) \to \mathcal{K}(G/\!\!/H)$ on (1.1).

We call the Ψ^H the *modular H-fixed-points functors.* These functors already exist at the level of additive categories $\mathrm{perm}(G;k)^\natural \to \mathrm{perm}(G/\!\!/H;k)^\natural$, where they agree with the classical Brauer quotient, although our construction is quite different. See Remark 5.8. These Ψ^H also recover motivic functors considered by Bachmann in [Bac16, Corollary 5.48]. Equipped with those Ψ^H, let us return to $\mathrm{Spc}(\mathcal{K}(G))$.

The spectrum. Each tt-functor Ψ^H induces a continuous map on spectra

$$\psi^H := \mathrm{Spc}(\Psi^H): \quad \mathrm{Spc}(\mathcal{K}(G/\!\!/H)) \longrightarrow \mathrm{Spc}(\mathcal{K}(G)). \tag{2.8}$$

In particular $\mathrm{Spc}(\mathcal{K}(G))$ receives via this map ψ^H the cohomological open $V_{G/\!\!/H}$ of the Weyl group of H:

$$V_{G/\!\!/H} = \mathrm{Spc}(\mathrm{D_b}(k(G/\!\!/H))) \hookrightarrow \mathrm{Spc}(\mathcal{K}(G/\!\!/H)) \xrightarrow{\psi^H} \mathrm{Spc}(\mathcal{K}(G)). \tag{2.9}$$

Using this, we can describe the *set* underlying $\mathrm{Spc}(\mathcal{K}(G))$ in Theorem 7.16:

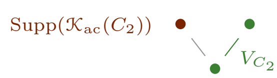
2.10. **Theorem.** *Every point of* $\mathrm{Spc}(\mathcal{K}(G))$ *is the image* $\psi^H(\mathfrak{p})$ *of a point* $\mathfrak{p} \in V_{G/\!\!/H}$ *for some* p*-subgroup* $H \leqslant G$*, in a unique way up to* G*-conjugation, i.e. we have* $\psi^H(\mathfrak{p}) = \psi^{H'}(\mathfrak{p}')$ *if and only if there exists* $g \in G$ *such that* $H^g = H'$ *and* $\mathfrak{p}^g = \mathfrak{p}'$.

In this description, the trivial subgroup $H = 1$ contributes the cohomological open V_G (since $\Psi^1 = \mathrm{Id}$). Its closed complement $\mathrm{Supp}(\mathcal{K}_{\mathrm{ac}}(G))$, introduced in (2.4), is covered by images of the modular fixed-points maps (2.9), for H running through all non-trivial p-subgroups of G. The main ingredient in proving Theorem 2.10 is our Conservativity Theorem 6.12 on the associated big categories:

2.11. **Theorem.** *The family of functors* $\{\mathcal{T}(G) \xrightarrow{\Psi^H} \mathcal{T}(G/\!\!/H) \twoheadrightarrow \mathrm{K\,Inj}(k(G/\!\!/H))\}_H$, *indexed by the (conjugacy classes of)* p*-subgroups* $H \leqslant G$*, is conservative.*

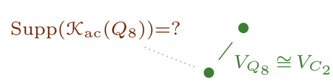
This determines the set $\mathrm{Spc}(\mathcal{K}(G))$. The topology of $\mathrm{Spc}(\mathcal{K}(G))$ involves new characters and we postpone its discussion to Part II.

Measuring progress by examples. Before the present work, we only knew the case of cyclic group C_p of order $p = 2$, where $\mathrm{Spc}(\mathcal{K}(C_2))$ is a 3-point space ([1])

(2.12) 

This was the starting point of our study of real Artin-Tate motives [BG22b, Theorem 3.14]. It appears independently in Dugger-Hazel-May [DHM24, Theorem 5.4].

The present paper gives a description of $\mathrm{Spc}(\mathcal{K}(G))$ for arbitrary finite groups G. We gather several examples in Section 8 to illustrate the progress made since (2.12), and also for later use in [BG23b]. Let us highlight the case of the quaternion group $G = Q_8$ (Example 8.12). By Quillen, we know that the cohomological open V_{Q_8} is the same as for its center $Z(Q_8) = C_2$, that is, the 2-point Sierpiński space displayed in green on the right-hand side of (2.12), and again below:

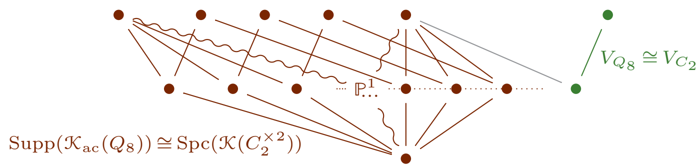


If intuition was solely based on (2.12) one could believe that $\mathrm{Spc}(\mathcal{K}(G))$ is just V_G with some discrete decoration for the acyclics, like the single (brown) point on the left-hand side of (2.12). The quaternion group offers a stark rebuttal.

[1] A line indicates specialization: The higher point is in the closure of the lower one.

Indeed, the spectrum $\mathrm{Spc}(\mathcal{K}(Q_8))$ is the following space:

(2.13)



Its support of acyclics (in brown) is actually way more complicated than the cohomological open itself: It has Krull dimension two and contains a copy of the projective line $\mathbb{P}^1_k$. In fact, the map ψ^{C_2} given by modular fixed-points identifies the closed piece $\mathrm{Supp}(\mathcal{K}_{\mathrm{ac}}(Q_8))$ with the whole spectrum for Q_8/C_2, which is a Klein-four. We discuss the latter in Example 8.10 where we also explain the meaning of $\mathbb{P}^1_{\because}$ and the undulated lines in (2.13).

3. Recollections and Koszul objects

3.1. *Recollection.* We refer to [Bal10b] for elements of tensor-triangular geometry. Recall simply that the *spectrum* of an essentially small tt-category $\mathcal{K}$ is $\mathrm{Spc}(\mathcal{K}) = \{\,\mathcal{P} \subsetneq \mathcal{K} \,|\, \mathcal{P} \text{ is a prime tt-ideal}\,\}$. For every object $x \in \mathcal{K}$, its support is $\mathrm{supp}(x) := \{\,\mathcal{P} \in \mathrm{Spc}(\mathcal{K}) \,|\, x \notin \mathcal{P}\,\}$. These form a basis of closed subsets for the topology.

3.2. *Recollection.* (Here k can be a commutative ring.) Recall our reference [BG21] for details on permutation modules. Linearizing a G-set X, we let $k(X)$ be the free k-module with basis X and G-action k-linearly extending the G-action on X. A *permutation kG-module* is a kG-module isomorphic to one of the form $k(X)$. These modules form an additive subcategory $\mathrm{Perm}(G;k)$ of $\mathrm{Mod}(kG)$, with all kG-linear maps. We write $\mathrm{perm}(G;k)$ for the full subcategory of finitely generated permutation kG-modules and $\mathrm{perm}(G;k)^\natural$ for its idempotent-completion.

We tensor kG-modules in the usual way, over k with diagonal G-action. The linearization functor $k(-)\colon G\text{-Sets} \longrightarrow \mathrm{Perm}(G;k)$ turns the cartesian product of G-sets into this tensor product. For every finite X, the module $k(X)$ is self-dual.

We consider the idempotent-completion $(-)^\natural$ of the homotopy category of bounded complexes in the additive category $\mathrm{perm}(G;k)$

$$\mathcal{K}(G) = \mathcal{K}(G;k) := \mathrm{K_b}(\mathrm{perm}(G;k))^\natural \cong \mathrm{K_b}(\mathrm{perm}(G;k)^\natural).$$

As $\mathrm{perm}(G;k)$ is an essentially small tensor-additive category, $\mathcal{K}(G)$ becomes an essentially small tensor triangulated category. As $\mathrm{perm}(G;k)$ is rigid so is $\mathcal{K}(G)$, with degreewise duals. Its tensor-unit $\mathbb{1} = k$ is the trivial kG-module $k = k(G/G)$.

The 'big' *derived category of permutation kG-modules* [BG21, Definition 3.6] is

$$\mathrm{DPerm}(G;k) = \mathrm{K}(\mathrm{Perm}(G;k))\big[\{G\text{-quasi-isos}\}^{-1}\big],$$

where a G-quasi-isomorphism $f\colon P \to Q$ is a morphism of complexes such that the induced morphism on H-fixed points f^H is a quasi-isomorphism for every subgroup $H \leqslant G$. It is also the localizing subcategory of $\mathrm{K}(\mathrm{Perm}(G;k))$ generated by $\mathcal{K}(G)$, and it follows that $\mathcal{K}(G) = \mathrm{DPerm}(G;k)^c$.

3.3. *Example.* For G trivial, the category $\mathcal{K}(1;k) = \mathrm{D_{perf}}(k)$ is that of perfect complexes over k (any ring) and $\mathrm{DPerm}(1;k)$ is the derived category of k.

3.4. *Remark.* The tt-category $\mathcal{K}(G)$ depends functorially on G and k. It is contravariant in the group. Namely if $\alpha\colon G \to G'$ is a homomorphism then restriction along α yields a tt-functor $\alpha^*\colon \mathcal{K}(G') \to \mathcal{K}(G)$. When α is the inclusion of a subgroup $G \leqslant G'$, we recover usual restriction

$$\mathrm{Res}^{G'}_{G} : \mathcal{K}(G') \to \mathcal{K}(G).$$

When α is a quotient $G \twoheadrightarrow G' = G/N$ for $N \trianglelefteq G$, we get *inflation*, denoted here ([2])

$$\mathrm{Infl}^{G/N}_{G} : \mathcal{K}(G/N; k) \to \mathcal{K}(G).$$

The covariance of $\mathcal{K}(G)$ in k is simply obtained by extension-of-scalars. All these functors are the 'compact parts' of similarly defined functors on DPerm.

Let us say a word of kG-linear morphisms between permutation modules.

3.5. *Recollection.* Let $H, K \leqslant G$ be subgroups. Then $\mathrm{Hom}_{kG}(k(G/H), k(G/K))$ admits a k-basis $\{f_g\}_{[g]}$ indexed by classes $[g] \in H\backslash G/K$. Namely, choosing a representative in each class $[g] \in H\backslash G/K$, one defines

$$f_g\colon \quad k(G/H) \underset{\eta}{\rightarrowtail} k(G/L) \underset{c_g}{\overset{\sim}{\to}} k(G/L^g) \underset{\epsilon}{\twoheadrightarrow} k(G/K) \tag{3.6}$$

where we set $L := H \cap {}^g\!K$, where η and ϵ are the usual maps using that $L \leqslant H$ and $L^g \leqslant K$ (thus η maps $[e]_H$ to $\sum_{\gamma \in H/L} \gamma$ and ϵ extends k-linearly the projection $G/L^g \twoheadrightarrow G/K$), and finally where the middle isomorphism c_g is

$$\begin{aligned} c_g\colon \qquad k(G/L) &\longrightarrow k(G/L^g) \\ [x]_L &\longmapsto [x \cdot g]_{L^g}\,. \end{aligned} \tag{3.7}$$

This is a standard computation, using the adjunction $\mathrm{Ind}^G_H \dashv \mathrm{Res}^G_H$ and the Mackey formula for $\mathrm{Res}^G_H(k(G/K)) \simeq \oplus_{[g]\in H\backslash G/K}\, k(H/H \cap {}^g\!K)$.

We can now begin our analysis of the spectrum of the tt-category $\mathcal{K}(G)$.

3.8. **Proposition.** *Let $G \leqslant G'$ be a subgroup of index invertible in k. Then the map $\mathrm{Spc}(\mathrm{Res}^{G'}_G)\colon \mathrm{Spc}(\mathcal{K}(G)) \to \mathrm{Spc}(\mathcal{K}(G'))$ is surjective.*

Proof. This is a standard argument. For a subgroup $G \leqslant G'$, the restriction functor $\mathrm{Res}^{G'}_G$ has a two-sided adjoint $\mathrm{Ind}^{G'}_G : \mathcal{K}(G) \to \mathcal{K}(G')$ such that the composite of the unit and counit of these adjunctions $\mathrm{Id} \to \mathrm{Ind}\,\mathrm{Res} \to \mathrm{Id}$ is multiplication by the index. If the latter is invertible, it follows that $\mathrm{Res}^{G'}_G$ is a faithful functor. The result now follows from [Bal18, Theorem 1.3]. □

3.9. **Corollary.** *Let k be a field of characteristic zero and G be a finite group. Then $\mathrm{Spc}(\mathcal{K}(G)) = *$ is a singleton.*

Proof. Direct from Proposition 3.8 since $\mathrm{Spc}(\mathcal{K}(1; k)) = \mathrm{Spc}(\mathrm{D_{perf}}(k)) = *$. □

3.10. *Remark.* In view of these reductions, the fun happens with coefficients in a field k of positive characteristic p dividing the order of G.

Let us now identify what the derived category tells us about $\mathrm{Spc}(\mathcal{K}(G))$.

---

[2] We avoid the traditional $\mathrm{Infl}^{G}_{G/N}$ notation which is not coherent with the restriction notation.

3.11. *Notation.* We can define a tt-ideal of $\mathcal{K}(G)=\mathrm{K_b}(\mathrm{perm}(G;k)^\natural)$ by

$$\mathcal{K}_{\mathrm{ac}}(G):=\big\{\,x\in\mathcal{K}(G)\,\big|\,x \text{ is } \textit{acyclic} \text{ as a complex of } kG\text{-modules}\,\big\}.$$

It is the kernel of the tt-functor $\Upsilon_G\colon \mathcal{K}(G)\to \mathrm{D_b}(kG):=\mathrm{D_b}(\mathrm{mod}(kG))$ induced by the inclusion $\mathrm{perm}(G;k)^\natural\hookrightarrow \mathrm{mod}(kG)$ of the additive category of p-permutation kG-modules inside the abelian category of all finitely generated kG-modules.

3.12. *Recollection.* The canonical functor induced by Υ_G on the Verdier quotient

$$\frac{\mathcal{K}(G)}{\mathcal{K}_{\mathrm{ac}}(G)}\longrightarrow \mathrm{D_b}(kG)$$

is an equivalence of tt-categories. This is [BG23a, Theorem 5.13]. In other words,

$$\Upsilon_G\colon \mathcal{K}(G)\twoheadrightarrow \mathrm{D_b}(kG) \tag{3.13}$$

realizes the derived category of finitely generated kG-modules as a localization of our $\mathcal{K}(G)$, away from the Thomason subset $\mathrm{Supp}(\mathcal{K}_{\mathrm{ac}}(G))$ of (2.4). Following Neeman-Thomason, the above localization (3.13) is the compact part of a finite localization of the corresponding 'big' tt-categories $\mathcal{T}(G)\twoheadrightarrow \mathrm{K\,Inj}(kG)$, the homotopy category of complexes of injectives. See [BG22a, Remark 4.21]. We return to this localization of big categories in Recollection 6.7.

We want to better understand the tt-ideal of acyclics $\mathcal{K}_{\mathrm{ac}}(G)$ and in particular show that it has closed support.

3.14. *Construction.* Let $H\leqslant G$ be a subgroup. We define a complex of kG-modules by tensor-induction (recall Convention 1.8)

$$\mathrm{kos}(H)=\mathrm{kos}_G(H):={}^{\otimes}\mathrm{Ind}_H^G(0\to k\xrightarrow{1}k\to 0)$$

where $0\to k\xrightarrow{1}k\to 0$ is non-trivial in homological degrees 1 and 0; hence $\mathrm{kos}(H)$ lives in degrees between $[G\!:\!H]$ and 0. Since H acts trivially on k, the action of G on $\mathrm{kos}(H)$ is the action of G by permutation of the factors $\otimes_{G/H}(0\to k\xrightarrow{1}k\to 0)$. This can be described as a Koszul complex. For every $0\leqslant d\leqslant [G:H]$, the complex $\mathrm{kos}(H)$ in degree d is the k-vector space $\Lambda^d(k(G/H))$ of dimension $\binom{[G:H]}{d}$. If we choose a numbering of the elements of $G/H=\{v_1,\ldots,v_{[G:H]}\}$ then $\mathrm{kos}(H)_d$ has a k-basis $\big\{\,v_{i_1}\wedge\cdots\wedge v_{i_d}\,\big|\,1\leqslant i_1<\cdots<i_d\leqslant [G:H]\,\big\}$. The canonical diagonal action of G permutes this basis but introduces signs when re-ordering the v_i's so that indices increase. When $p=2$ these signs are irrelevant. When $p>2$, every such 'sign-permutation' kG-module is isomorphic to an actual permutation kG-module (by changing some signs in the basis, see [BG23a, Lemma 3.8]).

3.15. **Proposition.** *Let $H\leqslant G$ be a subgroup. Then $\mathrm{kos}_G(H)$ is an acyclic complex of finitely generated permutation kG-modules which is concentrated in degrees between $[G\!:\!H]$ and 0 and such that it is k in degree 0 and $k(G/H)$ in degree 1.*

Proof. See Construction 3.14. Exactness is obvious since the underlying complex of k-modules is $(0\to k\to k\to 0)^{\otimes[G:H]}$. The values in degrees $0,1$ are immediate. □

3.16. *Example.* We have $\mathrm{kos}_G(G)=0$ in $\mathcal{K}(G)$. The complex $\mathrm{kos}_G(1)$ is an acyclic complex of permutation modules that was important in [BG23a, § 3]:

$$\mathrm{kos}_G(1)=\ \cdots 0\longrightarrow P_n\longrightarrow\cdots\longrightarrow P_2\longrightarrow kG\longrightarrow k\longrightarrow 0\cdots$$

3.17. **Lemma.** *Let $H \trianglelefteq G$ be a normal subgroup and $H \leqslant K \leqslant G$. Then* $\mathrm{kos}_G(K) \cong \mathrm{Infl}_G^{G/H}(\mathrm{kos}_{G/H}(K/H))$. *In particular,* $\mathrm{kos}_G(H) \cong \mathrm{Infl}_G^{G/H}(\mathrm{kos}_{G/H}(1))$.

Proof. The construction of $\mathrm{kos}_G(K) = \otimes_{G/K}(0 \to k \xrightarrow{1} k \to 0)$ depends only on the G-set G/K which is inflated from the G/H-set $(G/H)/(K/H)$. □

In fact, $\mathrm{kos}_G(H)$ is not only exact. It is split-exact on H. More generally:

3.18. **Lemma.** *For every subgroups $H, K \leqslant G$ and every choice of representatives in $K\backslash G/H$, we have a non-canonical isomorphism of complexes of kK-modules*

$$\mathrm{Res}^G_K(\mathrm{kos}_G(H)) \simeq \bigotimes_{[g]\in K\backslash G/H} \mathrm{kos}_K(K \cap {}^gH).$$

In particular, if $K \leqslant_G H$, we have $\mathrm{Res}^G_K(\mathrm{kos}_G(H)) = 0$ in $\mathcal{K}(K)$.

Proof. By the Mackey formula for tensor-induction, we have in $\mathrm{Ch_b}(\mathrm{perm}(K;k))$

$$\mathrm{Res}^G_K(\mathrm{kos}_G(H)) \simeq \bigotimes_{[g]\in K\backslash G/H} {}^{\otimes}\mathrm{Ind}^K_{K\cap {}^gH}\big({}^g\mathrm{Res}^H_{K\cap {}^gH}(0 \to k \xrightarrow{1} k \to 0)\big).$$

The result follows since $\mathrm{Res}(0 \to k \xrightarrow{1} k \to 0) = (0 \to k \xrightarrow{1} k \to 0)$. If $K \leqslant_G H$, the factor $\mathrm{kos}_K(K)$ appears in the tensor product and $\mathrm{kos}_K(K) = 0$ in $\mathcal{K}(K)$. □

We record a general technical argument that we shall use a couple of times.

3.19. **Lemma.** *Let $\mathcal{A}$ be a rigid tensor category and $s = (\cdots s_2 \to s_1 \to s_0 \to 0 \cdots)$ a complex concentrated in non-negative degrees. Let $x \in \mathrm{Ch_b}(\mathcal{A})$ be a bounded complex such that $s_1 \otimes x = 0$ in $\mathrm{K_b}(\mathcal{A})$. Then there exists $n \gg 0$ such that $s_0^{\otimes n} \otimes x$ belongs to the smallest thick subcategory $\langle s\rangle'$ of $\mathrm{K}(\mathcal{A})$ that contains s and is closed under tensoring with $\mathrm{K_b}(\mathcal{A}) \cup \{s\}$ in $\mathrm{K}(\mathcal{A})$. In particular, if $s \in \mathrm{K_b}(\mathcal{A})$ is itself bounded, then $s_0^{\otimes n} \otimes x$ belongs to the tt-ideal $\langle s\rangle$ generated by s in $\mathrm{K_b}(\mathcal{A})$.*

Proof. Let $u := s_{\geq 1}[-1]$ be the truncation of s such that $s = \mathrm{cone}(d : u \to s_0)$. Similarly we have $u = \mathrm{cone}(u_{\geq 1}[-1] \to s_1)$. Note that $u_{\geq 1}$ is concentrated in positive degrees. Since $x \otimes s_1 = 0$ we have $u \otimes x \cong u_{\geq 1} \otimes x$ in $\mathrm{K}(\mathcal{A})$ and thus

$$u^{\otimes n} \otimes x \cong (u_{\geq 1})^{\otimes n} \otimes x$$

for all $n \geq 0$. For n large enough there are no non-zero maps of complexes from $(u_{\geq 1})^{\otimes n} \otimes x$ to $s_0^{\otimes n} \otimes x$, simply because the former 'moves' further and further away to the left and x is bounded. So $d^{\otimes n} \otimes x \colon u^{\otimes n} \otimes x \longrightarrow s_0^{\otimes n} \otimes x$ is zero in $\mathcal{K}(\mathcal{A})$.

Let $\mathcal{L}$ be the tt-subcategory of $\mathrm{K}(\mathcal{A})$ generated by $\mathrm{K_b}(\mathcal{A}) \cup \{s\}$; then $\langle s\rangle'$ is a tt-ideal in $\mathcal{L}$, and similarly we write $\langle \mathrm{cone}(d^{\otimes n})\rangle'$ for the tt-ideal in $\mathcal{L}$ generated by $\mathrm{cone}(d^{\otimes n})$. By the argument above, we have $s_0^{\otimes n} \otimes x \in \langle \mathrm{cone}(d^{\otimes n})\rangle' \subseteq \langle s\rangle'$. □

3.20. **Corollary.** *Let $\mathcal{A}$ be a rigid tensor category and $\mathcal{J} \subseteq \mathrm{K_b}(\mathcal{A})$ a tt-ideal. Let $s \in \mathcal{J}$ be a (bounded) complex concentrated in non-negative degrees such that*

(1) $\mathrm{supp}(s_0) \supseteq \mathrm{supp}(\mathcal{J})$ *in* $\mathrm{Spc}(\mathrm{K_b}(\mathcal{A}))$ *(for instance if $s_0 = \mathbb{1}_{\mathcal{A}}$), and*

(2) $\mathrm{supp}(s_1) \cap \mathrm{supp}(\mathcal{J}) = \varnothing$, *meaning that $s_1 \otimes x = 0$ in $\mathrm{K_b}(\mathcal{A})$ for all $x \in \mathcal{J}$.*

Then s generates $\mathcal{J}$ as a tt-ideal in $\mathrm{K_b}(\mathcal{A})$, *that is,* $\mathrm{supp}(\mathcal{J}) = \mathrm{supp}(s)$ *in* $\mathrm{Spc}(\mathrm{K_b}(\mathcal{A}))$.

Proof. Let $x \in \mathcal{J}$. By (2), Lemma 3.19 gives us $s_0^{\otimes n} \otimes x \in \langle s\rangle$ for $n \gg 0$. Hence $\mathrm{supp}(s_0) \cap \mathrm{supp}(x) \subseteq \mathrm{supp}(s)$. By (1) we have $\mathrm{supp}(x) \subseteq \mathrm{supp}(s_0)$. Therefore $\mathrm{supp}(x) = \mathrm{supp}(s_0) \cap \mathrm{supp}(x) \subseteq \mathrm{supp}(s)$. In short $x \in \langle s\rangle$ for all $x \in \mathcal{J}$. □

We apply this to the object $s = \mathrm{kos}_G(H)$ of Construction 3.14.

3.21. **Proposition.** *For every subgroup $H \leqslant G$, the object $\mathrm{kos}_G(H)$ generates the tt-ideal $\mathrm{Ker}(\mathrm{Res}^G_H)$ of $\mathcal{K}(G)$.*

Proof. We apply Corollary 3.20 to $\mathcal{J} = \mathrm{Ker}(\mathrm{Res}^G_H)$ and $s = \mathrm{kos}_G(H)$. We have $s \in \mathcal{J}$ by Lemma 3.18. Conditions (1) and (2) hold since $s_0 = k$ and $s_1 = k(G/H)$ by Proposition 3.15 and Frobenius gives $s_1 \otimes \mathcal{J} = k(G/H) \otimes \mathcal{J} = \mathrm{Ind}^G_H \mathrm{Res}^G_H(\mathcal{J}) = 0$. □

We can apply the above discussion to $H = 1$ and $\mathcal{J} = \mathrm{Ker}(\mathrm{Res}^G_1) = \mathcal{K}_{\mathrm{ac}}(G)$.

3.22. **Proposition.** *The tt-functor $\Upsilon_G \colon \mathcal{K}(G) \twoheadrightarrow \mathrm{D_b}(kG)$ induces an* open *inclusion $\upsilon_G \colon V_G \hookrightarrow \mathrm{Spc}(\mathcal{K}(G))$ where $V_G = \mathrm{Spc}(\mathrm{D_b}(kG)) \cong \mathrm{Spec^h}(\mathrm{H}^\bullet(G,k))$. The closed complement of V_G is the support of $\mathrm{kos}_G(1) = {}^{\otimes}\mathrm{Ind}^G_1(0 \to k \xrightarrow{1} k \to 0)$.*

Proof. The homeomorphism $\mathrm{Spc}(\mathrm{D_b}(kG)) \cong \mathrm{Spec^h}(\mathrm{H}^\bullet(G,k))$ follows from the tt-classification [BCR97]; see [Bal10b, Theorem 57]. By Recollection 3.12, the map $\upsilon_G := \mathrm{Spc}(\Upsilon_G)$ is a homeomorphism onto its image, and the complement of this image is $\mathrm{supp}(\mathcal{K}_{\mathrm{ac}}(G)) = \mathrm{supp}(\mathrm{kos}_G(1))$, by Proposition 3.21 applied to $H = 1$. In particular, $\mathrm{supp}(\mathcal{K}_{\mathrm{ac}}(G))$ is a closed subset, not just a Thomason. □

3.23. *Remark.* The notation for the so-called *cohomological open* V_G has been chosen to evoke the classical *projective support variety* $\mathcal{V}_G(k) = \mathrm{Proj}(\mathrm{H}^\bullet(G,k)) \cong \mathrm{Spc}(\mathrm{stmod}(kG))$, which consists of V_G without its unique closed point, $\mathrm{H}^+(G;k)$.

We can also describe the kernel of restriction for classes of subgroups.

3.24. **Corollary.** *For every collection $\mathcal{H}$ of subgroups of G, we have an equality of tt-ideals in $\mathcal{K}(G)$*

$$\bigcap_{H \in \mathcal{H}} \mathrm{Ker}(\mathrm{Res}^G_H) = \Big\langle \bigotimes_{H \in \mathcal{H}} \mathrm{kos}_G(H) \Big\rangle.$$

Proof. This is direct from Proposition 3.21 and the general fact that $\langle x \rangle \cap \langle y \rangle = \langle x \otimes y \rangle$. (In the case of $\mathcal{H} = \varnothing$, the intersection is $\mathcal{K}(G)$ and the tensor is $\mathbb{1}$.) □

4. Restriction, induction and geometric fixed-points

In the previous section, we saw how much of $\mathrm{Spc}(\mathcal{K}(G))$ comes from $\mathrm{D_b}(kG)$. We now want to discuss how much is controlled by restriction to subgroups, to see how far the 'standard' strategy of [BS17] gets us.

4.1. *Remark.* The tt-categories $\mathcal{K}(G)$ and $\mathrm{D_b}(kG)$, as well as the Weyl groups $G/\!\!/H$ are functorial in G. To keep track of this, we adopt the following notational system.

Let $\alpha \colon G \to G'$ be a group homomorphism. We write $\alpha^* \colon \mathcal{K}(G') \to \mathcal{K}(G)$ for restriction along α, and similarly for $\alpha^* \colon \mathrm{D_b}(kG') \to \mathrm{D_b}(kG)$. When applying the contravariant $\mathrm{Spc}(-)$, we simply denote $\mathrm{Spc}(\alpha^*)$ by $\alpha_* \colon \mathrm{Spc}(\mathcal{K}(G)) \to \mathrm{Spc}(\mathcal{K}(G'))$ and similarly for $\alpha_* \colon V_G \to V_{G'}$ on the spectrum of derived categories.

As announced, Weyl groups $G/\!\!/H = (N_G H)/H$ of subgroups $H \leqslant G$ will play a role. Since $\alpha(N_G H) \leqslant N_{G'}(\alpha(H))$, every homomorphism $\alpha \colon G \to G'$ induces a homomorphism $\bar\alpha \colon G/\!\!/H \to G'/\!\!/\alpha(H)$. Combining with the above, these homomorphisms $\bar\alpha$ define functors $\bar\alpha^*$ and maps $\bar\alpha_*$. For instance, $\bar\alpha_* \colon V_{G/\!\!/H} \to V_{G'/\!\!/\alpha(H)}$ is the continuous map induced on $\mathrm{Spc}(\mathrm{D_b}(k(-)))$ by $\bar\alpha \colon G/\!\!/H \to G'/\!\!/\alpha(H)$.

Following tradition, we have special names when α is an inclusion, a quotient or a conjugation. For the latter, we choose the lightest notation possible.

(a) For *conjugation*, for a subgroup $G \leqslant G'$ and an element $x \in G'$, the isomorphism $c_x\colon G \xrightarrow{\sim} G^x$ induces a tt-functor $c_x^*\colon \mathcal{K}(G^x) \xrightarrow{\sim} \mathcal{K}(G)$ and a homeomorphism

$$(-)^x := (c_x)_* = \mathrm{Spc}(c_x^*)\colon \qquad \begin{array}{ccc} \mathrm{Spc}(\mathcal{K}(G)) & \xrightarrow{\sim} & \mathrm{Spc}(\mathcal{K}(G^x)) \\ \mathcal{P} & \longmapsto & \mathcal{P}^x. \end{array}$$

Note that if $x = g \in G$ belongs to G itself, the functor $c_g^*\colon \mathcal{K}(G) \to \mathcal{K}(G)$ is isomorphic to the identity and therefore we get the useful fact that

$$g \in G \qquad \Longrightarrow \qquad \mathcal{P}^g = \mathcal{P} \qquad \text{for all } \mathcal{P} \in \mathrm{Spc}(\mathcal{K}(G)). \tag{4.2}$$

Similarly we have a conjugation homeomorphism $\mathfrak{p} \mapsto \mathfrak{p}^x$ on the cohomological opens $V_G \xrightarrow{\sim} V_{G^x}$, which is the identity if $x \in G$. When $H \leqslant G$ is a further subgroup then conjugation yields homeomorphisms $V_{G/\!\!/H} \xrightarrow{\sim} V_{G^x/\!\!/H^x}$ still denoted $\mathcal{P} \mapsto \mathcal{P}^x$. Again, if $x = g \in N_G H$, so $[g]_H$ defines an element in $G/\!\!/H$, the equivalence $(c_g)_*\colon \mathrm{D_b}(G/\!\!/H) \xrightarrow{\sim} \mathrm{D_b}(G/\!\!/H)$ is isomorphic to the identity. Thus

$$g \in N_G(H) \qquad \Longrightarrow \qquad \mathfrak{p}^g = \mathfrak{p} \qquad \text{for all } \mathfrak{p} \in V_{G/\!\!/H}. \tag{4.3}$$

(b) For *restriction*, take α the inclusion $K \hookrightarrow G$ of a subgroup. We write

$$\rho_K = \rho_K^G := \mathrm{Spc}(\mathrm{Res}^G_K)\colon \;\; \mathrm{Spc}(\mathcal{K}(K)) \to \mathrm{Spc}(\mathcal{K}(G)) \tag{4.4}$$

and similarly for derived categories. When $H \leqslant K$ is a subgroup, we write $\bar\rho_K\colon V_{K/\!\!/H} \to V_{G/\!\!/H}$ for the map induced by restriction along $K/\!\!/H \hookrightarrow G/\!\!/H$. Beware that ρ_K is not necessarily injective, already on $V_K \to V_G$, as 'fusion' phenomena can happen: If $g \in G$ normalizes K, then $\mathcal{Q}$ and $\mathcal{Q}^g$ in V_K have the same image in V_G by (4.2) but are in general different in V_K.

(c) For *inflation*, let $N \trianglelefteq G$ be a normal subgroup and let $\alpha = \mathrm{proj}\colon G \twoheadrightarrow G/N$ be the quotient homomorphism. We write

$$\pi^{G/N} = \pi_G^{G/N} := \mathrm{Spc}(\mathrm{Infl}_G^{G/N})\colon \;\; \mathrm{Spc}(\mathcal{K}(G)) \to \mathrm{Spc}(\mathcal{K}(G/N)) \tag{4.5}$$

and similarly for derived categories. For $H \leqslant G$ a subgroup, we write $\bar\pi_G^{G/N}\colon V_{G/\!\!/H} \to V_{(G/N)/\!\!/(HN/N)}$ for the map induced by $\overline{\mathrm{proj}}\colon G/\!\!/H \to (G/N)/\!\!/(HN/N)$. (Note that this homomorphism is not always surjective, *e.g.* with $G = D_8$ and $N \simeq C_2^{\times 2}$.)

4.6. *Recollection.* One verifies that the $\mathrm{Res}^G_H \dashv \mathrm{Ind}^G_H$ adjunction is monadic, see for instance [Bal16, § 4], and that the associated monad $A_H \otimes -$ is separable, where $A_H := k(G/H) = \mathrm{Ind}^G_H k \in \mathrm{perm}(G;k)$. The ring structure on A_H is given by the usual unit $\eta\colon k \to k(G/H)$, mapping 1 to $\sum_{\gamma\in G/H}\gamma$, and the multiplication $\mu\colon A_H \otimes A_H \to A_H$ that is characterized by $\mu(\gamma\otimes\gamma) = \gamma$ and $\mu(\gamma\otimes\gamma') = 0$ for all $\gamma \neq \gamma'$ in G/H. The ring A_H is separable and commutative. The tt-category $\mathrm{Mod}(A_H) = \mathrm{Mod}_{\mathcal{K}(G)}(A_H)$ of A_H-modules in $\mathcal{K}(G)$ identifies with $\mathcal{K}(H)$, in such a way that extension-of-scalars to A_H (*i.e.* along η) coincides with restriction Res^G_H. Similarly, extension-of-scalars along the isomorphism $c_{g^{-1}}\colon A_{H^g} \xrightarrow{\sim} A_H$, being an equivalence, is the inverse of its adjoint, that is $((c_{g^{-1}})^*)^{-1} = c_g^*$, hence is the conjugation tt-functor $c_g^*\colon \mathcal{K}(H^g) \xrightarrow{\sim} \mathcal{K}(H)$ of Remark 4.1.

4.7. **Proposition.** *The continuous map* $\rho_H\colon \mathrm{Spc}(\mathcal{K}(H)) \to \mathrm{Spc}(\mathcal{K}(G))$ *of* (4.4) *is a closed map and for every* $y \in \mathcal{K}(H)$*, we have* $\rho_H(\mathrm{supp}(y)) = \mathrm{supp}(\mathrm{Ind}^G_H(y))$

in $\mathrm{Spc}(\mathcal{K}(G))$*. In particular,* $\mathrm{Im}(\rho_H)=\mathrm{supp}(k(G/H))$*. Moreover, there is a coequalizer of topological spaces (independent of the choices of representatives* g*)*

$$\coprod_{[g]\in H\backslash G/H}\mathrm{Spc}(\mathcal{K}(H\cap {}^gH)) \ \rightrightarrows\ \mathrm{Spc}(\mathcal{K}(H)) \xrightarrow{\rho_H} \mathrm{supp}(k(G/H))$$

where the two left horizontal maps are, on the $[g]$*-component, induced by the restriction functor and by conjugation by* g *followed by restriction, respectively.*

Proof. We invoke [Bal16, Theorem 3.19]. In particular, we have a coequalizer

$$\mathrm{Spc}(\mathrm{Mod}(A_H\otimes A_H))\rightrightarrows \mathrm{Spc}(\mathrm{Mod}(A_H))\to \mathrm{supp}(A_H) \tag{4.8}$$

where the two left horizontal maps are induced by the canonical ring morphisms $A_H\otimes\eta$ and $\eta\otimes A_H\colon A_H\to A_H\otimes A_H$. For any choice of representatives $[g]\in H\backslash G/H$ the Mackey isomorphism

$$\bigoplus_{[g]\in H\backslash G/H} A_{H\cap {}^gH} \xrightarrow{\sim} A_H\otimes A_H$$

maps $[x]_{H\cap {}^gH}$ to $[x]_H\otimes[x\cdot g]_H$. We can then plug this identification in (4.8). The second homomorphism $\eta\otimes A_H$ followed by the projection onto the factor indexed by $[g]$ becomes the composite $A_H\xrightarrow{c_{g^{-1}}} A_{{}^gH}\overset{\eta}{\rightarrowtail} A_{H\cap {}^gH}$. See Recollection 4.6. ☐

4.9. **Corollary.** *For* $\mathcal{P},\mathcal{P}'\in\mathrm{Spc}(\mathcal{K}(H))$ *we have* $\rho_H(\mathcal{P})=\rho_H(\mathcal{P}')$ *in* $\mathrm{Spc}(\mathcal{K}(G))$ *if and only if there exists* $g\in G$ *and* $\mathcal{Q}\in\mathrm{Spc}(\mathcal{K}(H\cap {}^gH))$ *such that*

$$\mathcal{P}=\rho^H_{H\cap {}^gH}(\mathcal{Q}) \qquad \textit{and} \qquad \mathcal{P}'=\big(\rho^{{}^gH}_{H\cap {}^gH}(\mathcal{Q})\big)^g$$

using Remark 4.1 for the notation $(-)^g\colon \mathrm{Spc}(\mathcal{K}({}^gH))\xrightarrow{\sim}\mathrm{Spc}(\mathcal{K}(H))$*.*

Proof. This is [Bal16, Corollary 3.12], which implies the set-theoretic part of the coequalizer of Proposition 4.7. ☐

We single out a particular case.

4.10. **Corollary.** *If* $H\leqslant Z(G)$ *is central in* G *(for example, if* G *is abelian) then restriction induces a closed immersion* $\rho_H\colon \mathrm{Spc}(\mathcal{K}(H))\hookrightarrow\mathrm{Spc}(\mathcal{K}(G))$*.* ☐

4.11. *Remark.* In view of Proposition 4.7, the image of the map induced by restriction $\mathrm{Im}(\rho_H)=\mathrm{supp}(k(G/H))$ coincides with the support of the tt-ideal generated by $\mathrm{Ind}_H^G(\mathcal{K}(H))$. Following the construction of the *geometric* fixed-points functor $\Phi^G\colon \mathrm{SH^c}(G)\to\mathrm{SH^c}$ in topology, we can consider the Verdier quotient

$$\tilde{\mathcal{K}}(G):=\frac{\mathcal{K}(G)}{\langle \mathrm{Ind}_H^G(\mathcal{K}(H))\mid H\lneqq G\rangle}$$

obtained by modding-out, in tensor-triangular fashion, everything induced from all proper subgroups H. This tt-category $\tilde{\mathcal{K}}(G)$ has a smaller spectrum than $\mathcal{K}(G)$, namely the 'geometric open' of the preamble, the complement in $\mathrm{Spc}(\mathcal{K}(G))$ of the closed subset $\cup_{H\lneqq G}\mathrm{Im}(\rho_H)$ covered by proper subgroups. This method has worked nicely in [BS17, BGH20, PSW22] because, in those instances, this Verdier quotient is equivalent to the non-equivariant version of the tt-category under consideration. However, this fails for $\tilde{\mathcal{K}}(G)$, for instance $\tilde{\mathcal{K}}(C_2)$ is *not* equivalent to $\mathcal{K}(1)=\mathrm{D_b}(k)$:

$$\frac{\mathrm{SH^c}(G)}{\langle \mathrm{Ind}_H^G(\mathrm{SH^c}(H))\mid H\lneqq G\rangle}\cong \mathrm{SH^c} \qquad \text{but} \qquad \frac{\mathcal{K}(G)}{\langle \mathrm{Ind}_H^G(\mathcal{K}(H))\mid H\lneqq G\rangle}\not\cong\mathcal{K}(1).$$

For small groups, for instance for cyclic p-groups C_{p^n}, the tt-category $\tilde{\mathcal{K}}(G)$ is reasonably complicated and one could still compute $\mathrm{Spc}(\mathcal{K}(G))$ through an analysis of $\tilde{\mathcal{K}}(G)$. However, the higher the p-rank, the harder it becomes to control $\tilde{\mathcal{K}}(G)$.

One can already see the germ of the problem with $G = C_2$, see (2.12):

$$\mathrm{Spc}(\mathcal{K}(C_2)) = \qquad \begin{array}{ccc} \mathcal{M}(C_2) & & \mathcal{M}(1) \\ & \diagdown \quad \diagup & \\ & \mathcal{P} & \end{array}$$

We have given names to the three primes. The only proper subgroup is $H = 1$ and the image of $\rho_1 = \mathrm{Spc}(\mathrm{Res}_1)$ is simply the single closed point $\{\mathcal{M}(1)\} = \mathrm{supp}(kC_2)$. Chopping off this induced part, leaves us with the open $\mathrm{Spc}(\tilde{\mathcal{K}}(C_2)) = \{\mathcal{M}(C_2), \mathcal{P}\}$. So geometric fixed points $\Phi^{C_2} : \mathcal{K}(C_2) \to \tilde{\mathcal{K}}(C_2)$ detects both of these points. (This also proves that $\tilde{\mathcal{K}}(G) \neq \mathcal{K}(1) = \mathrm{D_b}(k)$ since $\mathrm{D_b}(k)$ would have only one point in its spectrum.) However there is no need for a tt-functor detecting $\mathcal{M}(C_2)$ *and* $\mathcal{P}$ *again*, since $\mathcal{P}$ is already in the cohomological open V_{C_2} detected by $\mathrm{D_b}(kC_2)$. In other words, geometric fixed points see *too much*, not too little: The target category $\tilde{\mathcal{K}}(G)$ is too complicated in general. And as the group grows, this issue only gets worse, as the reader can check with Klein-four in Example 8.10.

In conclusion, we need tt-functors better tailored to the task, namely tt-functors that detect just what is missing from V_G. In the case of C_2, we expect a tt-functor to $\mathrm{D_b}(k)$, to catch $\mathcal{M}(C_2)$, but for larger groups the story gets more complicated and involves more complex subquotients of G, as we explain in the next section.

5. Modular fixed-points functors

Motivated by Remark 4.11, we want to find a replacement for geometric fixed points in the setting of modular representation theory. In a nutshell, our construction amounts to taking classical Brauer quotients [Bro85, § 1] on the level of permutation modules and then passing to the tt-categories $\mathcal{K}(G)$ and $\mathcal{T}(G)$. We follow a somewhat different route than [Bro85] though, more in line with the construction of the geometric fixed-points discussed in Remark 4.11. We hope some readers will benefit from our exposition.

It is here important that $\mathrm{char}(k) = p$ is positive.

5.1. *Warning.* A tt-functor $\Psi^H : \mathcal{K}(G) \to \mathcal{K}(G /\!\!/ H)$ such that $\Psi^H(k(X)) \cong k(X^H)$, as in (2.6), cannot exist unless H is a p-subgroup. Indeed, if $P \leqslant G$ is a p-Sylow then since $[G\!:\!P]$ is invertible in k, the unit $\mathbb{1} = k$ is a direct summand of $k(G/P)$ in $\mathcal{K}(G)$. A tt-functor Ψ^H cannot map $\mathbb{1}$ to zero. Thus $\Psi^H(k(G/P)) = k((G/P)^H)$ must be non-zero, forcing $(G/P)^H \neq \varnothing$. If $[g] \in G/P$ is fixed by H then $H^g \leqslant P$ and therefore H must be a p-subgroup. (If $\mathrm{char}(k) = 0$ this would force $H = 1$.)

5.2. *Recollection.* A collection $\mathcal{F}$ of subgroups of G is called a *family* if it is closed under conjugation and subgroups. For instance, given $H \leqslant G$, we have the family

$$\mathcal{F}_H = \big\{\, K \leqslant G \,\big|\, (G/K)^H = \varnothing \,\big\} = \big\{\, K \leqslant G \,\big|\, H \not\leqslant_G K \,\big\}.$$

For $N \trianglelefteq G$ a normal subgroup, it is $\mathcal{F}_N = \big\{\, K \leqslant G \,\big|\, N \not\leqslant K \,\big\}$.

In view of Warning 5.1, we must focus attention on p-subgroups. The following standard lemma would not be true without the characteristic p hypothesis.

5.3. **Lemma.** *Let $N \trianglelefteq G$ be a normal p-subgroup. Let $H, K \leqslant G$ be subgroups such that $N \leqslant H$ and $N \nleqslant K$. Then every kG-linear homomorphism that factors as $f\colon k(G/H) \xrightarrow{\ell} k(G/K) \xrightarrow{m} k$ must be zero.*

Proof. By Recollection 3.5 and k-linearity, we can assume that m is the augmentation and that $\ell = \epsilon \circ c_g \circ \eta$ as in (3.6), where $g \in G$ is some element, where we set $L = H \cap {}^gK$ and where $\epsilon\colon k(G/L^g) \twoheadrightarrow k(G/K)$, $c_g\colon k(G/L) \xrightarrow{\sim} k(G/L^g)$ and $\eta\colon k(G/H) \rightarrowtail k(G/L)$ are the usual maps, using $L \leqslant H$ and $L^g \leqslant K$. The composite $m \circ \epsilon \circ c_g$ is an augmentation map again, hence our map f is the composite

$$f\colon \quad k(G/H) \overset{\eta}{\rightarrowtail} k(G/L) \overset{\epsilon}{\twoheadrightarrow} k.$$

So f maps $[e]_H$ to $\sum_{\gamma \in H/L} 1 = |H/L|$ in k. Now, the p-group $N \leqslant H$ acts on the set H/L by multiplication on the left. This action has no fixed point, for otherwise we would have $N \leqslant_H L \leqslant_G K$ and thus $N \leqslant K$, a contradiction. Therefore the N-set H/L has order divisible by p. So $|H/L| = 0$ in k and $f = 0$ as claimed. □

5.4. **Proposition.** *Let $N \trianglelefteq G$ be a normal p-subgroup. Then the permutation category of the quotient G/N is an additive quotient of the permutation category of G. More precisely, consider $\operatorname{proj}(\mathcal{F}_N) = \operatorname{add}^\natural \left\{ k(G/K) \,\middle|\, K \in \mathcal{F}_N \right\}$, the closure of $\left\{ k(G/K) \,\middle|\, N \nleqslant K \right\}$ under direct sum and summands in $\operatorname{perm}(G;k)^\natural$. Consider the additive quotient of $\operatorname{perm}(G;k)^\natural$ by this $\otimes$-ideal.* (3) *Then the composite*

$$\operatorname{perm}(G/N;k)^\natural \xrightarrow{\operatorname{Infl}_G^{G/N}} \operatorname{perm}(G;k)^\natural \xrightarrow{\text{quot}} \frac{\operatorname{perm}(G;k)^\natural}{\operatorname{proj}(\mathcal{F}_N)} \tag{5.5}$$

is an equivalence of tensor categories.

Proof. By the Mackey formula and since $\mathcal{F}_N$ is a family, $\operatorname{proj}(\mathcal{F}_N)$ is a tensor ideal, hence quot is a tensor-functor. Inflation $\operatorname{Infl}_G^{G/N}\colon \operatorname{perm}(G/N;k)^\natural \to \operatorname{perm}(G;k)^\natural$ is also a tensor-functor. It is moreover fully faithful with essential image the subcategory $\operatorname{add}^\natural \left\{ k(G/H) \,\middle|\, N \leqslant H \right\}$. So we need to show that the composite

$$\operatorname{add}^\natural \left\{ k(G/H) \,\middle|\, N \leqslant H \right\} \hookrightarrow \operatorname{perm}(G;k)^\natural \twoheadrightarrow \frac{\operatorname{perm}(G;k)^\natural}{\operatorname{add}^\natural \left\{ k(G/K) \,\middle|\, N \nleqslant K \right\}}$$

is an equivalence. Both functors in the composite are full. The composite is faithful by Lemma 5.3, rigidity, additivity and the Mackey formula. Essential surjectivity is then easy (idempotent-completion is harmless since the functor is fully-faithful). □

5.6. *Construction.* Let $N \trianglelefteq G$ be a normal p-subgroup. The composite of the additive quotient functor with the inverse of the equivalence of Proposition 5.4 yields a tensor-functor on the categories of p-permutation modules

$$\Psi^N\colon \operatorname{perm}(G;k)^\natural \twoheadrightarrow \frac{\operatorname{perm}(G;k)^\natural}{\operatorname{proj}(\mathcal{F}_N)} \xrightarrow{\sim} \operatorname{perm}(G/N;k)^\natural. \tag{5.7}$$

Applying the above degreewise, we get a tt-functor on homotopy categories $\mathrm{K_b}(-)$

$$\Psi^N = \Psi^{N;G}\colon \mathcal{K}(G) \longrightarrow \mathcal{K}(G/N).$$

---

[3] Keep the same objects as $\operatorname{perm}(G;k)^\natural$ and define Hom groups by modding out all maps that factor via objects of $\operatorname{proj}(\mathcal{F}_N)$, as in the ordinary construction of the stable module category.

5.8. *Remark.* Following up on Remark 4.11, we have constructed Ψ^N by modding-out in *additive* fashion this time, everything induced from subgroups not containing N. We did it on the 'core' additive category and only then passed to homotopy categories. Such a construction would not make sense on bounded derived categories, as Ψ^N has no reason to preserve acyclic complexes.

The classical Brauer quotient seems different at first sight. It is typically defined at the level of individual kG-modules M by a formula like

$$\operatorname{coker}\big(\oplus_{Q\lneq N} M^Q \xrightarrow{(\operatorname{Tr}_Q^N)_Q} M^N\big). \tag{5.9}$$

A priori, this definition uses the ambient abelian category of modules and one then needs to verify that it preserves permutation modules, the tensor structure, etc. Our approach is a categorification of (5.9): Proposition 5.4 recovers the category $\operatorname{perm}(G/N;k)^\natural$ as a tensor-additive quotient of $\operatorname{perm}(G;k)^\natural$, at the categorical level, not at the individual module level. Amusingly, one can verify that it yields the same answer (Proposition 5.12) – a fact that we shall not use at all.

We relax the condition that the p-subgroup is normal in the standard way.

5.10. *Definition.* Let $H \leqslant G$ be an arbitrary p-subgroup. We define the *modular (or Brauer) H-fixed-points functor* by the composite

$$\Psi^{H;G}: \quad \mathcal{K}(G) \xrightarrow{\operatorname{Res}^G_{N_G H}} \mathcal{K}(N_G H) \xrightarrow{\Psi^{H;N_G H}} \mathcal{K}(G/\!\!/H)$$

where $N_G H$ is the normalizer of H in G and $G/\!\!/H = (N_G H)/H$ its Weyl group. The second functor comes from Construction 5.6. Note that $\Psi^{H;G}$ is computed degreewise, applying the functors $\operatorname{Res}^G_{N_G H}$ and $\Psi^{H;N_G H}$ at the level of $\operatorname{perm}(-;k)^\natural$.

5.11. *Remark.* We prefer the phrase 'modular fixed-points' to 'Brauer fixed-points', out of respect for L. E. J. Brouwer and his fixed points. It also fits nicely in the flow: naive fixed-points, geometric fixed-points, modular fixed-points. Finally, the phrase 'Brauer quotient' would be unfortunate, as $\Psi^H: \mathcal{K}(G) \to \mathcal{K}(G/\!\!/H)$ is *not* a quotient of categories in any reasonable sense.

Let us verify that our Ψ^H linearize the H-fixed-points of G-sets, as promised.

5.12. **Proposition.** *Let $H \leqslant G$ be a p-subgroup. The following square commutes up to isomorphism:*

$$\begin{array}{ccccc}
G\text{-sets} & \xrightarrow{k(-)} & \operatorname{perm}(G;k)^\natural & \hookrightarrow & \mathcal{K}(G) \\
\big\downarrow{\scriptstyle (-)^H} & & \big\downarrow{\scriptstyle \Psi^H} & & \big\downarrow{\scriptstyle \Psi^H} \\
(G/\!\!/H)\text{-sets} & \xrightarrow{k(-)} & \operatorname{perm}(G/\!\!/H;k)^\natural & \hookrightarrow & \mathcal{K}(G/\!\!/H).
\end{array}$$

In particular, for every $K \leqslant G$, we have an isomorphism of $k(G/\!\!/H)$-modules

$$\Psi^H(k(G/K)) \cong k((G/K)^H) = k(N_G(H,K)/K). \tag{5.13}$$

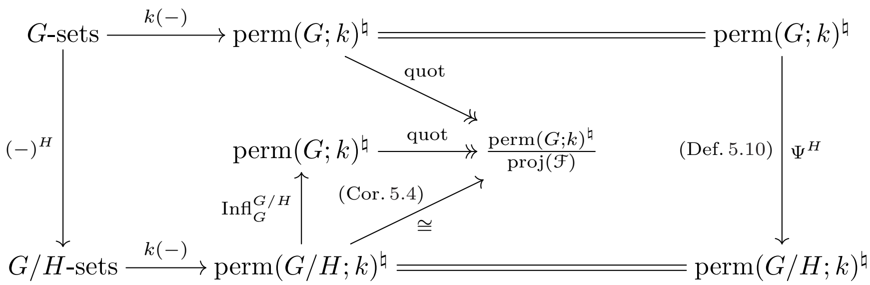
This module is non-zero if and only if H is subconjugate to K in G.

Proof. We only need to prove the commutativity of the left-hand square. As restriction to a subgroup commutes with linearization, we can assume that $H \trianglelefteq G$ is normal. Let X be a G-set. Consider its G-subset X^H (which is truly inflated from G/H). Inclusion yields a morphism in $\operatorname{perm}(G;k)$, natural in X,

$$f_X: k(X^H) \to k(X). \tag{5.14}$$

We claim that this morphism becomes an isomorphism in the quotient $\frac{\mathrm{perm}(G;k)^\natural}{\mathrm{proj}(\mathcal{F}_H)}$. By additivity, we can assume that $X = G/K$ for $K \leqslant G$. It is a well-known exercise that $(G/K)^H = N_G(H,K)/K$, which in the normal case $H \trianglelefteq G$ boils down to G/K or $\varnothing$, depending on whether $H \leqslant K$ or not, *i.e.* whether $K \notin \mathcal{F}_H$ or $K \in \mathcal{F}_H$. In both cases, f_X becomes an isomorphism (an equality or $0 \xrightarrow{\sim} k(G/K)$, respectively) in the quotient by $\mathrm{proj}(\mathcal{F}_H)$. Hence the claim.

Let us now discuss the commutativity of the following diagram



The module $k(X^H)$ in (5.14) can be written more precisely as $k(\mathrm{Infl}_G^{G/H}(X^H)) \cong \mathrm{Infl}_G^{G/H} k(X^H)$. So the first part of the proof shows that the left-hand 'hexagon' of the diagram commutes, *i.e.* the two functors G-sets $\to \frac{\mathrm{perm}(G;k)^\natural}{\mathrm{proj}(\mathcal{F})}$ are isomorphic. The result follows by definition of Ψ^H, recalled on the right-hand side. □

Here is how modular fixed points act on restriction.

5.15. **Proposition.** *Let $\alpha : G \to G'$ be a homomorphism and $H \leqslant G$ a p-subgroup. Set $H' = \alpha(H) \leqslant G'$. Then the following square commutes up to isomorphism*

$$\begin{array}{ccc} \mathcal{K}(G') & \xrightarrow{\alpha^*} & \mathcal{K}(G) \\ {\scriptstyle \Psi^{H',G'}}\downarrow & & \downarrow{\scriptstyle \Psi^{H;G}} \\ \mathcal{K}(G' /\!\!/ H') & \xrightarrow{\bar{\alpha}^*} & \mathcal{K}(G /\!\!/ H). \end{array}$$

Proof. Exercise. This already holds at the level of $\mathrm{perm}(-;k)^\natural$. □

5.16. **Corollary.** *Let $N \trianglelefteq G$ be a normal p-subgroup. Then the composite functor $\Psi^N \circ \mathrm{Infl}_G^{G/N} : \mathcal{K}(G/N) \to \mathcal{K}(G) \to \mathcal{K}(G/N)$ is isomorphic to the identity. Consequently, the map $\mathrm{Spc}(\Psi^H)$ is a split injection retracted by $\mathrm{Spc}(\mathrm{Infl}_G^{G/H})$.*

Proof. Apply Proposition 5.15 to $\alpha\colon G \twoheadrightarrow G/N$ and $H = N$, and thus $H' = 1$. The second statement is just contravariance of $\mathrm{Spc}(-)$. □

Composition of two 'nested' modular fixed-points functors almost gives another modular fixed-points functor. We only need to beware of Weyl groups.

5.17. **Proposition.** *Let $H \leqslant G$ be a p-subgroup and $\bar{K} = K/H$ a p-subgroup of $G/\!\!/H$, for $H \leqslant K \leqslant N_G H$. Then there is a canonical inclusion*

$$(G/\!\!/H)/\!\!/\bar{K} = (N_{G/\!\!/H}\bar{K})/\bar{K} \hookrightarrow (N_G K)/K = G/\!\!/K$$

and the following square commutes up to isomorphism

$$\begin{array}{ccc} \mathcal{K}(G) & \xrightarrow{\Psi^{H;G}} & \mathcal{K}(G/\!\!/H) \\ {\scriptstyle \Psi^{K;G}}\downarrow & & \downarrow{\scriptstyle \Psi^{\bar{K};G/\!\!/H}} \\ \mathcal{K}(G/\!\!/K) & \xrightarrow{\text{Res}} & \mathcal{K}\big((G/\!\!/H)/\!\!/\bar{K}\big). \end{array}$$

Proof. The inclusion comes from $N_{N_G(H)}K \hookrightarrow N_G K$ and the rest is an exercise. Again, everything already holds at the level of $\text{perm}(-;k)^\natural$. □

5.18. **Corollary.** *Let $H \leqslant K \leqslant G$ be two p-subgroups with $H \trianglelefteqslant G$ normal. Then $(G/H)/\!\!/(K/H) \cong G/\!\!/K$ and the following diagram commutes up to isomorphism*

$$\begin{array}{ccc} \mathcal{K}(G) & \xrightarrow{\Psi^{H;G}} & \mathcal{K}(G/H) \\ & {\scriptstyle \Psi^{K;G}}\searrow & \downarrow{\scriptstyle \Psi^{K/H;\,G/H}} \\ & & \mathcal{K}\big(G/\!\!/K\big). \end{array}$$

Proof. The surjectivity of the canonical inclusion $G/H/\!\!/(K/H) \hookrightarrow G/\!\!/K$ of Proposition 5.17 holds since H is normal in G. The result follows. □

5.19. *Remark.* We have essentially finished the proof of Proposition 2.7. It only remains to verify that there are variants of the constructions and results of this section for the big categories of Recollection 3.2. For a normal p-subgroup $N \trianglelefteqslant G$, the canonical functor on big additive categories

$$\text{Add}^\natural(\{\, k(G/H) \,|\, N \leqslant H \,\}) \to \frac{\text{Perm}(G;k)^\natural}{\text{Proj}(\mathcal{F}_N)} \tag{5.20}$$

is an equivalence of tensor categories, where

$$\text{Proj}(\mathcal{F}_N) = \text{Add}^\natural \left\{ k(G/K) \,\middle|\, N \not\leqslant K \right\}$$

is the closure of $\text{proj}(\mathcal{F}_N)$ under coproducts and summands. Since the tensor product commutes with coproducts, $\text{Proj}(\mathcal{F}_N)$ is again a $\otimes$-ideal in $\text{Perm}(G;k)^\natural$. Fullness and essential surjectivity of (5.20) are easy, and faithfulness reduces to the finite case by compact generation. (A map $f : P \to Q$ in $\text{Perm}(G;k)$ is zero if and only if all composites $P' \xrightarrow{\alpha} P \xrightarrow{f} Q$ are zero, for P' finitely generated. Such a composite necessarily factors through a finitely generated direct summand of Q, etc.) As a consequence, the analogue of Proposition 5.4 also holds for big categories.

Let us write $\mathcal{S}(G)$ for $\text{K}(\text{Perm}(G;k)) = \text{K}(\text{Perm}(G;k)^\natural)$, which is a compactly generated tt-category with compact unit. (Compactly generated is not obvious: see [BG21, Remark 5.12].) By the above discussion, the modular fixed-points functor with respect to a p-subgroup $H \leqslant G$ extends to the big categories $\mathcal{S}(-)$:

$$\Psi^H = \Psi^{H;G}\colon\ \mathcal{S}(G) \xrightarrow{\text{Res}^G_{N_G H}} \mathcal{S}(N_G H) \to \text{K}\left(\frac{\text{Perm}(N_G H;k)}{\text{Proj}(\mathcal{F}_H)}\right) \xleftarrow[\sim]{\text{Infl}^{G/\!\!/H}_{N_G H}} \mathcal{S}(G/\!\!/H).$$

Note that Ψ^H is a tensor triangulated functor that commutes with coproducts and that maps $\mathcal{K}(G)$ into $\mathcal{K}(G/\!\!/H)$. It follows that it restricts to $\Psi^H\colon \text{DPerm}(G;k) \to \text{DPerm}(G/\!\!/H;k)$. The analogues of Propositions 5.12, 5.15 and 5.17 and Corollaries 5.16 and 5.18 all continue to hold for both $\mathcal{S}(-)$ and $\text{DPerm}(-;k)$.

This finishes our exposition of modular fixed-points functors Ψ^H on derived categories of permutation modules. We now start using them to analyze the tt-geometry. First, we apply them to the Koszul complexes $\mathrm{kos}_G(K)$ of Construction 3.14.

5.21. **Lemma.** *Let $H, K \leqslant G$ be two subgroups, with H a p-subgroup.*
(a) *If $H \not\leqslant_G K$, then $\Psi^H(\mathrm{kos}_G(K))$ generates $\mathcal{K}(G/\!\!/H)$ as a tt-ideal.*
(b) *If $H \lneq_G K$, then $\Psi^H(\mathrm{kos}_G(K))$ is acyclic in $\mathcal{K}(G/\!\!/H)$.*
(c) *If $H \sim_G K$, then $\Psi^H(\mathrm{kos}_G(K))$ generates $\mathcal{K}_{\mathrm{ac}}(G/\!\!/H)$ as a tt-ideal.*

Proof. For (a), we have $N_G(H,K) = \varnothing$ and thus $\Psi^H(k(G/K)) = 0$ by Proposition 5.12. It follows that $\Psi^H(\mathrm{kos}_G(K)) = (\cdots \to * \to 0 \to k \to 0)$ by Proposition 3.15. Thus the $\otimes$-unit $\mathbb{1}_{\mathcal{K}(G/\!\!/H)} = k[0]$ is a direct summand of $\Psi^H(\mathrm{kos}_G(K))$.

For (b) and (c), by invariance under conjugation, we can assume that $H \leqslant K$. Let $N := N_G H$ be the normalizer of H. We have by Lemma 3.18 that

$$\Psi^{H;G}(\mathrm{kos}_G(K)) = \Psi^{H;N}\,\mathrm{Res}^G_N(\mathrm{kos}_G(K)) \simeq \bigotimes_{[g]\in N\backslash G/K} \Psi^{H;N}\big(\mathrm{kos}_N(N\cap{}^gK)\big). \tag{5.22}$$

For the index $g = e$ (or simply $g \in N_G K$), we can use $H \trianglelefteq N \cap K$ and compute

$$\begin{aligned}\Psi^{H;N}(\mathrm{kos}_N(N\cap K)) &\cong \Psi^{H;N}(\mathrm{Infl}^{N/H}_N \mathrm{kos}_{N/H}((N\cap K)/H)) && \text{by Lemma 3.17}\\ &\cong \mathrm{kos}_{N/H}((N\cap K)/H) && \text{by Corollary 5.16.}\end{aligned}$$

As this object is acyclic in $\mathcal{K}(N/H)$ so is the tensor in (5.22). Hence (b). Continuing in the special case (c) with $H = K$, we have $(N\cap K)/H = 1$ and the above $\mathrm{kos}_{N/H}(1)$ generates $\mathcal{K}_{\mathrm{ac}}(N/H)$ by Proposition 3.21. It suffices to show that all the other factors in the tensor product (5.22) generate the whole $\mathcal{K}(G/\!\!/H)$. This follows from Part (a) applied to the group N; indeed when $g \in G \smallsetminus N$ we have $H \not\leqslant_N N \cap {}^gH$ (as $H \leqslant_N N\cap{}^gH$ and $H \trianglelefteq N$ would imply $H = {}^gH$). ☐

6. Conservativity via modular fixed-points

In this section, we explain why the spectrum of $\mathcal{K}(G)$ is controlled by modular fixed-points functors Ψ^H together with the localizations $\Upsilon_G\colon \mathcal{K}(G) \twoheadrightarrow \mathrm{D_b}(kG)$. It stems from a conservativity result on the 'big' category $\mathcal{T}(G) = \mathrm{DPerm}(G;k)$, namely Theorem 6.12, for which we need some preparation.

6.1. **Lemma.** *Suppose that G is a p-group. Let $H \leqslant G$ be a subgroup and let $\bar{G} = G/\!\!/H$ be its Weyl group. The modular H-fixed-points functor $\Psi^H\colon \mathrm{perm}(G;k)^\natural \to \mathrm{perm}(\bar{G};k)^\natural$ induces a ring homomorphism*

$$\Psi^H\colon \mathrm{End}_{kG}(k(G/H)) \longrightarrow \mathrm{End}_{k\bar{G}}(k(\bar{G})). \tag{6.2}$$

This homomorphism is surjective with nilpotent kernel: $(\ker(\Psi^H))^n = 0$ *for* $n \gg 1$. *More precisely, it suffices to take $n \in \mathbb{N}$ such that* $\mathrm{Rad}(kG)^n = 0$.

Proof. The reader can check this with Brauer quotients. We outline the argument. By (5.13) we have $\Psi^H(k(G/H)) \cong k(N_G(H,H)/H) = k(\bar{G})$, so the problem is well-stated. Recollection 3.5 provides k-bases for both vector spaces in (6.2), namely

$$\{\, f_g = \epsilon \circ c_g \circ \eta \,\}_{[g]\in H\backslash G/H} \qquad \text{and} \qquad \{\, c_{\bar{g}} \,\}_{\bar{g}\in\bar{G}}$$

using the notation of (3.6) and (3.7). The homomorphism Ψ^H in (6.2) respects those bases. Even better, it is a bijection from the part of the first basis indexed by $H\backslash(N_G H)/H$ onto the second basis, and it maps the rest of the first basis to zero. Indeed, when $g \in N_G H$, we have $f_g = c_g$ and $\Psi^H(f_g) = \Psi^H(c_g) = c_{\bar{g}}$ for $\bar{g} = [g]_H$. On the other hand, when $g \in G \smallsetminus N_G H$ then $\Psi^H(f_g) = 0$, by the factorization (3.6) and the fact that $\Psi^H(k(G/L)) = 0$ for $L = H \cap {}^g H$ with $g \notin N_G H$, using again (5.13). Hence (6.2) is surjective and $\ker(\Psi^H)$ has basis $\{f_g = \epsilon \circ c_g \circ \eta\}_{[g] \in H\backslash G/H,\, g \notin N_G H}$. One easily verifies that such an f_g has image contained in $\mathrm{Rad}(kG) \cdot k(G/H)$, using that $H \cap {}^g H$ is strictly smaller than H. Composing n such generators $f_{g_1} \circ \cdots \circ f_{g_n}$ then maps $k(G/H)$ into $\mathrm{Rad}(kG)^n \cdot k(G/H)$ which is eventually zero for $n \gg 1$, since G is a p-group. $\square$

We now isolate a purely additive result that we shall of course apply to the case where Ψ is a modular fixed-points functor.

6.3. **Lemma.** *Let $\Psi \colon \mathcal{A} \to \mathcal{D}$ be an additive functor between additive categories. Let $\mathcal{B}, \mathcal{C} \subseteq \mathcal{A}$ be full additive subcategories such that:*

(1) *Every object of $\mathcal{A}$ decomposes as $B \oplus C$ with $B \in \mathcal{B}$ and $C \in \mathcal{C}$.*
(2) *The functor Ψ vanishes on $\mathcal{C}$, that is, $\Psi(\mathcal{C}) = 0$.*
(3) *The restricted functor $\Psi_{\restriction \mathcal{B}} \colon \mathcal{B} \to \mathcal{D}$ is full with nilpotent kernel.* ([4])

Let $X \in \mathrm{Ch}(\mathcal{A})$ be a complex such that $\Psi(X)$ is contractible in $\mathrm{Ch}(\mathcal{D})$. Then X is homotopy equivalent to a complex in $\mathrm{Ch}(\mathcal{C})$.

Proof. Decompose every $X_i = B_i \oplus C_i$ in $\mathcal{A}$, using (1), for all $i \in \mathbb{Z}$. We are going to build a complex on the objects C_i in such a way that $X_\bullet$ becomes homotopy equivalent to $C_\bullet$ in $\mathrm{Ch}(\mathcal{A}^\natural)$, where $\mathcal{A}^\natural$ is the idempotent-completion of $\mathcal{A}$. As both $X_\bullet$ and $C_\bullet$ belong to $\mathrm{Ch}(\mathcal{A})$, this proves the result. By (2), the complex $\cdots \to \Psi(B_i) \to \Psi(B_{i-1}) \to \cdots$ is isomorphic to $\Psi(X)$, hence it is contractible. So each $\Psi(B_i)$ decomposes in $\mathcal{D}^\natural$ as $D_i \oplus D_{i-1}$ in such a way that the differential $\Psi(B_i) = D_i \oplus D_{i-1} \longrightarrow \Psi(B_{i-1}) = D_{i-1} \oplus D_{i-2}$ is just $\left(\begin{smallmatrix} 0 & 1 \\ 0 & 0 \end{smallmatrix}\right)$. Since $\Psi_{\restriction \mathcal{B}} \colon \mathcal{B} \to \mathcal{D}$ is full with nilpotent kernel by (3), we can lift idempotents. In other words, we can decompose each B_i in the idempotent-completion $\mathcal{B}^\natural$ (hence in $\mathcal{A}^\natural$) as

$$B_i \simeq B_i' \oplus B_i''$$

with $\Psi(B_i') \simeq D_i$ and $\Psi(B_i'') \simeq D_{i-1}$ in a compatible way with the decomposition in $\mathcal{D}^\natural$. This means that when we write the differentials in X in components in $\mathcal{A}^\natural$

$$\cdots \to X_i = B_i' \oplus B_i'' \oplus C_i \xrightarrow{\left(\begin{smallmatrix} * & b_i & * \\ * & * & * \\ * & * & * \end{smallmatrix}\right)} X_{i-1} = B_{i-1}' \oplus B_{i-1}'' \oplus C_{i-1} \to \cdots$$

the component $b_i \colon B_i'' \to B_{i-1}'$ maps to the isomorphism $\Psi(B_i'') \simeq D_{i-1} \simeq \Psi(B_{i-1}')$ in $\mathcal{D}^\natural$. Hence b_i is already an isomorphism in $\mathcal{B}^\natural$ by (3) again. (Note that (3) passes to $\mathcal{B}^\natural \to \mathcal{D}^\natural$.) Using elementary operations on X_i and X_{i-1} we can replace X by an isomorphic complex in $\mathcal{A}^\natural$ of the form

$$(6.4) \quad \cdots \to X_{i+1} \to B_i' \oplus B_i'' \oplus C_i \xrightarrow{\left(\begin{smallmatrix} 0 & b_i & 0 \\ * & 0 & * \\ * & 0 & * \end{smallmatrix}\right)} B_{i-1}' \oplus B_{i-1}'' \oplus C_{i-1} \to X_{i-2} \to \cdots$$

This being a complex forces the 'previous' differential $X_{i+1} \to X_i$ to be of the form $\left(\begin{smallmatrix} * & * & * \\ 0 & 0 & 0 \\ * & * & * \end{smallmatrix}\right)$ and the 'next' differential $X_{i-1} \to X_{i-2}$ to be of the form $\left(\begin{smallmatrix} 0 & * & * \\ 0 & * & * \\ 0 & * & * \end{smallmatrix}\right)$. We can then remove from X a direct summand in $\mathrm{Ch}(\mathcal{A}^\natural)$ that is a homotopically trivial complex of the form $\cdots 0 \to B_i'' \overset{\sim}{\to} B_{i-1}' \to 0 \cdots$.

[4] There exists $n \gg 1$ such that if n composable morphisms $f_1, \ldots, f_n$ in $\mathcal{B}$ all go to zero in $\mathcal{D}$ under Ψ then their composite $f_n \circ \cdots \circ f_1$ is zero in $\mathcal{B}$.

The reader might be concerned about how to perform this reduction in all degrees at once, since we do not put boundedness conditions on X (thus preventing the 'obvious' induction argument). The solution is simple. Do the above for all differentials in *even* indices $i = 2j$. By elementary operations on X_{2j} and X_{2j-1} for all $j \in \mathbb{Z}$, we can replace X up to isomorphism into a complex whose *even* differentials are of the form (6.4). We then remove the contractible complexes $\cdots 0 \to B''_{2j} \overset{\sim}{\to} B'_{2j-1} \to 0 \cdots$. We obtain in this way a homotopy equivalent complex in $\mathcal{A}^\natural$ that we call $\tilde{X}$, where $B'_i, B''_i \in \mathcal{B}^\natural$ and $C_i \in \mathcal{C}$

$$\cdots \to B''_{2j+1} \oplus C_{2j+1} \xrightarrow{\left(\begin{smallmatrix} a_{2j+1} & * \\ * & * \end{smallmatrix}\right)} B'_{2j} \oplus C_{2j} \xrightarrow{\left(\begin{smallmatrix} * & * \\ * & * \end{smallmatrix}\right)} B''_{2j-1} \oplus C_{2j-1} \to \cdots \tag{6.5}$$

in which the even differentials go to zero under Ψ, by the above construction. In particular the homotopy trivial complex $\Psi(\tilde{X}) \simeq \Psi(X)$ in $\mathcal{D}^\natural$ has the form $\cdots \xrightarrow{0} \Psi(B''_{2j+1}) \xrightarrow{\Psi(a_{2j+1})} \Psi(B'_{2j}) \xrightarrow{0} \cdots$ hence its odd-degree differentials $\Psi(a_{2j+1})$ are isomorphisms. It follows that $a_{2j+1} \colon B''_{2j+1} \to B'_{2j}$ is itself an isomorphism by (3) again. Using new elementary operations (again in all degrees), we change the odd-degree differentials of the complex $\tilde{X}$ in (6.5) into diagonal ones and we remove the contractible summands $0 \to B''_{2j+1} \overset{\sim}{\to} B'_{2j} \to 0$ as before, to get a complex consisting only of the C_i in each degree $i \in \mathbb{Z}$. In summary, we have shown that X is homotopy equivalent to a complex $C \in \mathrm{Ch}(\mathcal{C})$ inside $\mathrm{Ch}(\mathcal{A}^\natural)$, as announced. □

6.6. *Remark.* Of course, it would be silly to discuss conservativity of the functors $\{\Psi^H\}_{H \leqslant G}$ since among them we find $\Psi^1 = \mathrm{Id}$. The interesting result appears when each Ψ^H is used in conjunction with the derived category of $G/\!\!/H$, or, in 'big' form, its homotopy category of injectives. Let us remind the reader.

6.7. *Recollection.* In [BG22a], we prove that the homotopy category of injective RG-modules, with coefficients in any regular ring R (*e.g.* our field k), is a localization of $\mathrm{DPerm}(G; R)$. In our case, we have an inclusion $J_G \colon \mathrm{K}\,\mathrm{Inj}(kG) \hookrightarrow \mathrm{DPerm}(G; k)$, inside $\mathrm{K}(\mathrm{Perm}(G; k))$, and this inclusion admits a left adjoint Υ_G

$$\begin{array}{c} \mathrm{DPerm}(G;k) \\ \Upsilon_G \big\downarrow \;\big\uparrow J_G \\ \mathrm{K}\,\mathrm{Inj}(kG). \end{array} \tag{6.8}$$

This realizes the finite localization of $\mathrm{DPerm}(G; k)$ with respect to the subcategory $\mathcal{K}_{\mathrm{ac}}(G) \subseteq \mathcal{K}(G) = \mathrm{DPerm}(G; k)^c$. In particular, Υ_G preserves compact objects and yields the equivalence $\Upsilon_G \colon \mathcal{K}(G)/\mathcal{K}_{\mathrm{ac}}(G) \cong \mathrm{D_b}(kG) \cong \mathrm{K}\,\mathrm{Inj}(kG)^c$ of (3.13), also denoted Υ_G for this reason. Note that $\Upsilon_G \circ J_G \cong \mathrm{Id}$ as usual with localizations.

Let $P \leqslant G$ be a subgroup. Observe that induction Ind_P^G preserves injectives so that $J_G \circ \mathrm{Ind}_P^G \cong \mathrm{Ind}_P^G \circ J_P$. Taking left adjoints, we see that

$$\mathrm{Res}^G_P \circ \Upsilon_G \cong \Upsilon_P \circ \mathrm{Res}^G_P . \tag{6.9}$$

6.10. *Notation.* For each p-subgroup $H \leqslant G$, we are interested in the composite

$$\check{\Psi}^H = \check{\Psi}^{H;G} \colon \quad \mathrm{DPerm}(G; k) \xrightarrow{\Psi^{H;G}} \mathrm{DPerm}(G/\!\!/H; k) \xrightarrow{\Upsilon_{G/\!\!/H}} \mathrm{K}\,\mathrm{Inj}(k(G/\!\!/H))$$

of the modular H-fixed-points functor followed by localization to the homotopy category of injectives (6.8). We use the same notation on compacts

$$\check{\Psi}^H = \check{\Psi}^{H;G}\colon \quad \mathcal{K}(G;k) \xrightarrow{\Psi^{H;G}} \mathcal{K}(G/\!\!/H;k) \xrightarrow{\Upsilon_{G/\!\!/H}} \mathrm{D_b}(k(G/\!\!/H)). \tag{6.11}$$

We are now ready to prove the first important result of the paper.

6.12. **Theorem** (Conservativity)**.** *Let G be a finite group. The above family of functors $\check{\Psi}^H\colon \mathcal{T}(G) \to \mathrm{K\,Inj}(k(G/\!\!/H))$, indexed by all the (conjugacy classes of) p-subgroups $H \leqslant G$, collectively detects vanishing of objects of* $\mathrm{DPerm}(G;k)$*.*

Proof. Let $P \leqslant G$ be a p-Sylow subgroup. For every subgroup $H \leqslant P$, we have $P/\!\!/H \hookrightarrow G/\!\!/H$ and $\check{\Psi}^{H;P} \circ \mathrm{Res}^G_P$ can be computed as $\mathrm{Res}^{G/\!\!/H}_{P/\!\!/H} \circ \check{\Psi}^{H;G}$ thanks to Proposition 5.15 and (6.9). On the other hand, Res^G_P is (split) faithful, as $\mathrm{Ind}^G_P \circ \mathrm{Res}^G_P$ admits the identity as a direct summand. Hence it suffices to prove the theorem for the group P, *i.e.* we can assume that G is a p-group.

Let G be a p-group and $\mathcal{F}$ be a family of subgroups (Recollection 5.2). We say that a complex X in $\mathrm{Ch}(\mathrm{Perm}(G;k))$ is *of type* $\mathcal{F}$ if every X_i is $\mathcal{F}$-free, *i.e.* a coproduct of $k(G/K)$ for $K \in \mathcal{F}$. So every complex is of type $\mathcal{F}_{\mathrm{all}} = \{\text{all } H \leqslant G\}$. Saying that X is of type $\mathcal{F}_1 = \varnothing$ means $X = 0$. We want to prove that if X defines an object in $\mathrm{DPerm}(G;k)$ and $\check{\Psi}^H(X) = 0$ for all $H \leqslant G$ then X is homotopy equivalent to a complex X' of type $\mathcal{F}_1 = \varnothing$. We proceed by a form of 'descending induction' on $\mathcal{F}$. Namely, we prove:

Claim: *Let $X \in \mathrm{DPerm}(G;k)$ be a complex of type $\mathcal{F}$ for some family $\mathcal{F}$ and let $H \in \mathcal{F}$ be a maximal element of $\mathcal{F}$ for inclusion. If $\check{\Psi}^H(X) = 0$ then $X \cong X'$ is homotopy equivalent to a complex $X' \in \mathrm{Ch}(\mathrm{Perm}(G;k))$ of type $\mathcal{F}' \subsetneq \mathcal{F}$.*

By the above discussion, proving this claim proves the theorem. Explicitly, we are going to prove this claim for $\mathcal{F}' = \mathcal{F} \cap \mathcal{F}_H$, that is, $\mathcal{F}$ with all conjugates of H removed. By maximality of H in $\mathcal{F}$, every $K \in \mathcal{F}$ is either conjugate to H or in $\mathcal{F}'$. Of course, for H' conjugate to H we have $k(G/H') \simeq k(G/H)$ in $\mathrm{Perm}(G;k)$.

We apply Lemma 6.3 for $\mathcal{A} = \mathrm{Add}\left\{ k(G/K) \,\middle|\, K \in \mathcal{F} \right\}$, $\mathcal{B} = \mathrm{Add}(k(G/H))$, $\mathcal{C} = \mathrm{Add}\left\{ k(G/K) \,\middle|\, K \in \mathcal{F}' \right\}$, $\mathcal{D} = \mathrm{Perm}(G/\!\!/H;k)$ and the functor $\Psi = \Psi^H$ naturally. Let us check the hypotheses of Lemma 6.3. Regrouping the terms $k(G/K)$ into those for which K is conjugate to H and those not conjugate to H, we get Hypothesis (1). Hypothesis (2) follows immediately from (5.13) since $(G/K)^H = \varnothing$ for every $K \in \mathcal{F}'$. Finally, Hypothesis (3) follows from Lemma 6.1 and additivity. So it remains to show that $\Psi^H(X)$ is contractible. Since X is of type $\mathcal{F}$ and H is maximal, we see that $\Psi^H(X) \in \mathrm{Ch}(\mathrm{Inj}(k(G/\!\!/H)))$ and applying $\Upsilon_{G/\!\!/H}$ gives the same complex (up to homotopy). In other words, $\check{\Psi}^H(X) = 0$ forces $\Psi^H(X)$ to be contractible and we can indeed get the above Claim from Lemma 6.3. ☐

7. The spectrum as a set

In this section, we obtain the description of all points of $\mathrm{Spc}(\mathcal{K}(G))$, as well as some elements of its topology. We start with a general fact, which is now folklore.

7.1. **Proposition.** *Let $F\colon \mathcal{T} \to \mathcal{S}$ be a coproduct-preserving tt-functor between 'big' tt-categories. Suppose that F is conservative. Then F detects $\otimes$-nilpotence of morphisms $f\colon x \to Y$ in $\mathcal{T}$, whose source $x \in \mathcal{T}^c$ is compact,* i.e. *if $F(f) = 0$ in $\mathcal{S}$*

then there exists $n \gg 1$ such that $f^{\otimes n} = 0$ in $\mathcal{T}$. In particular, $F\colon \mathcal{T}^c \to \mathcal{S}^c$ detects nilpotence of morphisms and therefore $\mathrm{Spc}(F)\colon \mathrm{Spc}(\mathcal{S}^c) \to \mathrm{Spc}(\mathcal{T}^c)$ is surjective.

Proof. Using rigidity of compacts, we can assume that $x = \mathbb{1}$. Given a morphism $f\colon \mathbb{1} \to Y$ we can construct in $\mathcal{T}$ the homotopy colimit $Y^\infty := \mathrm{hocolim}_n Y^{\otimes n}$ under the transition maps $f \otimes \mathrm{id}\colon Y^{\otimes n} \to Y^{\otimes(n+1)}$. Let $f^\infty\colon \mathbb{1} \to Y^\infty$ be the resulting map. Now since $F(f) = 0$ it follows that $F(Y^\infty) = 0$ in $\mathcal{S}$, as it is a sequential homotopy colimit of zero maps. By conservativity of F, we get $Y^\infty = 0$ in $\mathcal{T}$. Since $\mathbb{1}$ is compact, the vanishing of $f^\infty\colon \mathbb{1} \to \mathrm{hocolim}\, Y^{\otimes n}$ must already happen at a finite stage, that is, the map $f^{\otimes n}\colon \mathbb{1} \to Y^{\otimes n}$ is zero for $n \gg 1$, as claimed. The second statement follows from this, together with [Bal18, Theorem 1.4]. □

Combined with our Conservativity Theorem 6.12 we get:

7.2. **Corollary.** *The family of functors $\check{\Psi}^H\colon \mathcal{K}(G) \to \mathrm{D_b}(k(G/\!\!/H))$, indexed by conjugacy classes of p-subgroups $H \leqslant G$, detects $\otimes$-nilpotence. So the induced map*

$$\coprod_{H \in \mathrm{Sub}_p(G)} \mathrm{Spc}(\mathrm{D_b}(k(G/\!\!/H))) \twoheadrightarrow \mathrm{Spc}(\mathcal{K}(G))$$

is surjective. □

7.3. *Definition.* Let $H \leqslant G$ be a p-subgroup. We write (under Convention 1.8)

$$\psi^H = \psi^{H;G} := \mathrm{Spc}(\Psi^H)\colon\ \mathrm{Spc}(\mathcal{K}(G/\!\!/H)) \to \mathrm{Spc}(\mathcal{K}(G))$$

for the map induced by the modular H-fixed-points functor. We write

$$\check{\psi}^H = \check{\psi}^{H;G} := \mathrm{Spc}(\check{\Psi}^H)\colon\ \mathrm{Spc}(\mathrm{D_b}(G/\!\!/H)) \to \mathrm{Spc}(\mathcal{K}(G))$$

for the map induced by the tt-functor $\check{\Psi}^H = \Upsilon_{G/\!\!/H} \circ \Psi^H$ of (6.11). In other words, $\check{\psi}^H$ is the composite of the inclusion of Proposition 3.22 with the above ψ^H

$$\check{\psi}^H :\quad V_{G/\!\!/H} = \mathrm{Spc}(\mathrm{D_b}(k(G/\!\!/H))) \overset{\upsilon_{G/\!\!/H}}{\hookrightarrow} \mathrm{Spc}(\mathcal{K}(G/\!\!/H)) \xrightarrow{\psi^H} \mathrm{Spc}(\mathcal{K}(G)).$$

7.4. *Definition.* Let $H \leqslant G$ be a p-subgroup and $\mathfrak{p} \in V_{G/\!\!/H}$ a 'cohomological' prime over the Weyl group of H in G. We define a point in $\mathrm{Spc}(\mathcal{K}(G))$ by

$$\mathcal{P}(H,\mathfrak{p}) = \mathcal{P}_G(H,\mathfrak{p}) := \check{\psi}^H(\mathfrak{p}) = (\check{\Psi}^H)^{-1}(\mathfrak{p}).$$

Corollary 7.2 tells us that every point of $\mathrm{Spc}(\mathcal{K}(G))$ is of the form $\mathcal{P}(H,\mathfrak{p})$ for some p-subgroup $H \leqslant G$ and some cohomological point $\mathfrak{p} \in V_{G/\!\!/H}$. Different subgroups and different cohomological points can give the same $\mathcal{P}(H,\mathfrak{p})$. See Theorem 7.16.

7.5. *Remark.* Although we shall not use it, we can unpack the definitions of $\mathcal{P}_G(H,\mathfrak{p})$ for the nostalgic reader. Let us start with the bijection $V_G = \mathrm{Spc}(\mathrm{D_b}(kG)) \cong \mathrm{Spec^h}(\mathrm{H}^\bullet(G,k))$. Let $\mathfrak{p}^\bullet \subset \mathrm{H}^\bullet(G;k) = \mathrm{End}^\bullet_{\mathrm{D_b}(kG)}(\mathbb{1})$ be a homogeneous prime ideal of the cohomology. The corresponding prime $\mathfrak{p}$ in $\mathrm{D_b}(kG)$ can be described as

$$\mathfrak{p} = \big\{\, x \in \mathrm{D_b}(kG) \,\big|\, \exists\, \zeta \in \mathrm{H}^\bullet(G;k) \text{ such that } \zeta \notin \mathfrak{p}^\bullet \text{ and } \zeta \otimes x = 0 \,\big\}.$$

Consequently, the prime $\mathcal{P}_G(H,\mathfrak{p})$ of Definition 7.4 is equal to

$$\big\{\, x \in \mathcal{K}(G) \,\big|\, \exists\, \zeta \in \mathrm{H}^\bullet(G/\!\!/H;k) \smallsetminus \mathfrak{p}^\bullet \text{ such that } \zeta \otimes \Psi^H(x) = 0 \text{ in } \mathrm{D_b}(k(G/\!\!/H)) \,\big\}.$$

7.6. *Remark.* By Proposition 5.15 and functoriality of $\mathrm{Spc}(-)$, the primes $\mathcal{P}_G(H,\mathfrak{p})$ are themselves functorial in G. To wit, if $\alpha : G \to G'$ is a group homomorphism and H is a p-subgroup of G then $\alpha(H)$ is a p-subgroup of G' and we have

$$\alpha_*(\mathcal{P}_G(H,\mathfrak{p})) = \mathcal{P}_{G'}(\alpha(H), \bar{\alpha}_*\mathfrak{p}) \tag{7.7}$$

in $\mathrm{Spc}(\mathcal{K}(G'))$, where $\alpha_*\colon \mathrm{Spc}(\mathcal{K}(G)) \to \mathrm{Spc}(\mathcal{K}(G'))$ and $\bar{\alpha}_*\colon V_{G/\!\!/H} \to V_{G'/\!\!/\alpha(H)}$ are as in Remark 4.1. We single out the usual suspects. Fix $H \leqslant G$ a p-subgroup.

(a) For *conjugation*, let $G \leqslant G'$ and $x \in G'$. We get $\mathcal{P}_G(H,\mathfrak{p})^x = \mathcal{P}_{G^x}(H^x,\mathfrak{p}^x)$ for every $\mathfrak{p} \in V_{G/\!\!/H}$. In particular, when x belongs to G itself, we get by (4.2)

$$g \in G \qquad \Longrightarrow \qquad \mathcal{P}_G(H,\mathfrak{p}) = \mathcal{P}_G(H^g,\mathfrak{p}^g). \tag{7.8}$$

(b) For *restriction*, let $K \leqslant G$ be a subgroup containing H and let $\mathfrak{p} \in V_{K/\!\!/H}$ be a cohomological point over the Weyl group of H in K. Then we have

$$\rho_K(\mathcal{P}_K(H,\mathfrak{p})) = \mathcal{P}_G(H,\bar{\rho}_K(\mathfrak{p})), \tag{7.9}$$

in $\mathrm{Spc}(\mathcal{K}(G))$, where the maps $\rho_K = (\mathrm{Res}^G_K)_*\colon \mathrm{Spc}(\mathcal{K}(K)) \to \mathrm{Spc}(\mathcal{K}(G))$ and $\bar{\rho}_K\colon V_{K/\!\!/H} \to V_{G/\!\!/H}$ are spelled out around (4.4).

(c) For *inflation*, let $N \trianglelefteq G$ be a normal subgroup. Set $\bar{G} = G/N$ and $\bar{H} = HN/N$. Then for every $\mathfrak{p} \in V_{G/\!\!/H}$, we have

$$\pi^{\bar{G}}(\mathcal{P}_G(H,\mathfrak{p})) = \mathcal{P}_{\bar{G}}(\bar{H},\bar{\pi}^{\bar{G}}\,\mathfrak{p}), \tag{7.10}$$

in $\mathrm{Spc}(\mathcal{K}(\bar{G}))$, where the maps $\pi^{\bar{G}} = (\mathrm{Infl}^{\bar{G}}_G)_*\colon \mathrm{Spc}(\mathcal{K}(G)) \to \mathrm{Spc}(\mathcal{K}(\bar{G}))$ and $\bar{\pi}^{\bar{G}}\colon V_{G/\!\!/H} \to V_{\bar{G}/\!\!/\bar{H}}$ are spelled out around (4.5).

Our primes also behave nicely under modular fixed-points maps:

7.11. **Proposition.** *Let $H \leqslant G$ be a p-subgroup and let $H \leqslant K \leqslant N_G H$ defining a 'further' p-subgroup $K/H \leqslant G/\!\!/H$. Then for every $\mathfrak{p} \in V_{(G/\!\!/H)/\!\!/(K/H)}$, we have*

$$\psi^{H;G}(\mathcal{P}_{G/\!\!/H}(K/H,\mathfrak{p})) = \mathcal{P}_G(K,\bar{\rho}(\mathfrak{p}))$$

in $\mathrm{Spc}(\mathcal{K}(G))$, where $\bar{\rho} = \mathrm{Spc}(\overline{\mathrm{Res}}^{G/\!\!/K}_{(G/\!\!/H)/\!\!/(K/H)})\colon V_{(G/\!\!/H)/\!\!/(K/H)} \longrightarrow V_{G/\!\!/K}$. In particular, if $H \trianglelefteq G$ is normal, we have

$$\psi^{H;G}(\mathcal{P}_{G/H}(K/H,\mathfrak{p})) = \mathcal{P}_G(K,\mathfrak{p})$$

in $\mathrm{Spc}(\mathcal{K}(G))$, using that $\mathfrak{p} \in V_{(G/H)/\!\!/(K/H)} = V_{G/\!\!/K}$.

Proof. This is immediate from Proposition 5.17 and Corollary 5.18. ☐

The relation between Koszul objects and modular fixed-points functors, obtained in Lemma 5.21, can be reformulated in terms of the primes $\mathcal{P}_G(H,\mathfrak{p})$.

7.12. **Lemma.** *Let $H \leqslant G$ be a p-subgroup and $\mathfrak{p} \in V_{G/\!\!/H}$. Let $K \leqslant G$ be a subgroup and $\mathrm{kos}_G(K)$ be the Koszul object of Construction 3.14. Then $\mathrm{kos}_G(K) \in \mathcal{P}_G(H,\mathfrak{p})$ if and only if $H \leqslant_G K$. (Note that the latter condition does not depend on $\mathfrak{p}$.)*

Proof. We have seen in Lemma 5.21 (b) that if $H \leqslant_G K$ then $\check{\Psi}^H(\mathrm{kos}_G(K)) = 0$ in $\mathrm{D_b}(k(G/\!\!/H))$, in which case $\mathrm{kos}_G(K) \in (\check{\Psi}^H)^{-1}(0) \subseteq (\check{\Psi}^H)^{-1}(\mathfrak{p}) = \mathcal{P}_G(H,\mathfrak{p})$ for every $\mathfrak{p}$. Conversely, we have seen in Lemma 5.21 (a) that if $H \not\leqslant_G K$ then $\check{\Psi}^H(\mathrm{kos}_G(K))$ generates $\mathrm{D_b}(k(G/\!\!/H))$, hence is not contained in any cohomological point $\mathfrak{p}$, in which case $\mathrm{kos}_G(K) \notin (\check{\Psi}^H)^{-1}(\mathfrak{p}) = \mathcal{P}_G(H,\mathfrak{p})$. ☐

7.13. **Corollary.** *If $\mathcal{P}_G(H,\mathfrak{p}) \subseteq \mathcal{P}_G(H',\mathfrak{p}')$ then $H' \leqslant_G H$. Therefore if $\mathcal{P}_G(H,\mathfrak{p}) = \mathcal{P}_G(H',\mathfrak{p}')$ then H and H' are conjugate in G.*

Proof. Apply Lemma 7.12 to $K = H$ twice, for H being once H and once H'. ☐

7.14. **Proposition.** *Let $H \leqslant G$ be a p-subgroup. Then the map $\check{\psi}^H\colon V_{G/\!\!/H} \to \mathrm{Spc}(\mathcal{K}(G))$ is injective, that is, $\mathcal{P}_G(H,\mathfrak{p}) = \mathcal{P}_G(H,\mathfrak{p}')$ implies $\mathfrak{p} = \mathfrak{p}'$.*

Proof. Let $N = N_G H$. By assumption we have $\rho_N^G(\mathcal{P}_N(H,\mathfrak{p})) = \rho_N^G(\mathcal{P}_N(H,\mathfrak{p}'))$. By Corollary 4.9, there exists $g \in G$ and a prime $\mathcal{Q} \in \mathrm{Spc}(\mathcal{K}(N \cap {}^gN))$ such that

$$\mathcal{P}_N(H,\mathfrak{p}) = \rho^N_{N\cap {}^gN}(\mathcal{Q}) \quad \text{and} \quad \mathcal{P}_N(H,\mathfrak{p}') = \big(\rho^{{}^gN}_{N\cap {}^gN}(\mathcal{Q})\big)^g. \tag{7.15}$$

By Corollary 7.2 for the group $N \cap {}^gN$, there exists a p-subgroup $L \leqslant N \cap {}^gN$ and some $\mathfrak{q} \in V_{(N\cap {}^gN)/\!\!/L}$ such that $\mathcal{Q} = \mathcal{P}_{N\cap {}^gN}(L,\mathfrak{q})$. By (7.7) we know where such a prime $\mathcal{P}_{N\cap {}^gN}(L,\mathfrak{q})$ goes under the maps $\rho = \mathrm{Spc}(\mathrm{Res})$ of (7.15) and, for the second one, we also know what happens under conjugation by Remark 7.6 (a). Applying these properties to the above relations (7.15) we get

$$\mathcal{P}_N(H,\mathfrak{p}) = \mathcal{P}_N(L,\mathfrak{q}') \quad \text{and} \quad \mathcal{P}_N(H,\mathfrak{p}') = \mathcal{P}_N(L^g,\mathfrak{q}'')$$

for suitable cohomological points $\mathfrak{q}' \in V_{N/\!\!/L}$ and $\mathfrak{q}'' \in V_{N/\!\!/L^g}$ that we do not need to unpack. By Corollary 7.13 applied to the group N, we must have $H \sim_N L$ and $H \sim_N L^g$. But since $H \trianglelefteq N$, this forces $H = L = L^g$ and therefore $g \in N_G H = N$. In that case, returning to (7.15), we have $N \cap N^g = N = N^g$ and therefore

$$\mathcal{P}_N(H,\mathfrak{p}) = \mathcal{Q} \quad \text{and} \quad \mathcal{P}_N(H,\mathfrak{p}') = \mathcal{Q}^g = \mathcal{Q}$$

where the last equality uses $g \in N$ and (4.2). Hence $\mathcal{P}_N(H,\mathfrak{p}) = \mathcal{Q} = \mathcal{P}_N(H,\mathfrak{p}')$. As H is normal in N the map $\psi^{H;N}\colon \mathrm{Spc}(\mathcal{K}(N/H)) \longrightarrow \mathrm{Spc}(\mathcal{K}(N))$ is split injective by Corollary 5.16, and we conclude that $\mathfrak{p} = \mathfrak{p}'$. ☐

We can now summarize our description of the set $\mathrm{Spc}(\mathcal{K}(G))$.

7.16. **Theorem.** *Every point in $\mathrm{Spc}(\mathcal{K}(G))$ is of the form $\mathcal{P}_G(H,\mathfrak{p})$ as in Definition 7.4, for some p-subgroup $H \leqslant G$ and some point $\mathfrak{p} \in V_{G/\!\!/H}$ of the cohomological open of the Weyl group of H in G. Moreover, we have $\mathcal{P}_G(H,\mathfrak{p}) = \mathcal{P}_G(H',\mathfrak{p}')$ if and only if there exists $g \in G$ such that $H' = H^g$ and $\mathfrak{p}' = \mathfrak{p}^g$.*

Proof. The first statement follows from Corollary 7.2. For the second statement, the "if"-direction follows from (7.8). For the "only if"-direction assume $\mathcal{P}_G(H,\mathfrak{p}) = \mathcal{P}_G(H',\mathfrak{p}')$. By Corollary 7.13, this forces $H \sim_G H'$. Using (7.8), we can replace H' by H^g and assume that $\mathcal{P}_G(H,\mathfrak{p}) = \mathcal{P}_G(H,\mathfrak{p}')$ for $\mathfrak{p},\mathfrak{p}' \in V_{G/\!\!/H}$. We can then conclude by Proposition 7.14. ☐

Here is an example of support, for the Koszul objects of Construction 3.14.

7.17. **Corollary.** *Let $K \leqslant G$. Then $\mathrm{supp}(\mathrm{kos}_G(K)) = \{\, \mathcal{P}(H,\mathfrak{p}) \,|\, H \not\leqslant_G K \,\}$.*

Proof. Since all primes are of the form $\mathcal{P}(H,\mathfrak{p})$, it is a simple contraposition on Lemma 7.12, for $\mathcal{P}(H,\mathfrak{p}) \in \mathrm{supp}(\mathrm{kos}_G(K)) \Leftrightarrow \mathrm{kos}_G(K) \notin \mathcal{P}(H,\mathfrak{p}) \Leftrightarrow H \not\leqslant_G K$. ☐

We can use this result to identify the image of ψ^H. First, in the normal case:

7.18. **Proposition.** *Let $H \trianglelefteq G$ be a normal p-subgroup. Then the continuous map*

$$\psi^H = \mathrm{Spc}(\Psi^H)\colon\ \mathrm{Spc}(\mathcal{K}(G/H)) \to \mathrm{Spc}(\mathcal{K}(G))$$

is a closed immersion, retracted by $\mathrm{Spc}(\mathrm{Infl}_G^{G/H})$. Its image is the closed subset

$$\mathrm{Im}(\psi^H) = \{\, \mathcal{P}_G(L,\mathfrak{p}) \,|\, H \leqslant L \in \mathrm{Sub}_p G,\ \mathfrak{p} \in V_{G/\!\!/L} \,\} = \cap_{K \ngeq H} \mathrm{supp}(\mathrm{kos}_G(K)) \tag{7.19}$$

Furthermore, this image of ψ^H is also the support of the object

$$\bigotimes_{K\in\mathcal{F}_H} \mathrm{kos}_G(K) \tag{7.20}$$

and it is also the support of the tt-ideal $\cap_{K\in\mathcal{F}_H} \mathrm{Ker}(\mathrm{Res}^G_K)$.

Proof. By Corollary 5.16, the map ψ^H has a continuous retraction hence is a closed immersion as soon as we know that its image is closed. So let us prove (7.19).

By Proposition 7.11 and the fact that all points are of the form $\mathcal{P}(L,\mathfrak{p})$, the image of ψ^H is the subset $\{\,\mathcal{P}_G(L,\mathfrak{p})\,|\,H\leqslant L,\ \mathfrak{p}\in V_{G/\!\!/L}\,\}$. Here we use $H\trianglelefteq G$.

Corollary 7.17 tells us that every such point $\mathcal{P}(L,\mathfrak{p})$ belongs to the support of $\mathrm{kos}_G(K)$ as long as $L\not\leqslant_G K$, which clearly holds if $H\leqslant L$ and $H\not\leqslant K$. Therefore $\mathrm{Im}(\psi^H)\subseteq\cap_{K\in\mathcal{F}_H}\mathrm{supp}(\mathrm{kos}_G(K))$.

Conversely, let $\mathcal{P}(L,\mathfrak{p})\in\cap_{K\in\mathcal{F}_H}\mathrm{supp}(\mathrm{kos}_G(K))$ and let us show that $H\leqslant L$. If *ab absurdo*, $H\not\leqslant L$ then $L\in\mathcal{F}_H$ is one of the indices K that appear in the intersection $\cap_{K\in\mathcal{F}_H}\mathrm{supp}(\mathrm{kos}_G(K))$. In other words, $\mathcal{P}(L,\mathfrak{p})\in\mathrm{supp}(\mathrm{kos}_G(L))$. By Corollary 7.17, this means $L\not\leqslant_G L$, which is absurd. Hence the result.

The 'furthermore part' follows: The first claim is (7.19) since $\mathrm{supp}(x)\cap\mathrm{supp}(y)=\mathrm{supp}(x\otimes y)$ and the second claim follows from Corollary 3.24. (For $H=1$, the result does not tell us much, as $\psi^1=\mathrm{id}$ and $\otimes_\varnothing=\mathbb{1}$.) ☐

Let us extend the above discussion to not necessarily normal subgroups H.

7.21. *Notation.* Let $H\leqslant G$ be an arbitrary subgroup. We define an object of $\mathcal{K}(G)$

$$\mathrm{zul}_G(H):=\mathrm{Ind}^G_{N_GH}\Bigg(\bigotimes_{\{\,K\leqslant N_GH\,|\,H\not\leqslant K\,\}}\mathrm{kos}_{N_G(H)}(K)\Bigg).\tag{7.22}$$

(Note that we use plain induction here, not tensor-induction as in Construction 3.14.) If $H\trianglelefteq G$ is normal this $\mathrm{zul}_G(H)$ is simply the object displayed in (7.20).

7.23. **Corollary.** *Let $H\leqslant G$ be a p-subgroup. Then the continuous map*

$$\psi^{H;G}=\mathrm{Spc}(\Psi^{H;G})\colon\ \mathrm{Spc}(\mathcal{K}(G/\!\!/H))\to\mathrm{Spc}(\mathcal{K}(G))$$

is a closed map, whose image is $\mathrm{supp}(\mathrm{zul}_G(H))$ *where* $\mathrm{zul}_G(H)$ *is as in* (7.22).

Proof. By definition $\Psi^{H;G}=\Psi^{H;N_GH}\circ\mathrm{Res}^G_{N_GH}$. We know the map induced on spectra by the second functor $\Psi^{H;N_GH}$ by Proposition 7.18 and we can describe what happens under the closed map Spc(Res) by Proposition 4.7. ☐

We record the answer to a question stated in the Introduction (Section 2):

7.24. **Corollary.** *The support of the tt-ideal of acyclics $\mathcal{K}_{\mathrm{ac}}(G)$ is the union of the images of the modular H-fixed-points maps ψ^H, for non-trivial p-subgroups $H\leqslant G$.*

Proof. The points of $\mathrm{Spc}(\mathcal{K}(G))$ are of the form $\mathcal{P}(H,\mathfrak{p})$. Such primes belong to $V_G=\{\,\mathcal{P}(1,\mathfrak{q})\,|\,\mathfrak{q}\in V_G\,\}$ if and only if H is trivial. The complement is then $\mathrm{Supp}(\mathcal{K}_{\mathrm{ac}}(G))$. Hence $\mathrm{Supp}(\mathcal{K}_{\mathrm{ac}}(G))\subseteq\cup_{H\neq1}\mathrm{Im}(\psi^H)$. Conversely, for every p-subgroup $H\neq1$, the object $\mathrm{zul}_G(H)$ of Corollary 7.23 is acyclic, since the tensor is non-empty and any $\mathrm{kos}_{N_GH}(K)$ is acyclic. So $\mathrm{Im}(\psi^H)\subseteq\mathrm{Supp}(\mathcal{K}_{\mathrm{ac}}(G))$. ☐

Let us now describe all closed points of $\mathrm{Spc}(\mathcal{K}(G))$.

7.25. *Remark.* Recall that in tt-geometry closed points $\mathcal{M}\in\mathrm{Spc}(\mathcal{K})$ are exactly the minimal primes for inclusion. Also every prime contains a minimal one.

For instance, the tt-category $\mathrm{D_b}(kG)$ is local, with a unique closed point $0=\mathrm{Ker}(\mathrm{D_b}(kG)\to\mathrm{D_b}(k))$. (In terms of homogeneous primes in $\mathrm{Spec^h}(\mathrm{H}^\bullet(G,k))$ the zero tt-ideal $\mathfrak{p}=0$ corresponds to the closed point $\mathfrak{p}^\bullet=\mathrm{H}^+(G,k)$.)

7.26. *Definition.* Let $H \leqslant G$ be a p-subgroup. (This definition only depends on the conjugacy class of H in G.) By Proposition 5.15, the following diagram commutes

$$\begin{array}{ccc} \mathcal{K}(G) & \xrightarrow{\mathrm{Res}^G_H} & \mathcal{K}(H) \\ {\scriptstyle \Psi^{H;G}}\downarrow & \searrow^{\mathbb{F}^H} & \downarrow{\scriptstyle \Psi^{H;H}} \\ \mathcal{K}(G/\!\!/H) & \xrightarrow[\mathrm{Res}^{G/\!\!/H}_1]{} & \mathcal{K}(1) = \mathrm{D_b}(k). \end{array} \tag{7.27}$$

We baptize $\mathbb{F}^H = \mathbb{F}^{H;G}$ the diagonal. Its kernel is one of the primes of Definition 7.4

$$\mathcal{M}(H) = \mathcal{M}_G(H) := \mathrm{Ker}(\mathbb{F}^H) = \mathcal{P}_G(H,0) \tag{7.28}$$

where $0 \in \mathrm{Spc}(\mathrm{D_b}(k(G/\!\!/H)))$ is the zero tt-ideal, *i.e.* the unique closed point of the cohomological open $V_{G/\!\!/H}$ of the Weyl group. (See Remark 7.25.) We can think of $\mathbb{F}^H\colon \mathcal{K}(G) \to \mathrm{D_b}(k)$ as a tt-residue field functor at the (closed) point $\mathcal{M}(H)$.

7.29. *Example.* For $H = 1$, we have $\mathcal{M}(1) = \mathrm{Ker}\left(\mathrm{Res}^G_1 : \mathcal{K}(G) \to \mathrm{D_b}(k)\right) = \mathcal{K}_{\mathrm{ac}}(G)$. In other words, $\mathcal{M}(1) = \Upsilon_G^{-1}(0)$ is the image under the open immersion $\upsilon_G\colon V_G \hookrightarrow \mathrm{Spc}(\mathcal{K}(G))$ of Proposition 3.22 of the unique closed point $0 \in V_G$ of Remark 7.25. In general, a closed point of an open is not necessarily closed in the ambient space. Here $\mathcal{M}(1)$ is closed since by definition $\{\mathcal{M}(1)\} = \mathrm{Im}(\rho^G_1)$ where $\rho^G_1 = \mathrm{Spc}(\mathrm{Res}^G_1)$. By Proposition 4.7, we know that $\mathrm{Im}(\rho^G_1) = \mathrm{supp}(k(G))$ is closed.

7.30. *Example.* For $H = G$ a p-group, we can give generators of the closed point

$$\mathcal{M}(G) = \langle k(G/K) \mid K \neq G \rangle.$$

As $\mathcal{M}(G) = \ker(\Psi^G : \mathcal{K}(G) \to \mathrm{D_b}(k))$, inclusion $\supseteq$ follows from Proposition 5.12. For $\subseteq$, let $X \in \mathcal{M}(G)$ be a complex that vanishes under Ψ^G. Splitting the modules X_n in each homological degree n into a trivial (*i.e.* a k-vector space with trivial action) and non-trivial permutation modules, Lemma 6.3 shows that X is homotopy equivalent to a complex in the additive category generated by $k(G/K)$, $K \neq G$.

7.31. **Corollary.** *The closed points of* $\mathrm{Spc}(\mathcal{K}(G))$ *are exactly the tt-primes* $\mathcal{M}_G(H)$ *of* (7.28) *for the p-subgroups $H \leqslant G$. Furthermore, we have $\mathcal{M}_G(H) = \mathcal{M}_G(H')$ if and only if H is conjugate to H' in G.*

Proof. Let us first verify that $\mathcal{M}_G(H)$ is closed for every $H \leqslant G$. For $H = 1$, we checked it in Example 7.29. For $H \neq 1$, we have $\mathcal{M}_G(H) = \mathcal{P}_G(H,0) = \Psi^H(\mathcal{M}_{G/\!\!/H}(1))$. This gives the result since $\mathcal{M}_{G/\!\!/H}(1)$ is closed in $\mathrm{Spc}(\mathcal{K}(G/\!\!/H))$, by Example 7.29 again, and since ψ^H is a closed map by Corollary 7.23.

Now, every point $\mathfrak{p} \in V_{G/\!\!/H}$ admits 0 in its closure in $\mathrm{Spc}(\mathrm{D_b}(k(G/\!\!/H))) = V_{G/\!\!/H}$. (See Remark 7.25.) By continuity of $\check{\psi}^H\colon V_{G/\!\!/H} \to \mathrm{Spc}(\mathcal{K}(G))$, it follows that $\check{\psi}^H(0) = \mathcal{M}_G(H)$ belongs to the closure of $\check{\psi}^H(\mathfrak{p}) = \mathcal{P}_G(H,\mathfrak{p})$, which proves that the $\mathcal{M}_G(H)$ are the only closed points.

We already saw that $\mathcal{P}(H,0) = \mathcal{P}(H',0)$ implies $H \sim_G H'$, in Theorem 7.16. ☐

We wrap up this section on the spectrum by discussing the strata defined by modular fixed-points.

7.32. **Proposition.** *For every p-subgroup $H \leqslant G$, consider the subset*

$$V_G(H) := \mathrm{Im}(\check{\psi}^H) = \check{\psi}^H(V_{G/\!\!/H})$$

of $\operatorname{Spc}(\mathcal{K}(G))$*. Then* $\mathcal{M}_G(H)$ *is the unique closed point of* $\operatorname{Spc}(\mathcal{K}(G))$ *that belongs to* $V_G(H)$*. We have a set-partition indexed by conjugacy classes of* p*-subgroups*

$$\operatorname{Spc}(\mathcal{K}(G)) = \coprod_{H\in(\operatorname{Sub}_p G)/G} V_G(H) \tag{7.33}$$

where each $V_G(H)$ *is open in its closure.*

Proof. The partition is immediate from Theorem 7.16. Each subset $V_G(H) = \{\mathcal{P}(H,\mathfrak{p}) \,|\, \mathfrak{p} \in V_{G/\!\!/H}\}$ is a subset of the closed set $\operatorname{Im}(\psi^H)$. By Corollary 7.24 and Proposition 7.11, the complement of $V_G(H)$ in $\operatorname{Im}(\psi^H)$ consists of the images $\operatorname{Im}(\psi^K)$ for every 'further' p-group K, *i.e.* such that $H \lneqq K \leqslant N_G(H)$ and these are closed by Corollary 7.23. Thus $V_G(H)$ is an open in the closed subset $\operatorname{Im}(\psi^H)$. $\square$

7.34. *Remark.* We can use (7.33) to define a map $\operatorname{Spc}(\mathcal{K}(G)) \to (\operatorname{Sub}_p G)/G$. Corollary 7.13 tells us that this map is continuous for the (Alexandrov) topology on $(\operatorname{Sub}_p G)/G$ whose open subsets are the ones stable under subconjugacy.

Moreover, for $H \leqslant G$ a p-subgroup, the square

$$\begin{array}{ccc} \operatorname{Spc}(\mathcal{K}(G/\!\!/H)) & \xrightarrow[\psi^H]{} & \operatorname{Spc}(\mathcal{K}(G)) \\ \downarrow & & \downarrow \\ (\operatorname{Sub}_p(G/\!\!/H))/(G/\!\!/H) & \hookrightarrow & (\operatorname{Sub}_p G)/G \end{array}$$

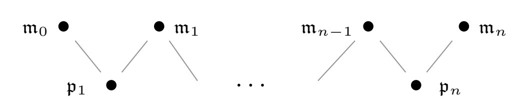
commutes, where the bottom horizontal arrow is the canonical inclusion that sends $H \leqslant K \leqslant N_G(H)$ to K. This follows from Proposition 7.11. Consequently, while ψ^H might not be injective in general, we still have $(\psi^H)^{-1}(V_G(H)) = V_{G/\!\!/H}$.

8. Examples

Although the full treatment of the topology of $\operatorname{Spc}(\mathcal{K}(G))$ will require the additional technology of Part II, we can already present the answer for small groups. Some of the most interesting phenomena are already visible once we reach p-rank two in Example 8.10. Let us start with the easy examples.

8.1. *Notation.* Fix an integer $n \geq 0$ and consider the following space $\mathbb{W}^n$ consisting of $2n+1$ points, with specialization relations pointing upward as usual:

(8.2) $\mathbb{W}^n =$



The closed subsets of $\mathbb{W}^n$ are simply the specialization-closed subsets, *i.e.* those that contain a $\mathfrak{p}_i$ only if they contain $\mathfrak{m}_{i-1}$ and $\mathfrak{m}_i$. So the $\mathfrak{m}_i$ are closed points and the $\mathfrak{p}_i$ are generic points of the n irreducible V-shaped closed subsets $\{\mathfrak{m}_{i-1}, \mathfrak{p}_i, \mathfrak{m}_i\}$.

8.3. **Proposition.** *Let* $G = C_{p^n}$ *be a cyclic* p*-group. Then* $\operatorname{Spc}(\mathcal{K}(C_{p^n}))$ *is homeomorphic to the space* $\mathbb{W}^n$ *of* (8.2)*.*

More precisely, if we denote by $1 = N_n < N_{n-1} < \cdots < N_0 = G$ *the* $n+1$ *subgroups of* C_{p^n} ([5])*, then the points* $\mathfrak{p}_i$ *and* $\mathfrak{m}_i$ *in* $\operatorname{Spc}(\mathcal{K}(G))$ *are given by*

$$\mathfrak{m}_i = (\check{\Psi}^{N_i})^{-1}(0) \qquad \textit{and} \qquad \mathfrak{p}_i = (\check{\Psi}^{N_i})^{-1}(\mathrm{D}_{\mathrm{perf}}(k(G/N_i)))$$

where $\check{\Psi}^N = \Upsilon_{G/N} \circ \Psi^N \colon \mathcal{K}(G) \to \mathcal{K}(G/N) \twoheadrightarrow \mathrm{D}_{\mathrm{b}}(k(G/N))$ *is the tt-functor* (6.11)*.*

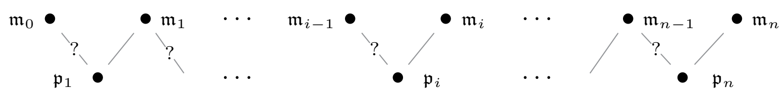
[5] The numbering of the N_i keeps track of the index, that is, $G/N_i \cong C_{p^i}$. This choice will allow simple formulas for inflation and fixed-points, and for procyclic groups in Part III [BG23b].

Proof. By Proposition 7.32, we have a partition of the spectrum in subsets

$$\mathrm{Spc}(\mathcal{K}(G)) = \coprod_{i=0}^{n} V_G(N_i) = \coprod_{i=0}^{n} \mathrm{Im}(\check{\psi}^{N_i})$$

and each $V_G(N_i)$ is homeomorphic to $\mathrm{Spc}(\mathrm{D_b}(kG/N_i)) = V_{G/N_i}$. For $i > 0$, each $V_G(N_i)$ is a Sierpiński space $\{\mathfrak{p}_i \rightsquigarrow \mathfrak{m}_i = \mathcal{M}(N_i)\}$, while $V_G(N_0)$ is a singleton set $\{\mathfrak{m}_0 := \mathcal{M}(G)\}$. In other words, we know the set $\mathrm{Spc}(\mathcal{K}(G))$ has the announced $2n+1$ points and the unmarked specializations $\mathfrak{p}_i \rightsquigarrow \mathfrak{m}_i$ below

(8.4)



We need to elucidate the topology. Since all $\mathfrak{m}_i = \mathcal{M}(N_i)$ are closed (Corollary 7.31), we only need to see where each $\mathfrak{p}_i$ specializes for $1 \leqslant i \leqslant n$. By Corollary 7.13, the point $\mathfrak{p}_i = \mathcal{P}(N_i, \mathfrak{p})$ can only specialize to a $\mathcal{P}(N_j, \mathfrak{q})$ for $N_j \geq N_i$, that is, to the points $\mathfrak{m}_j$ or $\mathfrak{p}_j$ for $j \leqslant i$. On the other hand, direct inspection using (5.13) shows that $\mathrm{supp}(k(G/N_{i-1})) = \{\, \mathfrak{m}_j \,|\, j \geq i-1 \,\} \cup \{\, \mathfrak{p}_j \,|\, j \geq i \,\}$. This closed subset contains $\mathfrak{p}_i$ hence its closure. Combining those two observations, we have

$$\overline{\{\mathfrak{p}_i\}} \subseteq \{\, \mathfrak{m}_j, \mathfrak{p}_j \,|\, j \leqslant i \,\} \cap \big(\{\mathfrak{m}_{i-1}\} \cup \{\, \mathfrak{m}_j, \mathfrak{p}_j \,|\, j \geq i \,\}\big) = \{\mathfrak{m}_{i-1}, \mathfrak{p}_i, \mathfrak{m}_i\}.$$

If any of the $\overline{\{\mathfrak{p}_i\}}$ was smaller than $\{\mathfrak{m}_{i-1}, \mathfrak{p}_i, \mathfrak{m}_i\}$, that is, if one of the specialization relations $\mathfrak{p}_i \rightsquigarrow \mathfrak{m}_{i-1}$ marked with '?' in (8.4) did not hold, then $\mathrm{Spc}(\mathcal{K}(G))$ would be a disconnected space. This would force the rigid tt-category $\mathcal{K}(G)$ to be the product of two tt-categories, which is clearly absurd, *e.g.* because $\mathrm{End}_{\mathcal{K}(G)}(\mathbb{1}) = k$. □

With this identification, we can record the maps ψ^H of Definition 7.3 and the maps ρ_K and $\pi^{G/N}$ of Remark 7.6, that relate different cyclic p-groups.

8.5. **Lemma.** *Let $n \geq 0$. We identify $\mathrm{Spc}(\mathcal{K}(C_{p^n}))$ with $\mathbb{W}^n$ as in Proposition 8.3.*
(a) *Let $0 \leqslant i \leqslant n$ and $H = N_i = C_{p^{n-i}} \leqslant C_{p^n}$, so that $C_{p^n}/H \cong C_{p^i}$. The map $\psi^H \colon \mathbb{W}^i \to \mathbb{W}^n$ induced by modular fixed points Ψ^H is the inclusion*

$$\psi \colon \mathbb{W}^i \hookrightarrow \mathbb{W}^n.$$

that catches the left-most points: $\mathfrak{p}_\ell \mapsto \mathfrak{p}_\ell$ and $\mathfrak{m}_\ell \mapsto \mathfrak{m}_\ell$.
(b) *Let $0 \leqslant j \leqslant n$ and $K = C_{p^j} \leqslant C_{p^n}$. The map $\rho_K \colon \mathbb{W}^j \to \mathbb{W}^n$ induced by restriction Res_K is the inclusion*

$$\rho : \mathbb{W}^j \hookrightarrow \mathbb{W}^n$$

that catches the right-most points: $\mathfrak{m}_\ell \mapsto \mathfrak{m}_{\ell+n-j}$ and $\mathfrak{p}_\ell \mapsto \mathfrak{p}_{\ell+n-j}$.
(c) *Let $0 \leqslant m \leqslant n$. Inflation along $C_{p^n} \twoheadrightarrow C_{p^m}$ induces on spectra the map*

$$\pi \colon \mathbb{W}^n \twoheadrightarrow \mathbb{W}^m$$

that retracts ψ and sends everything else to $\mathfrak{m}_m$, that is, for all $0 \leqslant \ell \leqslant n$

$$\pi(\mathfrak{p}_\ell) = \begin{cases} \mathfrak{p}_\ell & \text{if } \ell \leqslant m \\ \mathfrak{m}_m & \text{otherwise} \end{cases} \qquad \text{and} \qquad \pi(\mathfrak{m}_\ell) = \begin{cases} \mathfrak{m}_\ell & \text{if } \ell \leqslant m \\ \mathfrak{m}_m & \text{otherwise.} \end{cases}$$

Proof. Part (a) follows from Proposition 7.11, while parts (b) and (c) follow from Remark 7.6. □

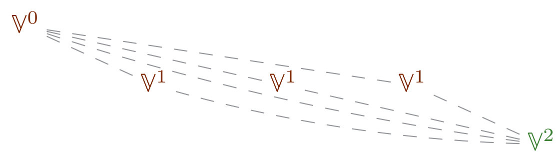
Let us now move to higher p-rank.

8.6. *Example.* Let $E = (C_p)^{\times r}$ be the elementary abelian p-group of rank r. We know that $V_E = \mathrm{Spc}(\mathrm{D_b}(kE)) \cong \mathrm{Spec^h}(\mathrm{H}^\bullet(E,k))$ is homeomorphic to the space

$$\mathbb{V}^r := \mathrm{Spec^h}(k[x_1,\ldots,x_r]), \tag{8.7}$$

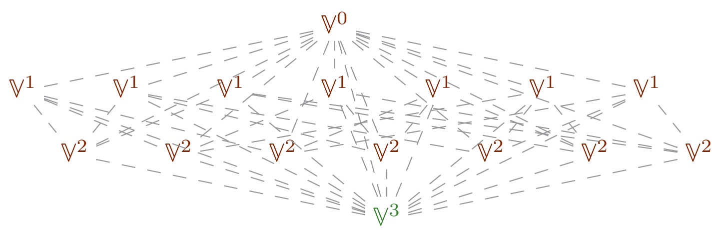
that is, projective space $\mathbb{P}_k^{r-1}$ with one closed point 'on top'. For instance, $\mathbb{V}^0$ is a single point and $\mathbb{V}^1$ is a 2-point Sierpiński space. The example of $r = 1$ (see Proposition 8.3 for $n = 1$) is not predictive of what happens in higher rank. Indeed, by Proposition 7.32, the closed complement $\mathrm{Supp}(\mathcal{K}_{\mathrm{ac}}(E))$ is far from discrete in general. It contains $\frac{p^r-1}{p-1}$ copies of $\mathbb{V}^{r-1}$ and more generally $|\mathrm{Gr}_p(d,r)|$ copies of the d-dimensional $\mathbb{V}^d$ for $d = 0,\ldots,r-1$, where $|\mathrm{Gr}_p(d,r)|$ is the number of rank-d subgroups of $(C_p)^{\times r}$. Here is a 'low-resolution' picture for Klein-four $r = p = 2$:

(8.8)



The dashed lines indicate 'partial' specialization relations: *Some* points in the lower variety specialize to *some* points in the higher one; see Corollary 7.13. In rank 3, the similar 'low-resolution' picture of $\mathrm{Spc}(\mathcal{K}(C_2^{\times 3}))$, still for $p = 2$, looks as follows:

(8.9)

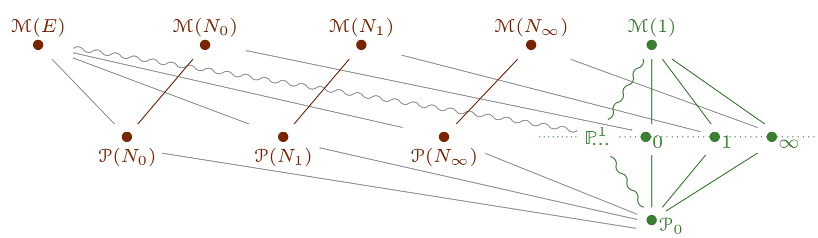


Each $\mathbb{V}^d$ has Krull dimension $d \in \{0,1,2,3\}$ and contains one of 16 closed points.

Let us now discuss the example of Klein-four and 'zoom-in' on (8.8) to display every point at its actual height, as well as all specialization relations.

8.10. *Example.* Let $G = C_2 \times C_2$ be the Klein four-group, in characteristic $p = 2$. In Example 16.16, we shall see that the spectrum $\mathrm{Spc}(\mathcal{K}(E))$ is exactly as follows:

(8.11)



In this picture, N_0, N_1 and N_∞ are the three cyclic subgroups of G. The colors match those of (8.8). The green part is the cohomological open $V_E \simeq \mathbb{V}^2$ as in (8.7), that is, a $\mathbb{P}^1$ with a closed point on top; we marked with $\bullet$ the closed point $\mathcal{M}(1)$, the three $\mathbb{F}_2$-rational points 0, 1, ∞ of $\mathbb{P}^1$ and its generic point $\mathcal{P}_0$; the notation $\mathbb{P}^1_{\because}$ and the dotted line indicate $\mathbb{P}^1 \smallsetminus \{0,1,\infty,\mathcal{P}_0\}$. The brown part is the support of the acyclics, namely the union of the $V_E(H)$ for non-trivial subgroups $H \leqslant E$ as in Proposition 7.32; it consists of three Sierpiński subspaces $\{\mathcal{P}(N_i) \rightsquigarrow \mathcal{M}(N_i)\} \simeq V_{E/N_i} \simeq \mathbb{V}^1$ and the singleton $\{\mathcal{M}(E)\} \simeq V_{E/E} \simeq \mathbb{V}^0$.

The specializations involving points of $\mathbb{P}^1_{\therefore}$ are displayed with undulated lines, indicating that all points share the same behavior. For instance, the gray undulated line indicates that *all* points of $\mathbb{P}^1_{\therefore}$ specialize to $\mathcal{M}(E)$. The proof of this critical fact will require the new tools of Part II.

8.12. *Example.* The spectrum of the quaternion group Q_8 is very similar to that of its quotient $E := Q_8/Z(Q_8) \cong C_2 \times C_2$, as we announced in (2.13). The center $Z := Z(Q_8) \cong C_2$ is the maximal elementary abelian 2-subgroup and it follows that $\mathrm{Res}^{Q_8}_Z$ induces a homeomorphism $V_{C_2} \xrightarrow{\sim} V_{Q_8}$. In other words, V_{Q_8} is again a Sierpiński space $\{\mathcal{P}, \mathcal{M}(1)\}$. On the other hand, the center Z is also the unique minimal non-trivial subgroup. It follows from Corollary 7.2 and Proposition 7.18 that $\mathrm{Supp}(\mathcal{K}_{\mathrm{ac}}(Q_8))$ is the image under the closed immersion ψ^Z of $\mathrm{Spc}(\mathcal{K}(Q_8/Z))$. It only remains to describe the specialization relations between the cohomological open V_{Q_8} and its closed complement $\mathrm{Supp}(\mathcal{K}_{\mathrm{ac}}(Q_8))$. Since $\mathcal{M}(1) \in V_{Q_8}$ is also a closed point in $\mathrm{Spc}(\mathcal{K}(Q_8))$, we only need to decide where the generic point $\mathcal{P}$ of V_{Q_8} specializes in $\mathrm{Spc}(\mathcal{K}(Q_8))$. Interestingly, $\mathcal{P}$ will not be generic in the whole of $\mathrm{Spc}(\mathcal{K}(Q_8))$. As $\mathcal{P}$ belongs to $\mathrm{Im}(\rho_Z)$, it suffices to determine $\rho_Z(\mathcal{M}_{C_2}(C_2))$. The preimage of $\mathrm{Im}(\rho_Z) = \mathrm{supp}(k(Q_8/Z))$ under ψ^Z is $\mathrm{supp}_E(\Psi^Z(k(Q_8/Z))) = \mathrm{supp}_E(k(E)) = \{\mathcal{M}_E(1)\}$. It follows that $\mathcal{P}$ specializes to exactly one point: $\psi^Z(\mathcal{M}_E(1)) = \mathcal{M}_{Q_8}(Z)$ as depicted in (2.13).

9. Stratification

It is by now well-understood how to deduce stratification in the presence of a noetherian spectrum and a conservative theory of supports. We follow the general method of Barthel-Heard-Sanders [BHS22, BHS21].

9.1. **Proposition.** *The spectrum* $\mathrm{Spc}(\mathcal{K}(G))$ *is a noetherian topological space.*

Proof. Recall that a space is noetherian if every open is quasi-compact. It follows that the continuous image of a noetherian space is noetherian. The claim now follows from Corollary 7.2. □

We start with the key technical fact. Recall that coproduct-preserving exact functors between compactly-generated triangulated categories have right adjoints by Brown-Neeman Representability. We apply this to Ψ^H.

9.2. **Lemma.** *Let* $N \trianglelefteq G$ *be a normal* p*-subgroup and* $\Psi^N_\rho \colon \mathrm{DPerm}(G/N;k) \to \mathrm{DPerm}(G;k)$ *the right adjoint of modular* N*-fixed points* $\Psi^N \colon \mathrm{DPerm}(G;k) \to \mathrm{DPerm}(G/N;k)$*. Then* $\Psi^N_\rho(\mathbb{1})$ *is isomorphic to a complex* s *in* $\mathrm{perm}(G;k)$*, concentrated in non-negative degrees*

$$s = \quad \big(\cdots \to s_n \to \cdots \to s_2 \to s_1 \to s_0 \to 0 \to 0 \cdots\big)$$

with $s_0 = k$ *and* $s_1 = \oplus_{H \in \mathcal{F}_N} k(G/H)$*, where* $\mathcal{F}_N = \{\, H \leqslant G \,|\, N \not\leqslant H \,\}$.

Proof. Following the recipe of Brown-Neeman Representability [Nee96], we give an explicit description of $\Psi^N_\rho(\mathbb{1})$ as the homotopy colimit in $\mathcal{T}(G)$ of a sequence of objects $x_0 = \mathbb{1} \xrightarrow{f_0} x_1 \xrightarrow{f_1} \cdots \to x_n \xrightarrow{f_n} x_{n+1} \to \cdots$ in $\mathcal{K}(G)$. This sequence is built together with maps $g_n\colon \Psi^N(x_n) \to \mathbb{1}$ in $\mathcal{K}(G/N)$ making the following commute

$$\begin{array}{ccccccc} \Psi^N(x_0) = \mathbb{1} & \xrightarrow{\Psi^N(f_0)} \cdots \longrightarrow & \Psi^N(x_n) & \xrightarrow{\Psi^N(f_n)} & \Psi^N(x_{n+1}) & \cdots \\ & {\scriptstyle g_0=\mathrm{id}} \searrow \quad \cdots & \downarrow {\scriptstyle g_n} & \swarrow {\scriptstyle g_{n+1}} & \cdots & \\ & & \mathbb{1} & & & \end{array} \tag{9.3}$$

Note that such g_n yield homomorphisms, natural in $t \in \mathrm{DPerm}(G;k)$, as follows

$$\alpha_{n,t}\colon \mathrm{Hom}_G(t,x_n) \xrightarrow{\Psi^N} \mathrm{Hom}_{G/N}(\Psi^N(t),\Psi^N(x_n)) \xrightarrow{(g_n)_*} \mathrm{Hom}_{G/N}(\Psi^N(t),\mathbb{1}) \tag{9.4}$$

where we abbreviate Hom_G for $\mathrm{Hom}_{\mathrm{DPerm}(G;k)}$. We are going to build our sequence of objects $x_0 \to x_1 \to \cdots$ and the maps g_n so that for each $n \geq 0$

$$\alpha_{n,t} \text{ is an isomorphism for every } t \in \left\{ \Sigma^i\, k(G/H) \,\middle|\, i < n,\ H \leqslant G \right\}. \tag{9.5}$$

It follows that, if we set $x_\infty = \mathrm{hocolim}_n\, x_n$ and $g_\infty\colon \Psi^N(x_\infty) \cong \mathrm{hocolim}_n\, \Psi^N(x_n) \to \mathbb{1}$ the colimit of the g_n, then the map

$$\alpha_t\colon \mathrm{Hom}_G(t,x_\infty) \xrightarrow{\Psi^N} \mathrm{Hom}_{G/N}(\Psi^N(t),\Psi^N(x_\infty)) \xrightarrow{(g_\infty)_*} \mathrm{Hom}_{G/N}(\Psi^N(t),\mathbb{1})$$

is an isomorphism for all $t \in \left\{ \Sigma^i\, k(G/H) \,\middle|\, i \in \mathbb{Z},\ H \leqslant G \right\}$. Since the $k(G/H)$ generate $\mathrm{DPerm}(G;k)$, it follows that α_t is an isomorphism for all $t \in \mathrm{DPerm}(G;k)$. Hence $x_\infty = \mathrm{hocolim}_n\, x_n$ is indeed the image of $\mathbb{1}$ by the right adjoint Ψ^N_ρ.

Let us construct these sequences x_n, f_n and g_n, for $n \geq 0$. In fact, every complex x_n will be concentrated in degrees between zero and n, so that (9.5) is trivially true for $n = 0$ (that is, for $i < 0$), both source and target of $\alpha_{n,t}$ being zero in that case. Furthermore, x_{n+1} will only differ from x_n in degree $n+1$, with f_n being the identity in degrees $\leqslant n$. So the verification of (9.5) for $n+1$ will boil down to checking the cases of $t = \Sigma^i\, k(G/H)$ for $i = n$.

As indicated, we set $x_0 = \mathbb{1}$ and $g_0 = \mathrm{id}$. We define x_1 by the exact triangle

$$s_1 \xrightarrow{\epsilon} \mathbb{1} \xrightarrow{f_0} x_1 \to \Sigma(s_1)$$

where $s_1 := \oplus_{H \in \mathcal{F}_N} k(G/H)$ and $\epsilon_H\colon k(G/H) \to k$ is the usual map. Note that $\Psi^N(s_1) = 0$ by (5.13), hence $\Psi^N(f_0)\colon \mathbb{1} \to \Psi^N(x_1)$ is an isomorphism. We call g_1 its inverse. One verifies that (9.5) holds for $n = 1$: For $t = k(G/H)$ with $H \in \mathcal{F}_N$, both the source and target of $\alpha_{1,t}$ are zero thanks to the definition of s_1. For the case where $H \geq N$, there are no non-zero homotopies for maps $k(G/H) \to x_1$ thanks to Lemma 5.3.

Let us construct x_{n+1} and g_{n+1} for $n \geq 1$. For every $H \leqslant G$ let $t = \Sigma^n(k(G/H))$ and choose generators $h_{H,1}, \ldots, h_{H,r_H} : t \to x_n$ of the k-module $\mathrm{Hom}_G(t,x_n)$, source of $\alpha_{n,t}$. Define $s_{n+1} = \oplus_{H \leqslant G} \oplus_{i=1}^{r_H} k(G/H)$ in $\mathrm{perm}(G;k)$, a sum of r_H copies of $k(G/H)$ for every $H \leqslant G$, and define $h_n\colon \Sigma^n(s_{n+1}) \to x_n$ as being $h_{H,i}$ on the i-th summand $\Sigma^n\, k(G/H)$. Define x_{n+1} as the cone of h_n in $\mathcal{K}(G)$:

$$\Sigma^n(s_{n+1}) \xrightarrow{h_n} x_n \xrightarrow{f_n} x_{n+1} \to \Sigma^{n+1}(s_{n+1}). \tag{9.6}$$

Note that x_{n+1} only differs from x_n in homological degree $n+1$ as announced. Since $n \geq 1$, we get $\mathrm{Hom}_{G/N}(\Psi^N(x_{n+1}),\mathbb{1}) \cong \mathrm{Hom}_{G/N}(\Psi^N(x_n),\mathbb{1})$ and there exists a unique $g_{n+1}\colon \Psi^N(x_{n+1}) \to \mathbb{1}$ making (9.3) commute. It remains to verify that $\alpha_{n+1,t}$ is an isomorphism for $t \in \left\{ \Sigma^n\, k(G/H) \,\middle|\, H \leqslant G \right\}$. Note that the target of this map is zero. Applying $\mathrm{Hom}_G(\Sigma^n\, k(G/H), -)$ to the exact triangle (9.6) shows that the source of $\alpha_{n+1,t}$ is also zero, by construction. Hence (9.5) holds for $n+1$.

This realizes the wanted sequence and therefore $\Psi_\rho^N(\mathbb{1}) \simeq \mathrm{hocolim}_n(x_n)$ has the following form:

$$\cdots \to s_n \to \cdots \to s_2 \to s_1 \to k \to 0 \to 0 \cdots$$

where $s_1 = \oplus_{H\in\mathcal{F}_N} k(G/H)$ and $s_n \in \mathrm{perm}(G;k)$ for all n. □

9.7. *Remark.* The above description of $\Psi_\rho^N(\mathbb{1})$ gives a formula for the right adjoint $\Psi_\rho^N\colon \mathrm{DPerm}(G/N;k) \to \mathrm{DPerm}(G;k)$ on all objects. Indeed, for every $t \in \mathrm{DPerm}(G/N;k)$, we have a canonical isomorphism in $\mathrm{DPerm}(G;k)$

$$\Psi_\rho^N(t) \cong \Psi_\rho^N(\Psi^N \mathrm{Infl}_G^{G/N}(t) \otimes \mathbb{1}) \cong \mathrm{Infl}_G^{G/N}(t) \otimes \Psi_\rho^N(\mathbb{1})$$

using that $\Psi^N \circ \mathrm{Infl}_G^{G/N} \cong \mathrm{Id}$ and the projection formula. In other words, the right adjoint Ψ_ρ^N is simply inflation tensored with the commutative ring object $\Psi_\rho^N(\mathbb{1})$.

9.8. **Lemma.** *Let $H \trianglelefteq G$ be a normal p-subgroup and $\Psi_\rho^H\colon \mathrm{DPerm}(G/H;k) \to \mathrm{DPerm}(G;k)$ the right adjoint of modular H-fixed points $\Psi^H\colon \mathrm{DPerm}(G;k) \to \mathrm{DPerm}(G/H;k)$. Then the object $\mathrm{zul}_G(H)$ displayed in* (7.20) *belongs to the localizing tt-ideal of $\mathrm{DPerm}(G;k)$ generated by $\Psi_\rho^H(\mathbb{1})$.*

Proof. By Proposition 7.18, we know that the tt-ideal generated by $\mathrm{zul}_G(H)$ is exactly $\cap_{K\in\mathcal{F}_H} \mathrm{Ker}\,\mathrm{Res}^G_K$. By Frobenius, the latter is the tt-ideal $\{\, x \in \mathcal{K}(G) \,|\, s_1 \otimes x = 0 \,\}$ where $s_1 = \oplus_{K\in\mathcal{F}_H} k(G/K)$ is the degree one part of the complex $s \simeq \Psi_\rho^H(\mathbb{1})$ of Lemma 9.2. We can now conclude by Lemma 3.19 applied to this complex s and $x = \mathrm{zul}_G(H)$ that x must belong to the localizing tensor-ideal of $\mathrm{DPerm}(G;k)$ generated by $\Psi_\rho^H(\mathbb{1})$. (Note that $s_0 = \mathbb{1}$ here.) □

Recall from Corollary 7.23 that the map ψ^H has closed image in $\mathrm{Spc}(\mathcal{K}(G))$.

9.9. **Proposition.** *Let $H \leqslant G$ be a p-subgroup and let $\Psi_\rho^H\colon \mathrm{DPerm}(G/\!\!/H;k) \to \mathrm{DPerm}(G;k)$ be the right adjoint of $\Psi^H\colon \mathrm{DPerm}(G;k) \to \mathrm{DPerm}(G/\!\!/H;k)$. Then the tt-ideal of $\mathcal{K}(G)$ supported on the closed subset $\mathrm{Im}(\psi^H)$ is contained in the localizing tt-ideal of $\mathrm{DPerm}(G;k)$ generated by $\Psi_\rho^H(\mathbb{1})$.*

Proof. Let $N = N_G H$. By definition, $\Psi^{H;G} = \Psi^{H;N} \circ \mathrm{Res}^G_N$ and therefore the right adjoint is $\Psi_\rho^{H;G} \cong \mathrm{Ind}_N^G \circ \Psi_\rho^{H;N}$. By Lemma 9.8, we can handle $H \trianglelefteq N$ hence we know (see also Proposition 7.18) that the generator $\mathrm{zul}_N(H)$ of the tt-ideal supported on $\mathrm{Im}(\psi^{H;N})$ belongs to $\mathrm{Loc}_\otimes(\Psi_\rho^{H;N}(\mathbb{1}))$ in $\mathrm{DPerm}(N;k)$. Applying Ind_N^G and using the fact that Res^G_N is surjective up to direct summands (by separability), we see that $\mathrm{zul}_G(H) \overset{\mathrm{def}}{=} \mathrm{Ind}_N^G(\mathrm{zul}_N(H))$ belongs to $\mathrm{Ind}_N^G(\mathrm{Loc}_\otimes(\Psi_\rho^{H;N}(\mathbb{1})) \subseteq \mathrm{Loc}_\otimes(\mathrm{Ind}_N^G \Psi_\rho^{H;N}(\mathbb{1})) = \mathrm{Loc}_\otimes(\Psi_\rho^{H;G}(\mathbb{1}))$ in $\mathrm{DPerm}(G;k)$. □

Let us now turn to stratification. By noetherianity, we can define a support for possibly non-compact objects in the 'big' tt-category under consideration, here $\mathrm{DPerm}(G;k)$, following Balmer-Favi [BF11, § 7]. We remind the reader.

9.10. *Recollection.* Every Thomason subset $Y \subseteq \mathrm{Spc}(\mathcal{K}(G))$ yields a so-called 'idempotent triangle' $e(Y) \to \mathbb{1} \to f(Y) \to \Sigma e(Y)$ in $\mathcal{T}(G) = \mathrm{DPerm}(G;k)$, meaning that $e(Y) \otimes f(Y) = 0$, hence $e(Y) \cong e(Y)^{\otimes 2}$ and $f(Y) \cong f(Y)^{\otimes 2}$. The left idempotent $e(Y)$ is the generator of $\mathrm{Loc}_\otimes(\mathcal{K}(G)_Y)$, the localizing tt-ideal of $\mathcal{T}(G)$ 'supported' on Y. The right idempotent $f(Y)$ realizes localization of $\mathcal{T}(G)$ 'away' from Y, that is, the localization on the complement Y^c.

By noetherianity, for every point $\mathcal{P} \in \operatorname{Spc}(\mathcal{K}(G))$, the closed subset $\overline{\{\mathcal{P}\}}$ is Thomason. Hence $\overline{\{\mathcal{P}\}} \cap (Y_{\mathcal{P}})^c = \{\mathcal{P}\}$, where $Y_{\mathcal{P}} := \operatorname{supp}(\mathcal{P}) = \{\, \mathcal{Q} \,|\, \mathcal{P} \not\subseteq \mathcal{Q} \,\}$ is always a Thomason subset. The idempotent $g(\mathcal{P})$ in $\mathcal{T}(G)$ is then defined as

$$g(\mathcal{P}) = e(\overline{\{\mathcal{P}\}}) \otimes f(Y_{\mathcal{P}}).$$

It is built to capture the part of $\operatorname{DPerm}(G;k)$ that lives both 'over $\overline{\{\mathcal{P}\}}$' (thanks to $e(\overline{\{\mathcal{P}\}})$) and 'over $Y_{\mathcal{P}}^c$' (thanks to $f(Y_{\mathcal{P}})$); in other words, $g(\mathcal{P})$ lives exactly 'at $\mathcal{P}$'. This idea originates in [HPS97]. It explains why the support is defined as

$$\operatorname{Supp}(t) = \{\, \mathcal{P} \in \operatorname{Spc}(\mathcal{K}(G)) \,|\, g(\mathcal{P}) \otimes t \neq 0 \,\}$$

for every (possibly non-compact) object $t \in \operatorname{DPerm}(G;k)$.

9.11. **Theorem.** *Let G be a finite group and let k be a field. Then the big tt-category $\mathcal{T}(G) = \operatorname{DPerm}(G;k)$ is stratified, that is, we have an order-preserving bijection*

$$\{\textit{Localizing tt-ideals } \mathcal{L} \subseteq \mathcal{T}(G)\} \xleftrightarrow{\sim} \{\textit{Subsets of } \operatorname{Spc}(\mathcal{K}(G))\}$$

given by sending a subcategory $\mathcal{L}$ to the union of the supports of its objects; its inverse sends a subset $Y \subseteq \operatorname{Spc}(\mathcal{K}(G))$ to $\mathcal{L}_Y := \{\, t \in \mathcal{T}(G) \,|\, \operatorname{Supp}(t) \subseteq Y \,\}$.

Proof. By induction on the order of the group, we can assume that the result holds for every proper subquotient $G/\!\!/H$ (with $H \neq 1$). By [BHS21, Theorem 3.21], noetherianity of the spectrum of compacts reduces stratification to proving *minimality* of $\operatorname{Loc}_\otimes(g(\mathcal{P}))$ for every $\mathcal{P} \in \operatorname{Spc}(\mathcal{K}(G))$. This means that $\operatorname{Loc}_\otimes(g(\mathcal{P}))$ admits no non-trivial localizing tt-ideal subcategory. If $\mathcal{P}$ belongs to the cohomological open $V_G = \operatorname{Spc}(\mathrm{D_b}(kG))$ then minimality at $\mathcal{P}$ in $\mathcal{T} = \operatorname{DPerm}(G;k)$ is equivalent to minimality at $\mathcal{P}$ in $\mathcal{T}(V_G) \cong \operatorname{K}\operatorname{Inj}(kG)$ by [BHS21, Proposition 5.2]. Since $\operatorname{K}\operatorname{Inj}(kG)$ is stratified by [BIK11], we have the result in that case.

Let now $\mathcal{P} \in \operatorname{Supp}(\mathcal{K}_{\mathrm{ac}}(G))$. By Corollary 7.24, we know that $\mathcal{P} = \mathcal{P}_G(H, \mathfrak{p})$ for some non-trivial p-subgroup $1 \neq H \leqslant G$ and some cohomological point $\mathfrak{p} \in V_{G/\!\!/H}$. (In the notation of Proposition 7.32, this means $\mathcal{P} \in V_G(H)$.) Suppose that $t \in \operatorname{Loc}_\otimes(g(\mathcal{P}))$ is non-zero. We need to show that $\operatorname{Loc}_\otimes(t) = \operatorname{Loc}_\otimes(g(\mathcal{P}))$, that is, we need to show that $g(\mathcal{P}) \in \operatorname{Loc}_\otimes(t)$.

Recall the tt-functor $\check{\Psi}^H \colon \operatorname{DPerm}(G;k) \to \operatorname{K}\operatorname{Inj}(kG/\!\!/H))$ from Notation 6.10. By general properties of BF-idempotents [BF11, Theorem 6.3], we have $\check{\Psi}^K(g(\mathcal{P})) = g((\check{\psi}^K)^{-1}(\mathcal{P}))$ in $\operatorname{K}\operatorname{Inj}(k(G/\!\!/K))$ for every $K \in \operatorname{Sub}_p G$. Since $\check{\psi}^K$ is injective by Proposition 7.14, the fiber $(\check{\psi}^K)^{-1}(\mathcal{P})$ is a singleton (namely $\mathfrak{p}$) if $K \sim H$ and is empty otherwise. It follows that for all $K \not\sim H$ we have $\check{\Psi}^K(g(\mathcal{P})) = 0$ and therefore $\check{\Psi}^K(t) = 0$ as well. Since t is non-zero, the Conservativity Theorem 6.12 forces the only remaining $\check{\Psi}^H(t)$ to be non-zero in $\operatorname{K}\operatorname{Inj}(k(G/\!\!/H))$. This forces $\Psi^H(t)$ to be non-zero in $\mathcal{T}(G/\!\!/H)$ as well, since $\check{\Psi}^H = \Upsilon_{G/\!\!/H} \circ \Psi^H$. This object $\Psi^H(t)$ belongs to $\operatorname{Loc}_\otimes(\Psi^H(g(\mathcal{P}))) = \operatorname{Loc}_\otimes(g((\psi^H)^{-1}(\mathcal{P})))$. Note that $\upsilon_{G/\!\!/H}(\mathfrak{p})$ is the only preimage of $\mathcal{P} = \mathcal{P}_G(H, \mathfrak{p})$ under ψ^H (see Remark 7.34). By induction hypothesis, this localizing tt-ideal $\operatorname{Loc}_\otimes(\Psi^H(g(\mathcal{P})))$ is minimal. And it contains our non-zero object $\Psi^H(t)$. Hence $\Psi^H(g(\mathcal{P})) \in \operatorname{Loc}_\otimes(\Psi^H(t))$. Applying the right adjoint Ψ^H_ρ, it follows that $\Psi^H_\rho \Psi^H(g(\mathcal{P})) \in \Psi^H_\rho(\operatorname{Loc}_\otimes(\Psi^H(t))) \subseteq \operatorname{Loc}_\otimes(t)$ where the last inclusion follows by the projection formula for $\Psi^H \dashv \Psi^H_\rho$. Hence by the projection formula again we have in $\mathcal{T}(G)$ that

$$\Psi^H_\rho(\mathbb{1}) \otimes g(\mathcal{P}) \in \operatorname{Loc}_\otimes(t).$$

But we proved in Proposition 9.9 that the localizing tt-ideal generated by $\Psi^H_\rho(\mathbb{1})$ contains $\mathcal{K}(G)_{\mathrm{Im}(\psi^H)}$ and in particular $e(\overline{\{\mathcal{P}\}})$ and a fortiori $g(\mathcal{P})$. In short, we have $g(\mathcal{P}) \cong g(\mathcal{P})^{\otimes 2} \in \mathrm{Loc}_\otimes(\Psi^H_\rho(\mathbb{1}) \otimes g(\mathcal{P})) \subseteq \mathrm{Loc}_\otimes(t)$ as needed to be proved. □

9.12. **Corollary.** *The Telescope Conjecture holds for* $\mathrm{DPerm}(G;k)$*. Every smashing tt-ideal* $\mathcal{S} \subseteq \mathrm{DPerm}(G;k)$ *is generated by its compact part:* $\mathcal{S} = \mathrm{Loc}_\otimes(\mathcal{S}^c)$*.*

Proof. This follows from noetherianity of $\mathrm{Spc}(\mathcal{K}(G))$ and stratification by [BHS21, Theorem 9.11]. □

Part II. Topology of the spectrum and twisted cohomology

10. Introduction to Part II

After identifying all the points in the spectrum $\mathrm{Spc}(\mathcal{K}(G))$ of the permutation tt-category (1.1) in Part I, we now want to describe the topology. This knowledge will give us the classification of thick $\otimes$-ideals in $\mathcal{K}(G)$.

The colimit theorem. To discuss the tt-geometry of $\mathcal{K}(G)$, it is instructive to keep in mind the bounded derived category of finitely generated kG-modules, $\mathrm{D_b}(kG)$, which is a localization of our $\mathcal{K}(G)$ by [BG23a, Theorem 5.13]. A theorem of Serre [Ser65], famously expanded by Quillen [Qui71], implies that $\mathrm{Spc}(\mathrm{D_b}(kG))$ is the colimit of the $\mathrm{Spc}(\mathrm{D_b}(kE))$, for E running through the elementary abelian p-subgroups of G; see [Bal16, § 4]. The indexing category for this colimit is an orbit category: Its morphisms keep track of conjugations and inclusions of subgroups.

In Part I, we proved that $\mathrm{Spc}(\mathcal{K}(G))$ is set-theoretically partitioned into spectra of derived categories $\mathrm{D_b}(k(G/\!\!/K))$ for certain subquotients of G, namely the Weyl groups $G/\!\!/K = (N_G K)/K$ of p-subgroups $K \leqslant G$. It is then natural to expect a more intricate analogue of Quillen's result for the tt-category $\mathcal{K}(G)$, in which subgroups are replaced by *subquotients*. This is precisely what we prove. The orbit category has to be replaced by a category $\mathcal{E}_p(G)$ whose objects are elementary abelian p-sections $E = H/K$, for p-subgroups $K \trianglelefteqslant H \leqslant G$. The morphisms in $\mathcal{E}_p(G)$ keep track of conjugations, inclusions *and quotients*. See Construction 11.1.

This allows us to formulate our reduction to elementary abelian groups:

10.1. **Theorem** (Theorem 11.10)**.** *There is a canonical homeomorphism*

$$\operatorname*{colim}_{E \in \mathcal{E}_p(G)} \mathrm{Spc}(\mathcal{K}(E)) \xrightarrow{\sim} \mathrm{Spc}(\mathcal{K}(G)).$$

The category $\mathcal{E}_p(G)$ has been considered before, *e.g.* in Bouc-Thévenaz [BT08]. Every morphism in $\mathcal{E}_p(G)$ is the composite of three special morphisms (Remark 11.3)

$$E \xrightarrow{\simeq} E' \to E'' \xrightarrow{!} E''' \tag{10.2}$$

where E' is a G-conjugate of E, where $E' \leqslant E''$ is a subgroup of E'' and where $E'' = E'''/N$ is a quotient of E''' (sic!). The tt-category $\mathcal{K}(E)$ is contravariant in $E \in \mathcal{E}_p(G)$ and the tt-functors corresponding to (10.2)

$$\mathcal{K}(E''') \xrightarrow{\Psi^N} \mathcal{K}(E'') \xrightarrow{\mathrm{Res}} \mathcal{K}(E') \xrightarrow{\simeq} \mathcal{K}(E) \tag{10.3}$$

yield the *modular N-fixed-points functor* Ψ^N introduced in Part I, and the standard restriction functor and conjugation isomorphism. As we saw, the Ψ^N are a type of

Brauer quotient that make sense on the homotopy category of permutation modules but do not exist on derived or stable categories. They distinguish our results and their proofs from the classical theory.

Twisted cohomology. The above discussion reduces the analysis of $\mathrm{Spc}(\mathcal{K}(G))$ to the case of elementary abelian p-groups E. As often in modular representation theory, this case is far from trivial and can be viewed as the heart of the matter.

So let E be an elementary abelian p-group. Our methods will rely on $\otimes$-invertible objects u_N in $\mathcal{K}(E)$ indexed by the set $\mathcal{N}(E) = \{\, N \lhd E \,|\, [E\!:\!N] = p \,\}$ of maximal subgroups. These objects are of the form $u_N = \big(0 \to k(E/N) \to k(E/N) \to k \to 0\big)$ for p odd and $u_N = \big(0 \to k(E/N) \to k \to 0\big)$ for $p = 2$. See Definition 12.3. We use these $\otimes$-invertibles u_N to construct a multi-graded ring

$$\mathrm{H}^{\bullet\bullet}(E) = \bigoplus_{s\in\mathbb{Z}} \bigoplus_{q\in\mathbb{N}^{\mathcal{N}(E)}} \mathrm{Hom}_{\mathcal{K}(E)}\big(\mathbb{1}, \mathbb{1}(q)[s]\big), \tag{10.4}$$

where $\mathbb{1}(q)$ is the $\otimes$-invertible $\bigotimes_{N\in\mathcal{N}(E)} u_N^{\otimes q(N)}$ for every tuple $q\colon \mathcal{N}(E) \to \mathbb{N}$, that we refer to as a 'twist'. Without these twists we would obtain the standard $\mathbb{Z}$-graded endomorphism ring $\mathrm{End}^\bullet(\mathbb{1}) := \oplus_{s\in\mathbb{Z}} \mathrm{Hom}(\mathbb{1}, \mathbb{1}[s])$ of $\mathbb{1}$ which, for $\mathrm{D_b}(kE)$, is the cohomology $\mathrm{H}^\bullet(E, k)$, but for $\mathcal{K}(E)$ is reduced to the field k and therefore rather uninteresting. We call $\mathrm{H}^{\bullet\bullet}(E)$ the *(permutation) twisted cohomology* of E. Some readers may appreciate the analogy with cohomology twisted by line bundles in algebraic geometry, or with Tate twists in motivic cohomology.

We can employ this multi-graded ring $\mathrm{H}^{\bullet\bullet}(E)$ to describe $\mathrm{Spc}(\mathcal{K}(E))$:

10.5. **Theorem** (Corollary 15.6)**.** *The space* $\mathrm{Spc}(\mathcal{K}(E))$ *identifies with an* open subspace *of the homogeneous spectrum of* $\mathrm{H}^{\bullet\bullet}(E)$ *via a canonical 'comparison map'.*

The comparison map in question generalizes the one of [Bal10a], which landed in the homogenous spectrum of $\mathrm{End}^\bullet(\mathbb{1})$ without twist. We also describe in Corollary 15.6 the open image of this map by explicit equations in $\mathrm{H}^{\bullet\bullet}(E)$.

Dirac geometry. If the reader is puzzled by the multi-graded ring $\mathrm{H}^{\bullet\bullet}(E)$, here is another approach based on a special open cover $\{U(H)\}_{H\leqslant E}$ of $\mathrm{Spc}(\mathcal{K}(E))$ indexed by the subgroups of E and introduced in Proposition 13.11. Its key property is that over each open $U(H)$ all the $\otimes$-invertible objects u_N are trivial: $(u_N)_{\restriction U(H)} \simeq \mathbb{1}[s]$ for some shift $s \in \mathbb{Z}$ depending on H and N. For the trivial subgroup $H = 1$, the open $U(1)$ is the 'cohomological open' of Part I, that corresponds to the image under $\mathrm{Spc}(-)$ of the localization $\mathcal{K}(E) \twoheadrightarrow \mathrm{D_b}(kE)$. See Proposition 13.14. At the other end, for $H = E$, we show in Proposition 13.17 that the open $U(E)$ is the 'geometric open' that corresponds to the localization of $\mathcal{K}(E)$ given by the geometric fixed-points functor. Compare Remark 4.11. For E of rank one, these two opens $U(1)$ and $U(E)$ are all there is to consider. But as the p-rank of E grows, there is an exponentially larger collection $\{U(H)\}_{H\leqslant E}$ of open subsets interpolating between $U(1)$ and $U(E)$. This cover $\{U(H)\}_{H\leqslant E}$ allows us to use the classical comparison map of [Bal10a] locally. It yields a homeomorphism between each $U(H)$ and the homogeneous spectrum of the $\mathbb{Z}$-graded endomorphism ring $\mathrm{End}^\bullet_{U(H)}(\mathbb{1})$ in the localization $\mathcal{K}(E)_{\restriction U(H)}$. In compact form, this can be rephrased as follows (a Dirac scheme is to a usual scheme what a $\mathbb{Z}$-graded ring is to a non-graded one):

10.6. **Theorem** (Corollary 15.4)**.** *The space* $\mathrm{Spc}(\mathcal{K}(E))$*, together with the sheaf of* $\mathbb{Z}$*-graded rings obtained locally from endomorphisms of the unit, is a Dirac scheme.*

Elementary abelian take-home. Let us ponder the $\mathbb{Z}$-graded endomorphism ring of the unit $\mathrm{End}^\bullet(\mathbb{1})$ for a moment longer. As we know, the ring $\mathrm{End}^\bullet_{\mathcal{K}(E)}(\mathbb{1}) = k$ is too small to provide geometric information. So we have developed two substitutes. Our first approach is to replace the usual $\mathbb{Z}$-graded ring $\mathrm{End}^\bullet(\mathbb{1})$ by a richer multi-graded ring involving twists. This leads us to twisted cohomology $\mathrm{H}^{\bullet\bullet}(E)$ and to Theorem 10.5. The second approach is to hope that the endomorphism ring $\mathrm{End}^\bullet(\mathbb{1})$, although useless globally, becomes rich enough to control the topology *locally* on $\mathrm{Spc}(\mathcal{K}(E))$, without leaving the world of $\mathbb{Z}$-graded rings. This is what we achieve in Theorem 10.6 thanks to the open cover $\{U(H)\}_{H\leqslant E}$. As can be expected, the two proofs are intertwined.

Touching ground. Combining Theorems 10.1 and 10.6 ultimately describes the topological space $\mathrm{Spc}(\mathcal{K}(G))$ for all G, in terms of homogeneous spectra of graded rings. In Sections 16 to 18 we improve and apply these results as follows.

In Section 16, we explain how to go from the 'local' rings $\mathrm{End}^\bullet_{U(H)}(\mathbb{1})$ over the open $U(H)$, for each subgroup $H \leqslant E$, to the 'global' topology of $\mathrm{Spc}(\mathcal{K}(E))$.

In Theorem 17.13, we give a finite presentation by generators and relations of the reduced k-algebra $(\mathrm{End}^\bullet_{U(H)}(\mathbb{1}))_{\mathrm{red}}$ generalizing the usual one for cohomology.

In Corollary 18.12, we express $\mathrm{Spc}(\mathcal{K}(G))$ for a general finite group G as the quotient of a disjoint union of $\mathrm{Spc}(\mathcal{K}(E))$ for the *maximal* elementary abelian p-sections E of G by *maximal* relations.

In Proposition 18.14, we prove that the irreducible components of $\mathrm{Spc}(\mathcal{K}(G))$ correspond to the maximal elementary abelian p-sections of G up to conjugation. It follows that the Krull dimension of $\mathrm{Spc}(\mathcal{K}(G))$ is the sectional p-rank of G, the maximal rank of elementary abelian p-sections. (For comparison, recall that for the derived category these irreducible components correspond to maximal elementary abelian p-subgroups, not sections, and the Krull dimension is the usual p-rank.)

And of course, we discuss more examples. Using our techniques, we compute $\mathrm{Spc}(\mathcal{K}(G))$ for some notable groups G, in particular Klein-four (Example 16.16) and the dihedral group (Example 18.17).

For the reader's convenience, we tried to keep Part II somewhat self-contained. Here is a quick summary of the main ingredients we need from Part I.

10.7. *Recollection.* The canonical localization $\Upsilon_G\colon \mathcal{K}(G) \twoheadrightarrow \mathrm{D_b}(kG)$ gives us an open piece $V_G := \mathrm{Spc}(\mathrm{D_b}(kG)) \cong \mathrm{Spec^h}(\mathrm{H}^\bullet(G,k))$ of the spectrum, that we call the 'cohomological open'. We write $\upsilon_G = \mathrm{Spc}(\Upsilon_G)\colon V_G \hookrightarrow \mathrm{Spc}(\mathcal{K}(G))$ for the inclusion. For every $H \in \mathrm{Sub}_p(G)$ we denote by $\Psi^H\colon \mathcal{K}(G) \to \mathcal{K}(G/\!\!/H)$ the *modular H-fixed-points* tt-functor constructed in Section 5. It is characterized by $\Psi^H(k(X)) \simeq k(X^H)$ on permutation modules and by the same formula degreewise on complexes. We write $\check{\Psi}^H = \Upsilon_{G/\!\!/H} \circ \Psi^H$ for the composite $\mathcal{K}(G) \to \mathcal{K}(G/\!\!/H) \twoheadrightarrow \mathrm{D_b}(k(G/\!\!/H))$ all the way down to the derived category of $G/\!\!/H$. For every $H \in \mathrm{Sub}_p(G)$, the tt-prime $\mathcal{M}(H) = \mathrm{Ker}(\check{\Psi}^H)$ is a closed point of $\mathrm{Spc}(\mathcal{K}(G))$. It is also $\mathcal{M}(H) = \mathrm{Ker}(\mathbb{F}^H)$ where $\mathbb{F}^H = \mathrm{Res}^{G/\!\!/H}_1 \circ \Psi^H\colon \mathcal{K}(G) \to \mathcal{K}(G/\!\!/H) \to \mathrm{D_b}(k)$. All closed points of $\mathrm{Spc}(\mathcal{K}(G))$ are of this form by Corollary 7.31. We write $\psi^H = \mathrm{Spc}(\Psi^H)\colon \mathrm{Spc}(\mathcal{K}(G/\!\!/H)) \to \mathrm{Spc}(\mathcal{K}(G))$ for the continuous map induced by Ψ^H and $\check{\psi}^H = \mathrm{Spc}(\check{\Psi}^H)\colon V_{G/\!\!/H} \overset{\upsilon}{\hookrightarrow} \mathrm{Spc}(\mathcal{K}(G/\!\!/H)) \xrightarrow{\psi^H} \mathrm{Spc}(\mathcal{K}(G))$ for its restriction to the cohomological open of $G/\!\!/H$. If we need to specify the ambient group we write $\psi^{H;G}$ for ψ^H, etc. We saw in Section 7 that ψ^H is a closed map, and a closed immersion if $H \trianglelefteqslant G$ is normal. Every prime $\mathcal{P} \in \mathrm{Spc}(\mathcal{K}(G))$

is of the form $\mathcal{P} = \mathcal{P}_G(H,\mathfrak{p}) := \check{\psi}^H(\mathfrak{p})$ for a p-subgroup $H \leqslant G$ and a point $\mathfrak{p} \in V_{G/\!\!/H}$ in the cohomological open of the Weyl group of H, in a unique way up to G-conjugation; see Theorem 7.16. Hence the pieces $V_G(H) := \check{\psi}(V_{G/\!\!/H})$ yield a partition $\mathrm{Spc}(\mathcal{K}(G)) = \sqcup_{H\in \mathrm{Sub}_p(G)/_G} V_G(H)$ into relatively open strata $V_G(H)$, homeomorphic to $V_{G/\!\!/H}$. The crux of the problem is to understand how these strata $V_G(H) \simeq V_{G/\!\!/H}$ attach together topologically, to build the space $\mathrm{Spc}(\mathcal{K}(G))$.

11. The colimit theorem

To reduce the determination of $\mathrm{Spc}(\mathcal{K}(G))$ to the elementary abelian case, we invoke the *category $\mathcal{E}_p(G)$ of elementary abelian p-sections* of a finite group G. Recall that a section of G is a pair (H,K) of subgroups with K normal in H.

11.1. *Construction.* We denote by $\mathcal{E}_p(G)$ the category whose objects are pairs (H,K) where $K \trianglelefteqslant H$ are p-subgroups of G such that H/K is elementary abelian. Morphisms $(H,K) \to (H',K')$ are defined to be elements $g \in G$ such that

$$K' \leqslant K^g \leqslant H^g \leqslant H'.$$

Composition of morphisms is defined by multiplication in G. Note that the rank of the elementary abelian group H/K increases or stays the same along any morphism $(H,K) \to (H',K')$ in this category.

11.2. *Examples.* Let us highlight three types of morphisms in $\mathcal{E}_p(G)$.

(a) We have an isomorphism $g\colon (H,K) \xrightarrow{\sim} (H^g,K^g)$ in $\mathcal{E}_p(G)$ for every $g \in G$. Intuitively, we can think of this as the group isomorphism $c_g\colon H/K \xrightarrow{\sim} H^g/K^g$.

(b) For every object (H',K') in $\mathcal{E}_p(G)$ and every subgroup $H \leqslant H'$ containing K', we have a well-defined object (H,K') and the morphism $1\colon (H,K') \to (H',K')$. Intuitively, we think of it as the inclusion $H/K' \hookrightarrow H'/K'$ of a subgroup.

(c) For (H,K) in $\mathcal{E}_p(G)$ and a subgroup $\bar{L} = L/K$ of H/K, for $K \leqslant L \leqslant H$, there is another morphism in $\mathcal{E}_p(G)$ associated to $1 \in G$, namely $1\colon (H,L) \to (H,K)$. This one does not correspond to an intuitive group homomorphism $H/L \dashrightarrow H/K$, as K is *smaller* than L. Instead, H/L is the quotient of H/K by $\bar{L} \trianglelefteqslant H/K$. This last morphism will be responsible for the modular $\bar{L}$-fixed-points functor.

11.3. *Remark.* Every morphism $g\colon (H,K) \to (H',K')$ in $\mathcal{E}_p(G)$ is a composition of three morphisms of the above types (a), (b) and (c) in the following canonical way:

$$\begin{matrix} H & & H^g & & H' & & H' \\ \triangledown & \overset{(a)}{\to} & \triangledown & \overset{(b)}{\to} & \triangledown & \overset{(c)}{\to} & \triangledown \\ K & & K^g & & K^g & & K' \end{matrix}$$

where the first is given by $g \in G$ and the last two are given by $1 \in G$.

11.4. *Construction.* To every object (H,K) in $\mathcal{E}_p(G)$, we associate the tt-category $\mathcal{K}(H/K) = \mathrm{K_b}(\mathrm{perm}(H/K;k))$. For every morphism $g\colon (H,K) \to (H',K')$ in $\mathcal{E}_p(G)$, we set $\bar{K} = K^g/K'$ and we define a functor of tt-categories:

$$\mathcal{K}(g)\colon \mathcal{K}(H'/K') \xrightarrow{\Psi^{\bar{K}}} \mathcal{K}(H'/K^g) \xrightarrow{\mathrm{Res}} \mathcal{K}(H^g/K^g) \xrightarrow{c_g^*} \mathcal{K}(H/K)$$

using that $(H'/K')/\bar{K} = H'/K^g$ for the modular fixed-points functor $\Psi^{\bar{K}}$, and using that H^g/K^g is a subgroup of H'/K^g for the restriction.

It follows from Proposition 5.15 and Corollary 5.18 that $\mathcal{K}(-)$ is a contravariant (pseudo) functor on $\mathcal{E}_p(G)$ with values in tt-categories:

$$\mathcal{K}\colon \mathcal{E}_p(G)^{\mathrm{op}} \longrightarrow \text{tt-Cat}\,. \tag{11.5}$$

We can compose this with $\mathrm{Spc}(-)$, which incidentally makes the coherence of the 2-isomorphisms accompanying (11.5) irrelevant, and obtain a covariant functor from $\mathcal{E}_p(G)$ to topological spaces. Let us compare this diagram of spaces (and its colimit) with the space $\mathrm{Spc}(\mathcal{K}(G))$. For each $(H,K)\in\mathcal{E}_p(G)$, we have a tt-functor

$$\mathcal{K}(G) \xrightarrow{\mathrm{Res}^G_H} \mathcal{K}(H) \xrightarrow{\Psi^K} \mathcal{K}(H/K) \tag{11.6}$$

which yields a natural transformation from the constant functor $(H,K)\mapsto\mathcal{K}(G)$ to the functor $\mathcal{K}\colon \mathcal{E}_p(G)^{\mathrm{op}}\to$ tt-Cat of (11.5). The above Ψ^K is $\Psi^{K;H}$. Since $H\leqslant N_G K$, the tt-functor (11.6) is also $\mathrm{Res}^{G/\!\!/K}_{H/K}\circ\Psi^{K;G}\colon \mathcal{K}(G)\to\mathcal{K}(G/\!\!/K)\to\mathcal{K}(H/K)$. Applying $\mathrm{Spc}(-)$ to this observation, we obtain a commutative square:

$$\begin{array}{ccc} \mathrm{Spc}(\mathcal{K}(H/K)) & \xrightarrow{\ \psi^{K;H}\ } & \mathrm{Spc}(\mathcal{K}(H)) \\ {\scriptstyle\rho_{H/K}}\downarrow & \searrow^{\varphi_{(H,K)}} & \downarrow{\scriptstyle\rho_H} \\ \mathrm{Spc}(\mathcal{K}(G/\!\!/K)) & \xrightarrow[\ \psi^{K;G}\ ]{} & \mathrm{Spc}(\mathcal{K}(G)) \end{array} \tag{11.7}$$

whose diagonal we baptize $\varphi_{(H,K)}$. In summary, we obtain a continuous map

$$\varphi\colon \operatorname*{colim}_{(H,K)\in\mathcal{E}_p(G)} \mathrm{Spc}(\mathcal{K}(H/K)) \to \mathrm{Spc}(\mathcal{K}(G)) \tag{11.8}$$

whose component $\varphi_{(H,K)}$ at (H,K) is the diagonal map in (11.7).

11.9. **Lemma.** (a) *Each of the maps* $\mathrm{Spc}(\mathcal{K}(g))\colon \mathrm{Spc}(\mathcal{K}(H/K))\to\mathrm{Spc}(\mathcal{K}(H'/K'))$ *in the colimit diagram* (11.8) *is a closed immersion.*

(b) *Each of the components* $\varphi_{(H,K)}\colon \mathrm{Spc}(\mathcal{K}(H/K))\to\mathrm{Spc}(\mathcal{K}(G))$ *of* (11.8) *is closed and preserves the dimension of points (i.e. the Krull dimension of their closure).*

Proof. These statements follow from two facts, see Recollection 10.7: When $N\trianglelefteq G$ is normal the map $\psi^N\colon \mathrm{Spc}(\mathcal{K}(G/N))\hookrightarrow\mathrm{Spc}(\mathcal{K}(G))$ is a closed immersion. When $H\leqslant G$ is any subgroup, the map $\rho_H\colon \mathrm{Spc}(\mathcal{K}(H))\to\mathrm{Spc}(\mathcal{K}(G))$ is closed, hence lifts specializations, and it moreover satisfies 'Incomparability' by [Bal16]. ☐

We are now ready to prove Theorem 10.1:

11.10. **Theorem.** *For any finite group G, the map φ in* (11.8) *is a homeomorphism.*

Proof. Each component $\varphi_{(H,K)}$ is a closed map and thus φ is a closed map. For surjectivity, by Recollection 10.7, we know that $\mathrm{Spc}(\mathcal{K}(G))$ is covered by the subsets $\psi^K(V_{G/\!\!/K})$, over all p-subgroups $K\leqslant G$. Hence it suffices to know that the $\mathrm{Im}(\rho_E)$ cover $V_{G/\!\!/K}=\mathrm{Spc}(\mathrm{D_b}(G/\!\!/K))$ as $E\leqslant G/\!\!/K$ runs through all elementary p-subgroups. (Such an E must be of the form H/K for an object $(H,K)\in\mathcal{E}_p(G)$.) This holds by a classical result of Quillen [Qui71]; see [Bal16, Theorem 4.10].

The key point is injectivity. Take $\mathcal{P}\in\mathrm{Spc}(\mathcal{K}(H/K))$ and $\mathcal{P}'\in\mathrm{Spc}(\mathcal{K}(H'/K'))$ with same image in $\mathrm{Spc}(\mathcal{K}(G))$. Write $\mathcal{P}=\mathcal{P}_{H/K}(L/K,\mathfrak{p})$ for suitable arguments ($K\leqslant L\leqslant H$, $\mathfrak{p}\in V_{H/L}$) and note that the map induced by $1\colon (H,L)\to(H,K)$ in $\mathcal{E}_p(G)$ sends $\mathcal{P}_{H/L}(1,\mathfrak{p})\in\mathrm{Spc}(\mathcal{K}(H/L))$ to $\mathcal{P}$. So we may assume

$L = K$. By Remark 7.6, the image of $\mathcal{P} = \mathcal{P}_{H/K}(1, \mathfrak{p})$ in $\mathrm{Spc}(\mathcal{K}(G))$ is $\mathcal{P}_G(K, \bar{\rho}(\mathfrak{p}))$ where $\bar{\rho}\colon V_{H/K} \to V_{G/\!\!/K}$ is induced by restriction. Similarly, we may assume $\mathcal{P}' = \mathcal{P}_{H'/K'}(1, \mathfrak{p}')$ for $\mathfrak{p}' \in V_{H'/K'}$ and we have $\mathcal{P}_G(K, \bar{\rho}(\mathfrak{p})) = \mathcal{P}_G(K', \bar{\rho}'(\mathfrak{p}'))$ in $\mathrm{Spc}(\mathcal{K}(G))$ and need to show that $\mathcal{P}$ and $\mathcal{P}'$ are identified in the colimit (11.8).

By Theorem 7.16, the relation $\mathcal{P}_G(K, \bar{\rho}(\mathfrak{p})) = \mathcal{P}_G(K', \bar{\rho}'(\mathfrak{p}'))$ can only hold because of G-conjugation, meaning that there exists $g \in G$ such that $K' = K^g$ and $\bar{\rho}'(\mathfrak{p}') = \bar{\rho}(\mathfrak{p})^g$ in $V_{G/\!\!/K'}$. Using the map $g\colon (H, K) \to (H^g, K^g)$ in $\mathcal{E}_p(G)$ we may replace $H, K, \mathfrak{p}$ by $H^g, K^g, \mathfrak{p}^g$ and reduce to the case $K = K'$. In other words, we have two points $\mathcal{P} = \mathcal{P}_{H/K}(1, \mathfrak{p}) \in \mathrm{Spc}(\mathcal{K}(H/K))$ and $\mathcal{P}' = \mathcal{P}_{H'/K}(1, \mathfrak{p}') \in \mathrm{Spc}(\mathcal{K}(H'/K))$ corresponding to two p-subgroups $H, H' \leqslant G$ containing the *same* subgroup K as a normal subgroup and two cohomological primes $\mathfrak{p} \in V_{H/K}$ and $\mathfrak{p}' \in V_{H'/K}$ such that $\bar{\rho}(\mathfrak{p}) = \bar{\rho}'(\mathfrak{p}')$ in $V_{G/\!\!/K}$ under the maps $\bar{\rho}$ and $\bar{\rho}'$ induced by restriction along $H/K \leqslant G/\!\!/K$ and $H'/K \leqslant G/\!\!/K$ respectively.

If we let $\bar{G} = G/\!\!/K = (N_G K)/K$, we have two elementary abelian p-subgroups $\bar{H} = H/K$ and $\bar{H}' = H'/K$ of $\bar{G}$, each with a point in their cohomological open, $\mathfrak{p} \in V_{\bar{H}}$ and $\mathfrak{p}' \in V_{\bar{H}'}$, and those two points have the same image in the cohomological open $V_{\bar{G}}$ of the 'ambient' group $\bar{G}$. By Quillen [Qui71] (or [Bal16, § 4]) again, we know that this coalescence must happen because of an element $\bar{g} \in \bar{G}$, that is, a $g \in N_G K$, and a prime $\mathfrak{q} \in V_{\bar{H} \cap {}^g\bar{H}'}$ that maps to $\mathfrak{p}$ and to $\mathfrak{p}'$ under the maps $V_{\bar{H} \cap {}^g\bar{H}'} \to V_{\bar{H}}$ and $V_{\bar{H} \cap {}^g\bar{H}'} \to V_{\bar{H}'}$ respectively. But our category $\mathcal{E}_p(G)$ contains all such conjugation-inclusion morphisms coming from the orbit category of G. Specifically, we have two morphisms $1\colon (H \cap {}^gH', K) \to (H, K)$ and $g\colon (H \cap {}^gH', K) \to (H', K)$ in $\mathcal{E}_p(G)$, under which the point $\mathcal{P}_{(H \cap {}^gH')/K}(1, \mathfrak{q})$ maps to $\mathcal{P}_{H/K}(1, \mathfrak{p}) = \mathcal{P}$ and $\mathcal{P}_{H'/K}(1, \mathfrak{p}') = \mathcal{P}'$ respectively. This shows that $\mathcal{P} = \mathcal{P}'$ in the domain of (11.8) as required. □

11.11. *Remark.* By Proposition 9.1, the space $\mathrm{Spc}(\mathcal{K}(G))$ is noetherian. Hence the topology is entirely characterized by the inclusion of primes. Now, suppose that $\mathcal{P}$ is the image under $\varphi_{(H,K)}\colon \mathrm{Spc}(\mathcal{K}(E)) \to \mathrm{Spc}(\mathcal{K}(G))$ of some $\mathcal{P}' \in \mathrm{Spc}(\mathcal{K}(E))$ for an elementary abelian subquotient $E = H/K$ corresponding to a section $(H, K) \in \mathcal{E}_p(G)$. Then the only way for another prime $\mathcal{Q} \in \mathrm{Spc}(\mathcal{K}(G))$ to belong to the closure of $\mathcal{P}$ is to be itself the image of some point $\mathcal{Q}'$ of $\mathrm{Spc}(\mathcal{K}(E))$ in the closure of $\mathcal{P}'$. This follows from Lemma 11.9. In other words, the question of inclusion of primes can also be reduced to the elementary abelian case.

12. Invertible objects and twisted cohomology

In this section we introduce a graded ring whose homogeneous spectrum helps us understand the topology on $\mathrm{Spc}(\mathcal{K}(G))$, at least for G elementary abelian. This graded ring, called the *twisted cohomology ring* (Definition 12.16), consists of morphisms between $\mathbb{1}$ and certain invertible objects. It all starts in the cyclic case.

12.1. *Example.* Let $C_p = \langle \sigma \mid \sigma^p = 1 \rangle$ be the cyclic group of prime order p, with a chosen generator. We write $kC_p = k[\sigma]/(\sigma^p - 1)$ as $k[\tau]/\tau^p$ for $\tau = \sigma - 1$. Then the coaugmentation and augmentation maps become:

$$\eta : k \xrightarrow{1 \mapsto \tau^{p-1}} kC_p \qquad \text{and} \qquad \epsilon : kC_p \xrightarrow{\tau \mapsto 0} k.$$

For p odd, we denote the first terms of the 'standard' minimal resolution of k by

$$u_p = (0 \to kC_p \xrightarrow{\tau} kC_p \xrightarrow{\epsilon} k \to 0).$$

We view this in $\mathcal{K}(C_p)$ with k in homological degree zero. One can verify directly that u_p is $\otimes$-invertible, with $u_p^{\otimes -1} = u_p^\vee \cong (0 \to k \xrightarrow{\eta} kC_p \xrightarrow{\tau} kC_p \to 0)$. Alternatively, one can use the conservative pair of functors $\mathbb{F}^H \colon \mathcal{K}(C_p) \to \mathrm{D_b}(k)$ for $H \in \{C_p\,, 1\}$, corresponding to the only closed points $\mathcal{M}(C_p)$ and $\mathcal{M}(1)$ of $\operatorname{Spc}(\mathcal{K}(C_p))$. Those functors map u_p to the $\otimes$-invertibles k and $k[2]$ in $\mathrm{D_b}(k)$, respectively.

For $p = 2$, we have a similar but shorter $\otimes$-invertible object in $\mathcal{K}(C_2)$

$$u_2 = (0 \to kC_2 \xrightarrow{\epsilon} k \to 0)$$

again with k in degree zero.

12.2. *Notation.* To avoid constantly distinguishing cases, we abbreviate

$$2' := \begin{cases} 2 & \text{if } p > 2 \\ 1 & \text{if } p = 2. \end{cases}$$

For any finite group G and any index-p normal subgroup N, we can inflate the $\otimes$-invertible u_p of Example 12.1 along $\pi \colon G \twoheadrightarrow G/N \simeq C_p$ to a $\otimes$-invertible in $\mathcal{K}(G)$.

12.3. *Definition.* Let $N \lhd G$ be a normal subgroup of index p. We define

$$u_N := \begin{cases} \cdots \to 0 \to k(G/N) \xrightarrow{\tau} k(G/N) \xrightarrow{\epsilon} k \to 0 \to \cdots & \text{if } p \text{ is odd} \\ \cdots \to 0 \to \quad 0 \quad \to k(G/N) \xrightarrow{\epsilon} k \to 0 \to \cdots & \text{if } p = 2 \end{cases} \tag{12.4}$$

with k in degree zero. We also define two morphisms

$$a_N \colon \mathbb{1} \to u_N \qquad \text{and} \qquad b_N \colon \mathbb{1} \to u_N[-2']$$

as follows. The morphism a_N is the identity in degree zero, independently of p:

$$\begin{array}{ccccccc} \mathbb{1} & = & \cdots \longrightarrow & 0 & \longrightarrow & k & \to 0 \to \cdots \\ {\scriptstyle a_N}\downarrow & & & \downarrow & & \downarrow{\scriptstyle 1} & \\ u_N & = & \cdots \to & k(G/N) & \xrightarrow[\epsilon]{} & k & \to 0 \to \cdots \end{array}$$

The morphism b_N is given by $\eta \colon k \to k(G/N)$ in degree zero, as follows:

$$\begin{array}{ccccc} \mathbb{1} & = & k & \longrightarrow & 0 \\ {\scriptstyle b_N}\downarrow & & \downarrow{\scriptstyle \eta} & & \downarrow \\ u_N[-1] & = & k(G/N) & \xrightarrow[\epsilon]{} & k \end{array} \qquad \text{and} \qquad \begin{array}{ccccccc} \mathbb{1} & = & k & \longrightarrow & 0 & \longrightarrow & 0 \\ {\scriptstyle b_N}\downarrow & & \downarrow{\scriptstyle \eta} & & \downarrow & & \downarrow \\ u_N[-2] & = & k(G/N) & \xrightarrow[\tau]{} & k(G/N) & \xrightarrow[\epsilon]{} & k \end{array}$$

where the target u_N is shifted once to the right for $p = 2$ (as in the left-hand diagram above) and shifted twice for $p > 2$ (as in the right-hand diagram).

When p is odd there is furthermore a third morphism $c_N \colon \mathbb{1} \to u_N[-1]$, that is defined to be $\eta \colon k \to k(G/N)$ in degree zero. This c_N will play a lesser role.

In statements made for all primes p, simply ignore c_N in the case $p = 2$ (or think $c_N = 0$). Here is an example of such a statement, whose meaning should now be clear: *The morphisms a_N and b_N, and c_N (for p odd), are inflated from G/N.*

12.5. *Remark.* Technically, u_N depends not only on an index-p subgroup $N \lhd G$ but also on the choice of a generator of G/N, to identify G/N with C_p. If one needs to make this distinction, one can write u_π for a chosen epimorphism $\pi \colon G \twoheadrightarrow C_p$. This does not change the isomorphism type of u_N, namely $\ker(\pi) = \ker(\pi')$ implies $u_\pi \cong u_{\pi'}$. (We expand on this topic in Remark 17.2.)

12.6. **Lemma.** *Let $N \lhd G$ be a normal subgroup of index p and let $q \geq 1$. Then there is a canonical isomorphism in $\mathcal{K}(G)$*

$$u_N^{\otimes q} \cong (\cdots 0 \to k(G/N) \xrightarrow{\tau} k(G/N) \xrightarrow{\tau^{p-1}} \cdots \xrightarrow{\tau} k(G/N) \xrightarrow{\epsilon} k \to 0 \cdots)$$

where the first $k(G/N)$ sits in homological degree $2' \cdot q$ and k sits in degree 0.

Proof. It is an exercise over the cyclic group C_p. Then inflate along $G \twoheadrightarrow G/N$. □

12.7. *Remark.* The morphism $b_N \colon \mathbb{1} \to u_N[-2']$ of Definition 12.3 is a quasi-isomorphism and the fraction

$$\zeta_N := (b_N[2'])^{-1} \circ a_N \colon \mathbb{1} \to u_N \leftarrow \mathbb{1}[2']$$

is a well-known morphism $\zeta_N \in \mathrm{Hom}_{\mathrm{D_b}(kG)}(\mathbb{1}, \mathbb{1}[2']) = \mathrm{H}^{2'}(G, k)$ in the derived category $\mathrm{D_b}(kG)$. For G elementary abelian, these ζ_N generate the cohomology k-algebra $\mathrm{H}^\bullet(G, k)$, on the nose for $p = 2$ and modulo nilpotents for p odd.

We sometimes write $\zeta_N^+ = \frac{a_N}{b_N}$ for ζ_N in order to distinguish it from the inverse fraction $\zeta_N^- := \frac{b_N}{a_N}$ that exists wherever a_N is inverted. Of course, when both a_N and b_N are inverted, we have $\zeta_N^- = (\zeta_N^+)^{-1} = \zeta_N^{-1}$.

12.8. *Remark.* The switch of factors $(12) \colon u_N \otimes u_N \cong u_N \otimes u_N$ can be computed directly to be the identity (over C_p, then inflate). Alternatively, it must be multiplication by a square-one element of $\mathrm{Aut}(\mathbb{1}) = k^\times$, hence ± 1. One can then apply the tensor-functor $\Psi^G \colon \mathcal{K}(G) \to \mathrm{D_b}(k)$, under which u_N goes to $\mathbb{1}$, to rule out -1.

It follows that for p odd, $u_N[-1]$ has switch -1, and consequently every morphism $\mathbb{1} \to u_N[-1]$ must square to zero. In particular $c_N \otimes c_N = 0$. This nilpotence explains why c_N will play no significant role in the topology.

We can describe the image under modular fixed-points functors of the $\otimes$-invertible objects u_N and of the morphisms a_N and b_N. (We leave c_N as an exercise.)

12.9. **Proposition.** *Let $H \trianglelefteq G$ be a normal p-subgroup. Then for every index-p normal subgroup $N \lhd G$, we have in $\mathcal{K}(G/H)$*

$$\Psi^H(u_N) \cong \begin{cases} u_{N/H} & \text{if } H \leqslant N \\ \mathbb{1} & \text{if } H \not\leqslant N \end{cases}$$

and under this identification

$$\Psi^H(a_N) = \begin{cases} a_{N/H} & \text{if } H \leqslant N \\ 1_{\mathbb{1}} & \text{if } H \not\leqslant N \end{cases} \quad \text{and} \quad \Psi^H(b_N) = \begin{cases} b_{N/H} & \text{if } H \leqslant N \\ 0 & \text{if } H \not\leqslant N. \end{cases}$$

Proof. Direct from Definition 12.3 and $\Psi^H(k(X)) \cong k(X^H)$ for $X = G/N$. □

For restriction, there is an analogous pattern but with the cases 'swapped'.

12.10. **Proposition.** *Let $H \leqslant G$ be a subgroup. Then for every index-p normal subgroup $N \lhd G$, we have in $\mathcal{K}(H)$*

$$\mathrm{Res}^G_H(u_N) \cong \begin{cases} \mathbb{1}[2'] & \text{if } H \leqslant N \\ u_{N\cap H} & \text{if } H \not\leqslant N \end{cases}$$

and under this identification

$$\mathrm{Res}^G_H(a_N) = \begin{cases} 0 & \text{if } H \leqslant N \\ a_{N\cap H} & \text{if } H \not\leqslant N \end{cases} \quad \text{and} \quad \mathrm{Res}^G_H(b_N) = \begin{cases} 1_{\mathbb{1}} & \text{if } H \leqslant N \\ b_{N\cap H} & \text{if } H \not\leqslant N. \end{cases}$$

Proof. Direct from Definition 12.3 and the Mackey formula for $\mathrm{Res}^G_H(k(G/N))$. □

We can combine the above two propositions and handle Ψ^H for non-normal H, since by definition $\Psi^{H;G} = \Psi^{H;N_G H} \circ \mathrm{Res}^G_{N_G H}$. Here is an application of this.

12.11. **Corollary.** *Let $H \leqslant G$ be a p-subgroup and $N \lhd G$ of index p. Recall the 'residue' tt-functor $\mathbb{F}^H = \mathrm{Res}_1 \circ \Psi^H \colon \mathcal{K}(G) \to \mathrm{D_b}(k)$ at the closed point $\mathcal{M}(H)$.*

(a) *If $H \not\leqslant N$ then $\mathbb{F}^H(a_N)$ is an isomorphism.*

(b) *If $H \leqslant N$ then $\mathbb{F}^H(b_N)$ is an isomorphism.*

Proof. We apply Proposition 12.10 for $N_G H \leqslant G$ and Proposition 12.9 for $H \lhd N_G H$. For (a), $H \not\leqslant N$ forces $N_G H \not\leqslant N$ and $H \not\leqslant N \cap N_G H$. Hence $\Psi^H(a_N) = \Psi^{H;N_G H} \mathrm{Res}_{N_G H}(a_N) = \Psi^{H;N_G H}(a_{N \cap N_G H}) = 1_{\mathbb{1}}$ is an isomorphism. Similarly for (b), if $N_G H \leqslant N$ then $\Psi^H(b_N)$ is an isomorphism and if $N_G H \not\leqslant N$ it is the quasi-isomorphism $b_{(N \cap N_G H)/H}$. Thus $\mathbb{F}^H(b_N)$ is an isomorphism in $\mathrm{D_b}(k)$. □

Let us now prove that the morphisms a_N and b_N, and c_N (for p odd), generate all morphisms from the unit $\mathbb{1}$ to tensor products of u_N's. This is a critical fact.

12.12. **Lemma.** *Let $N_1, \ldots, N_\ell$ be index-p normal subgroups of G and abbreviate $u_i := u_{N_i}$ for $i = 1, \ldots, \ell$ and similarly $a_i := a_{N_i}$ and $b_i := b_{N_i}$ and $c_i := c_{N_i}$ (see Definition 12.3). Let $q_1, \ldots, q_\ell \in \mathbb{N}$ be non-negative integers and $s \in \mathbb{Z}$. Then every morphism $f \colon \mathbb{1} \to u_1^{\otimes q_1} \otimes \cdots \otimes u_\ell^{\otimes q_\ell}[s]$ in $\mathcal{K}(G)$ is a k-linear combination of tensor products of (i.e. a 'polynomial' in) the morphisms a_i and b_i, and c_i (for p odd).*

Proof. We proceed by induction on ℓ. The case $\ell = 0$ is just $\mathrm{End}^\bullet_{\mathcal{K}(G)}(\mathbb{1}) = k$. Suppose $\ell \geq 1$ and the result known for $\ell - 1$. Up to reducing to $\ell - 1$, we can assume that the $N_1, \ldots, N_\ell$ are all distinct. Set for readability

$$v := u_1^{\otimes q_1} \otimes \cdots \otimes u_{\ell-1}^{\otimes q_{\ell-1}}[s], \qquad N := N_\ell, \qquad u := u_\ell = u_N \qquad \text{and} \qquad q := q_\ell$$

so that f is a morphism of the form

$$f \colon \mathbb{1} \to v \otimes u^{\otimes q}.$$

We then proceed by induction on $q \geq 0$. We assume the result known for $q-1$ (the case $q = 0$ holds by induction hypothesis on ℓ). The proof will now depend on p.

Suppose first that $p = 2$. Consider the exact triangle in $\mathcal{K}(G)$

$$\begin{array}{ccccccccccccc}
u^{\otimes(q-1)} & = & \cdots 0 & \longrightarrow & 0 & \longrightarrow & k(G/N) & \xrightarrow{\tau} \cdots \xrightarrow{\tau} & k(G/N) & \xrightarrow{\epsilon} & k & \to 0 \cdots \\
\downarrow{\scriptstyle a_N} & & & & \downarrow & & \| & & \| & & \| & \\
u^{\otimes q} & = & \cdots 0 & \to & k(G/N) & \xrightarrow{\tau} & k(G/N) & \xrightarrow{\tau} \cdots \xrightarrow{\tau} & k(G/N) & \xrightarrow{\epsilon} & k & \to 0 \cdots \\
\downarrow & & & & \| & & \downarrow & & \downarrow & & \downarrow & \\
k(G/N)[q] & = & \cdots 0 & \to & k(G/N) & \longrightarrow & 0 & \longrightarrow \cdots \longrightarrow & 0 & \longrightarrow & 0 & \to 0 \cdots
\end{array} \tag{12.13}$$

where k is in degree zero. (See Lemma 12.6.) Tensoring the above triangle with v and applying $\mathrm{Hom}_G(\mathbb{1}, -) := \mathrm{Hom}_{\mathcal{K}(G)}(\mathbb{1}, -)$ we get an exact sequence

(12.14)
$$\begin{array}{ccccc}
\mathrm{Hom}_G(\mathbb{1}, v \otimes u^{\otimes(q-1)}) & \xrightarrow{\cdot a_N} & \mathrm{Hom}_G(\mathbb{1}, v \otimes u^{\otimes q}) & \longrightarrow & \mathrm{Hom}_G(\mathbb{1}, v \otimes k(G/N)[q]) \\
& & \cup\!\!\mid & & \| \\
& & f & \longmapsto & f' \in \mathrm{Hom}_N(\mathbb{1}, \mathrm{Res}^G_N(v)[q])
\end{array}$$

Our morphism f belongs to the middle group. By adjunction, the right-hand term is $\mathrm{Hom}_N(\mathbb{1}, \mathrm{Res}^G_N(v)[q])$. Now since all $N_1, \ldots, N_\ell = N$ are distinct, we can apply Proposition 12.10 to compute $\mathrm{Res}^G_N(v)$ and we know by induction hypothesis (on ℓ)

that the image f' of our f in this group $\mathrm{Hom}_N(\mathbb{1}, \mathrm{Res}^G_N(v)[q])$ is a k-linear combination of tensor products of $a_{N_i\cap N}$ and $b_{N_j\cap N}$ for $1 \leqslant i, j \leqslant \ell - 1$, performed over the group N. We can perform the 'same' k-linear combination of tensor products of a_i's and b_j's over the group G, thus defining a morphism $f'' \in \mathrm{Hom}_G(\mathbb{1}, v[q])$. We can now multiply f'' with $b_N^{\otimes q} \colon \mathbb{1} \to u_N^{\otimes q}[-q]$ to obtain a morphism $f'' b_N^q$ in the same group $\mathrm{Hom}_G(\mathbb{1}, v \otimes u^{\otimes q})$ that contains f. Direct computation shows that the image of this $f'' b_N^q$ in $\mathrm{Hom}_N(\mathbb{1}, \mathrm{Res}_N(v)[s])$ is also equal to f'. The key point is that $b_N^{\otimes q}$ is simply $\eta \colon k \to k(G/N)$ in degree q and this η is also the unit of the $\mathrm{Res}^G_N \dashv \mathrm{Ind}^G_N$ adjunction. In other words, the difference $f - f'' b_N^q$ comes from the left-hand group $\mathrm{Hom}_G(\mathbb{1}, v \otimes u^{\otimes(q-1)})$ in the exact sequence (12.14), reading

$$f = f'' b_N^q + f''' a_N$$

for some $f''' \in \mathrm{Hom}_G(\mathbb{1}, v \otimes u^{\otimes(q-1)})$. By induction hypothesis (on q), f''' is a polynomial in a_i's and b_j's. Since f'' also was such a polynomial, so is f.

The proof for p odd follows a similar pattern of induction on q, with one complication. The cone of the canonical map $a_N : u_N^{\otimes(q-1)} \to u_N^{\otimes q}$ is not simply $k(G/N)$ in a single degree as in (12.13) but rather the complex

$$C := \big(\cdots \to 0 \to k(G/N) \xrightarrow{\ \tau\ } k(G/N) \to 0 \to \cdots\big)$$

with $k(G/N)$ in two consecutive degrees $2q$ and $2q-1$. So the exact sequence

$$\mathrm{Hom}_G(\mathbb{1}, v \otimes u^{\otimes(q-1)}) \xrightarrow{\cdot a_N} \mathrm{Hom}_G(\mathbb{1}, v \otimes u^{\otimes q}) \longrightarrow \mathrm{Hom}_G(\mathbb{1}, v \otimes C) \tag{12.15}$$

has a more complicated third term than the one of (12.14). That third term $\mathrm{Hom}_G(\mathbb{1}, v \otimes C)$ itself fits in its own exact sequence associated to the exact triangle $k(G/N)[2q-1] \xrightarrow{\tau} k(G/N)[2q-1] \to C \to k(G/N)[2q]$. Each of the terms $\mathrm{Hom}_G(\mathbb{1}, v \otimes k(G/N)[*]) \cong \mathrm{Hom}_N(\mathbb{1}, \mathrm{Res}^G_N(v)[*])$ can be computed as before, by adjunction. The image of f in $\mathrm{Hom}_N(\mathbb{1}, \mathrm{Res}^G_N(v)[2q])$ can again be lifted to a polynomial $f' b_N^q : \mathbb{1} \to v \otimes u^{\otimes q}$ so that the image of the difference $f - f' b_N^q$ in $\mathrm{Hom}_G(\mathbb{1}, v \otimes C)$ comes from some element in $\mathrm{Hom}_N(\mathbb{1}, \mathrm{Res}^G_N(v)[2q-1])$. That element may be lifted to a polynomial $f'' b_N^{q-1} c_N : \mathbb{1} \to v \otimes u^{\otimes q}$, and we obtain

$$f = f' b_N^q + f'' b_N^{q-1} c_N + f''' a_N$$

for some $f''' \in \mathrm{Hom}_G(\mathbb{1}, v \otimes u^{\otimes(q-1)})$ similarly as before. □

We can now assemble all the hom groups of Lemma 12.12 into a big graded ring.

12.16. *Definition.* We denote the set of all index-p normal subgroups of G by

$$\mathcal{N} = \mathcal{N}(G) := \big\{\, N \lhd G \;\big|\; [G\colon N] = p \,\big\}. \tag{12.17}$$

Let $\mathbb{N}^{\mathcal{N}} = \mathbb{N}^{\mathcal{N}(G)} = \{q \colon \mathcal{N}(G) \to \mathbb{N}\}$ be the monoid of *twists*, *i.e.* tuples of non-negative integers indexed by this finite set. Consider the $(\mathbb{Z} \times \mathbb{N}^{\mathcal{N}})$-graded ring

$$\mathrm{H}^{\bullet\bullet}(G) = \mathrm{H}^{\bullet\bullet}(G;k) := \bigoplus_{s\in\mathbb{Z}} \bigoplus_{q\in\mathbb{N}^{\mathcal{N}}} \mathrm{Hom}_{\mathcal{K}(G)}\Big(\mathbb{1}\,,\ \bigotimes_{N\in\mathcal{N}} u_N^{\otimes q(N)}[s]\Big). \tag{12.18}$$

Its multiplication is induced by the tensor product in $\mathcal{K}(G)$. We call $\mathrm{H}^{\bullet\bullet}(G)$ the *(permutation) twisted cohomology ring of G*. It is convenient to simply write

$$\mathbb{1}(q) = \bigotimes_{N\in\mathcal{N}} (u_N)^{\otimes q(N)} \tag{12.19}$$

for every twist $q \in \mathbb{N}^{\mathcal{N}(G)}$ and thus abbreviate $\mathrm{H}^{s,q}(G) = \mathrm{Hom}\big(\mathbb{1}, \mathbb{1}(q)[s]\big)$.

12.20. *Remark.* The graded ring $\mathrm{H}^{\bullet\bullet}(G)$ is graded-commutative by using only the parity of the shift, not the twist; see Remark 12.8. In other words, we have

$$h_1 \cdot h_2 = (-1)^{s_1 \cdot s_2}\, h_2 \cdot h_1 \qquad \text{when} \qquad h_i \in \mathrm{H}^{s_i,q_i}(G).$$

For instance, for p odd, when dealing with the morphisms a_N and b_N, which land in even shifts of the object u_N, we do not have to worry too much about the order. This explains the 'unordered' notation $\zeta_N = \frac{a_N}{b_N}$ used in Remark 12.7.

The critical Lemma 12.12 gives the main property of this construction:

12.21. **Theorem.** *The twisted cohomology ring* $\mathrm{H}^{\bullet\bullet}(G)$ *of Definition 12.16 is a k-algebra generated by the finitely many elements a_N and b_N, and c_N (for p odd), of Definition 12.3, over all $N \lhd G$ of index p. In particular* $\mathrm{H}^{\bullet\bullet}(G)$ *is noetherian.* □

12.22. *Example.* The reader can verify by hand that $\mathrm{H}^{\bullet\bullet}(C_2) = k[a_N, b_N]$, without relations, and that $\mathrm{H}^{\bullet\bullet}(C_p) = k[a_N, b_N, c_N]/\langle c_N^2\rangle$ for p odd, where in both cases $N = 1$ is the only $N \in \mathcal{N}(C_p)$. This example is deceptive, for the $\{a_N, b_N, c_N\}_{N\in\mathcal{N}}$ usually satisfy some relations, as the reader can already check for $G = C_2 \times C_2$ for instance. We systematically discuss these relations in Section 17.

We conclude this section with some commentary.

12.23. *Remark.* The name 'cohomology' in Definition 12.16 is used in the loose sense of a graded endomorphism ring of the unit in a tensor-triangulated category. However, since we are using the tt-category $\mathcal{K}(G)$ and not $\mathrm{D_b}(kG)$, the ring $\mathrm{H}^{\bullet\bullet}(G)$ is quite different from $\mathrm{H}^{\bullet}(G, k)$ in general. In fact, $\mathrm{H}^{\bullet\bullet}(G)$ could even be rather dull. For instance, if G is a non-cyclic simple group then $\mathcal{N}(G) = \varnothing$ and $\mathrm{H}^{\bullet\bullet}(G) = k$. We will make serious use of $\mathrm{H}^{\bullet\bullet}(G)$ in Section 15 to describe $\mathrm{Spc}(\mathcal{K}(G))$ for G elementary abelian. In that case, $\mathrm{H}^{\bullet}(G, k)$ is a localization of $\mathrm{H}^{\bullet\bullet}(G)$. See Example 14.13.

12.24. *Remark.* By Proposition 12.9, there is no 'collision' in the twists: If there is an isomorphism $\mathbb{1}(q)[s] \simeq \mathbb{1}(q')[s']$ in $\mathcal{K}(G)$ then we must have $q = q'$ in $\mathbb{N}^{\mathcal{N}}$ and $s = s'$ in $\mathbb{Z}$. The latter is clear from $\mathbb{F}^G(u_N) \cong \mathbb{1}$ in $\mathrm{D_b}(k)$, independently of N. We then conclude from $\mathbb{F}^N(\mathbb{1}(q)) \simeq \mathbb{1}[2'q(N)]$ in $\mathrm{D_b}(k)$, for each $N \in \mathcal{N}$.

12.25. *Remark.* We only use positive twists $q(N)$ in (12.18). The reader can verify that already for $G = C_p$ cyclic, the $\mathbb{Z}^2$-graded ring $\oplus_{(s,q)\in\mathbb{Z}^2} \mathrm{Hom}(\mathbb{1}, u_p^{\otimes q}[s])$ is not noetherian. See for instance [DHM24] for $p = 2$. However, negatively twisted elements tend to be nilpotent. So the $\mathbb{Z} \times \mathbb{Z}^{\mathcal{N}}$-graded version of $\mathrm{H}^{\bullet\bullet}(G)$ may yield the same topological information as our $\mathbb{Z} \times \mathbb{N}^{\mathcal{N}}$-graded one. We have not pushed this investigation of negative twists, as it brought no benefit to our analysis.

13. An open cover of the spectrum

In this section, we extract some topological information about $\mathrm{Spc}(\mathcal{K}(G))$ from the twisted cohomology ring $\mathrm{H}^{\bullet\bullet}(G)$ of Definition 12.16 and the maps a_N and b_N of Definition 12.3, associated to every index-p normal subgroup N in $\mathcal{N} = \mathcal{N}(G)$.

Recall from Construction 3.14 that we can use tensor-induction to associate to every subgroup $H \leqslant G$ a *Koszul object* $\mathrm{kos}_G(H) = {}^{\otimes}\mathrm{Ind}_H^G(0 \to k \xrightarrow{1} k \to 0)$. It generates in $\mathcal{K}(G)$ the tt-ideal $\mathrm{Ker}(\mathrm{Res}^G_H)$, see Proposition 3.21:

$$\langle \mathrm{kos}_G(H)\rangle_{\mathcal{K}(G)} = \mathrm{Ker}\big(\mathrm{Res}^G_H \colon \mathcal{K}(G) \to \mathcal{K}(H)\big). \tag{13.1}$$

13.2. **Lemma.** *Let $N \lhd G$ be a normal subgroup of index p. Then we have:*

(a) *In $\mathcal{K}(G)$, the object* $\mathrm{cone}(a_N)$ *generates the same thick subcategory as $k(G/N)$. In particular,* $\mathrm{supp}(\mathrm{cone}(a_N)) = \mathrm{supp}(k(G/N))$.

(b) *In $\mathcal{K}(G)$, the object* $\mathrm{cone}(b_N)$ *generates the same thick tensor-ideal as* $\mathrm{kos}_G(N)$. *In particular,* $\mathrm{supp}(\mathrm{cone}(b_N)) = \mathrm{supp}(\mathrm{kos}_G(N)) = \mathrm{supp}(\mathrm{Ker}(\mathrm{Res}^G_N))$.

Proof. For $p = 2$, we have $\mathrm{cone}(a_N) = k(G/N)[1]$ so the first case is clear. For p odd, we have $\mathrm{cone}(a_N)[-1] \simeq (0 \to k(G/N) \xrightarrow{\tau} k(G/N) \to 0) = \mathrm{cone}(\tau_{|k(G/N)})$. Hence $\mathrm{cone}(a_N) \in \mathrm{thick}(k(G/N))$. Conversely, since $\tau^p = 0$, the octahedron axiom inductively shows that $k(G/N) \in \mathrm{thick}(\mathrm{cone}(\tau_{|k(G/N)}))$. This settles (a).

For (b), the complex $s := \mathrm{cone}(b_N)[2']$ becomes split exact when restricted to N since it is inflated from an exact complex on G/N. In degree one we have $s_1 = k(G/N)$, whereas $s_0 = k$. Hence Corollary 3.20 tells us that the complex s generates the tt-ideal $\mathrm{Ker}(\mathrm{Res}^G_N \colon \mathcal{K}(G) \to \mathcal{K}(N))$. We conclude by (13.1). □

13.3. **Corollary.** *Let $N \lhd G$ be of index p. Then* $\mathrm{cone}(a_N) \otimes \mathrm{cone}(b_N) = 0$.

Proof. By Lemma 13.2 it suffices to show $k(G/N) \otimes \mathrm{kos}_G(N) = 0$. By Frobenius, this follows from $\mathrm{Res}^G_N(\mathrm{kos}_G(N)) = 0$, which holds by (13.1). □

We now relate the spectrum of $\mathcal{K}(G)$ to the homogeneous spectrum of $\mathrm{H}^{\bullet\bullet}(G)$, in the spirit of [Bal10a]. The comparison map of [Bal10a] is denoted by $\rho^{\bullet}$ but we prefer a more descriptive notation (and here, the letter ρ is reserved for $\mathrm{Spc}(\mathrm{Res})$).

13.4. **Proposition.** *There is a continuous 'comparison' map*

$$\mathrm{comp}_G \colon \mathrm{Spc}(\mathcal{K}(G)) \longrightarrow \mathrm{Spec}^{\mathrm{h}}(\mathrm{H}^{\bullet\bullet}(G))$$

mapping a tt-prime $\mathcal{P}$ to the ideal generated by those homogeneous $f \in \mathrm{H}^{\bullet\bullet}(G)$ whose cone does not belong to $\mathcal{P}$. It is characterized by the fact that for all f

$$\mathrm{comp}_G^{-1}(Z(f)) = \mathrm{supp}(\mathrm{cone}(f)) = \{\, \mathcal{P} \mid f \text{ is not invertible in } \mathcal{K}(G)/\mathcal{P} \,\} \tag{13.5}$$

where $Z(f) = \{\, \mathfrak{p} \mid f \in \mathfrak{p} \,\}$ is the closed subset of $\mathrm{Spec}^{\mathrm{h}}(\mathrm{H}^{\bullet\bullet}(G))$ *defined by f.*

Proof. The fact that the homogeneous ideal $\mathrm{comp}_G(\mathcal{P})$ is prime comes from [Bal10a, Theorem 4.5]. Equation (13.5) is essentially a reformulation of the definition. □

13.6. *Remark.* The usual notation for $Z(f)$ would be $V(f)$, and $D(f)$ for its open complement. Here, we already use V for V_G and for $V_G(H)$, and the letter D is certainly overworked in our trade. So we stick to $Z(f)$ and $Z(f)^c$.

13.7. *Notation.* In view of Proposition 13.4, for any f, the open subset of $\mathrm{Spc}(\mathcal{K}(G))$

$$\mathrm{open}(f) := \mathrm{open}(\mathrm{cone}(f)) = \{\, \mathcal{P} \mid f \text{ is invertible in } \mathcal{K}(G)/\mathcal{P} \,\} \tag{13.8}$$

is the preimage by $\mathrm{comp}_G \colon \mathrm{Spc}(\mathcal{K}(G)) \to \mathrm{Spec}^{\mathrm{h}}(\mathrm{H}^{\bullet\bullet}(G))$ of the principal open $Z(f)^c = \{\, \mathfrak{p} \mid f \notin \mathfrak{p} \,\}$. It is the open locus of $\mathrm{Spc}(\mathcal{K}(G))$ where f is invertible. In particular, our distinguished elements a_N and b_N (see Definition 12.3) give us the following open subsets of $\mathrm{Spc}(\mathcal{K}(G))$, for every $N \in \mathcal{N}(G)$:

$\mathrm{open}(a_N) = \mathrm{comp}_G^{-1}(Z(a_N)^c)$, the open where a_N is invertible, and

$\mathrm{open}(b_N) = \mathrm{comp}_G^{-1}(Z(b_N)^c)$, the open where b_N is invertible.

Since $(c_N)^2 = 0$ by Remark 12.8, we do not have much use for $\mathrm{open}(c_N) = \varnothing$.

13.9. **Corollary.** *With notation as above, we have for every $N \lhd G$ of index p*

$$\mathrm{open}(a_N) \cup \mathrm{open}(b_N) = \mathrm{Spc}(\mathcal{K}(G)).$$

Proof. We compute $\mathrm{open}(a_N) \cup \mathrm{open}(b_N) = \mathrm{open}(\mathrm{cone}(a_N)) \cup \mathrm{open}(\mathrm{cone}(b_N)) = \mathrm{open}(\mathrm{cone}(a_N) \otimes \mathrm{cone}(b_N)) = \mathrm{open}(0_{\mathcal{K}(E)}) = \mathrm{Spc}(\mathcal{K}(G))$, using Corollary 13.3. □

13.10. *Remark.* Every object u_N is not only $\otimes$-invertible in $\mathcal{K}(G)$ but actually locally trivial over $\mathrm{Spc}(\mathcal{K}(G))$, which is a stronger property in general tt-geometry. Indeed, Corollary 13.9 tells us that around each point of $\mathrm{Spc}(\mathcal{K}(G))$, either u_N becomes isomorphic to $\mathbb{1}$ via a_N, or u_N becomes isomorphic to $\mathbb{1}[2']$ via b_N. This holds for *one* invertible u_N. We now construct a fine enough open cover of $\mathrm{Spc}(\mathcal{K}(G))$ such that *every* u_N is trivialized on each open.

13.11. **Proposition.** *Let $H \leqslant G$ be a p-subgroup. Define an open of* $\mathrm{Spc}(\mathcal{K}(G))$ *by*

$$U(H) = U_G(H) := \bigcap_{\substack{N \in \mathcal{N} \\ H \not\leqslant N}} \mathrm{open}(a_N) \ \cap \bigcap_{\substack{N \in \mathcal{N} \\ H \leqslant N}} \mathrm{open}(b_N). \tag{13.12}$$

Then the closed point $\mathcal{M}(H) \in \mathrm{Spc}(\mathcal{K}(G))$ belongs to this open $U(H)$. Consequently $\{U(H)\}_{H \in \mathrm{Sub}_p(G)}$ is an open cover of $\mathrm{Spc}(\mathcal{K}(G))$.

Proof. The point $\mathcal{M}(H) = \mathrm{Ker}(\mathbb{F}^H)$ belongs to $U(H)$ by Corollary 12.11. It follows by general tt-geometry that $\{U(H)\}_H$ is a cover: Let $\mathcal{P} \in \mathrm{Spc}(\mathcal{K}(G))$; there exists a closed point in $\overline{\{\mathcal{P}\}}$, that is, some $\mathcal{M}(H)$ that admits $\mathcal{P}$ as a generalization; but then $\mathcal{M}(H) \in U(H)$ forces $\mathcal{P} \in U(H)$ since open subsets are generalization-closed. □

For a p-group, we now discuss $U(H)$ at the two extremes $H = 1$ and $H = G$.

13.13. *Recollection.* Let G be a p-group and $F = F(G) = \cap_{N \in \mathcal{N}(G)} N$ be its Frattini subgroup. So $F \lhd G$ and G/F is the largest elementary abelian quotient of G.

13.14. **Proposition.** *Let G be a p-group with Frattini subgroup F. The closed complement of the open $U_G(1)$ is the support of $\mathrm{kos}_G(F)$, i.e. the closed support of the tt-ideal $\mathrm{Ker}(\mathrm{Res}^G_F)$ of $\mathcal{K}(G)$. In particular, if G is elementary abelian then $U_G(1)$ is equal to the cohomological open $V_G = \mathrm{Spc}(\mathrm{D_b}(kG)) \cong \mathrm{Spec^h}(\mathrm{H}^\bullet(G,k))$.*

Proof. By definition, $U(1) = \cap_{N \in \mathcal{N}} \mathrm{open}(b_N)$. By Lemma 13.2, its closed complement is $\cup_{N \in \mathcal{N}} \mathrm{supp}(\mathrm{kos}_G(N))$. By Corollary 7.17, for every $K \leqslant G$

$$\mathrm{supp}(\mathrm{kos}_G(K)) = \big\{\, \mathcal{P}(H,\mathfrak{p}) \,\big|\, H \not\leqslant_G K \,\big\} \tag{13.15}$$

(taking all possible $\mathfrak{p} \in V_{G /\!\!/ H}$). It follows that our closed complement of $U(1)$ is

$$\cup_{N \in \mathcal{N}(G)} \mathrm{supp}(\mathrm{kos}_G(N)) \overset{(13.15)}{=} \big\{\, \mathcal{P}(H,\mathfrak{p}) \,\big|\, \exists\, N \in \mathcal{N}(G) \text{ such that } H \not\leqslant_G N \,\big\}$$
$$= \big\{\, \mathcal{P}(H,\mathfrak{p}) \,\big|\, H \not\leqslant \cap_{N \in \mathcal{N}(G)} N \,\big\} = \big\{\, \mathcal{P}(H,\mathfrak{p}) \,\big|\, H \not\leqslant F \,\big\} \overset{(13.15)}{=} \mathrm{supp}(\mathrm{kos}_G(F)).$$

The statement with $\mathrm{Ker}(\mathrm{Res}^G_F)$ then follows from (13.1). Finally, if G is elementary abelian then $F = 1$ and $\mathrm{Ker}(\mathrm{Res}^G_1) = \mathcal{K}_{\mathrm{ac}}(G)$ is the tt-ideal of acyclic complexes. The complement of its support is $\mathrm{Spc}(\mathcal{K}(G)/\mathcal{K}_{\mathrm{ac}}(G)) = \mathrm{Spc}(\mathrm{D_b}(kG)) = V_G$. □

In the above proof, we showed that $\cup_{N \in \mathcal{N}} \mathrm{supp}(\mathrm{kos}(N)) = \mathrm{supp}(\mathrm{kos}(F))$ thanks to the fact that $\cap_{N \in \mathcal{N}} N = F$. So the very same argument gives us:

13.16. **Corollary.** *Let G be a p-group and let $N_1, \ldots, N_r \in \mathcal{N}(G)$ be some index-p subgroups such that $N_1 \cap \cdots \cap N_r$ is the Frattini subgroup F. (This can be realized with r equal to the p-rank of G/F.) Then $U_G(1) = \cap_{i=1}^r \mathrm{open}(b_{N_i})$ already. Hence if $\mathcal{P} \in \mathrm{open}(b_{N_i})$ for all $i = 1, \ldots, r$ then $\mathcal{P} \in \mathrm{open}(b_N)$ for all $N \in \mathcal{N}(G)$.* □

Let us turn to the open $U_G(H)$ for the p-subgroup at the other end: $H = G$.

13.17. **Proposition.** *Let G be a p-group. Then the complement of the open $U_G(G)$ is the union of the images of the spectra $\mathrm{Spc}(\mathcal{K}(H))$ under the maps $\rho_H = \mathrm{Spc}(\mathrm{Res}_H)$, over all the* proper *subgroups $H \lneqq G$.*

Proof. By Lemma 13.2, the closed complement of $U(G) = \cap_{N\in\mathcal{N}} \mathrm{open}(a_N)$ equals $\cup_{N\in\mathcal{N}} \mathrm{supp}(k(G/N))$. For every $H \leqslant G$, we have $\mathrm{supp}(k(G/H)) = \mathrm{Im}(\rho_H)$; see Proposition 4.7 if necessary. This gives the result because restriction to any proper subgroup factors via some index-p subgroup, since G is a p-group. □

13.18. *Remark.* Let G be a p-group. This open complement $U(G)$ of $\cup_{H\lneqq G} \mathrm{Im}(\rho_H)$ could be called the 'geometric open'. Indeed, the localization functor

$$\Phi^G : \mathcal{K}(G) \twoheadrightarrow \frac{\mathcal{K}(G)}{\langle k(G/H) \mid H \lneqq G\rangle}$$

corresponding to $U(G)$ is analogous to the way the geometric fixed-points functor is constructed in topology. For more on this topic, see Remark 4.11.

13.19. *Remark.* For G not a p-group, the open $U(G)$ is not defined (we assume $H \in \mathrm{Sub}_p(G)$ in Proposition 13.11) and the 'geometric open' is void anyway as we have $\mathrm{Im}(\rho_P) = \mathrm{Spc}(\mathcal{K}(G))$ for any p-Sylow $P \lneqq G$. The strategy to analyze non-p-groups is to first descend to the p-Sylow, using that Res_P is faithful.

13.20. *Remark.* We saw in Proposition 13.17 that the complement of $U(G)$ is covered by the images of the closed maps $\rho_H = \mathrm{Spc}(\mathrm{Res}_H)$ for $H \lneqq G$. We could wonder whether another closed map into $\mathrm{Spc}(\mathcal{K}(G))$ covers $U(G)$ itself. The answer is the closed immersion $\psi^F : \mathrm{Spc}(\mathcal{K}(G/F)) \hookrightarrow \mathrm{Spc}(\mathcal{K}(G))$ induced by the modular fixed-points functor Ψ^F with respect to the Frattini subgroup $F \lhd G$. This can be deduced from the results of Section 11 or verified directly, as we now outline. Indeed, every prime $\mathcal{P} = \mathcal{P}_G(K, \mathfrak{p})$ for $K \leqslant G$ and $\mathfrak{p} \in V_{G/\!\!/K}$ comes by Quillen from some elementary abelian subgroup $E = H/K \leqslant G/\!\!/K = (N_G K)/K$. One verifies that unless $N_G K = G$ and $H = G$, the prime $\mathcal{P}$ belongs to the image of $\rho_{G'}$ for a proper subgroup G' of G. Thus if $\mathcal{P}$ belongs to $U(G)$, we must have $E = H/K = G/K$ for $K \trianglelefteq G$. Such a K must contain the Frattini and the result follows.

14. Twisted cohomology under tt-functors

Still for a general finite group G, we gather some properties of the twisted cohomology ring $\mathrm{H}^{\bullet\bullet}(G)$ introduced in Definition 12.16. We describe its behavior under specific tt-functors, namely restriction, modular fixed-points and localization onto the open subsets $U_G(H)$. Recall that $\mathcal{N} = \mathcal{N}(G) = \{ N \lhd G \mid [G\!:\!N] = p \}$.

14.1. *Remark.* Twisted cohomology $\mathrm{H}^{\bullet\bullet}(G)$ is graded over a monoid of the form $\mathbb{Z} \times \mathbb{N}^\ell$. The ring homomorphisms induced by the above tt-functors will be homogeneous *with respect to a certain homomorphism* γ on the corresponding grading monoids, meaning of course that the image of a homogeneous element of degree (s, q) is homogeneous of degree $\gamma(s, q)$. The 'shift' part (in $\mathbb{Z}$) is rather straightforward. The 'twist' part (in $\mathbb{N}^\ell$) will depend on the effect of said tt-functors on the u_N.

Let us start with modular fixed-points, as they are relatively easy.

14.2. *Construction.* Let $H \trianglelefteq G$ be a normal subgroup. By Proposition 12.9, the tt-functor $\Psi^H\colon \mathcal{K}(G) \to \mathcal{K}(G/H)$ maps every u_N for $N \not\geq H$ to $\mathbb{1}$, whereas it maps u_N for $N \geq H$ to $u_{N/H}$. This defines a homomorphism of grading monoids

$$\gamma = \gamma_{\Psi^H}\colon \mathbb{Z} \times \mathbb{N}^{\mathcal{N}(G)} \to \mathbb{Z} \times \mathbb{N}^{\mathcal{N}(G/H)} \tag{14.3}$$

given by $\gamma(s,q) = (s,\bar{q})$ where $\bar{q}(N/H) = q(N)$ for every $N/H \in \mathcal{N}(G/H)$. In other words, $q \mapsto \bar{q}$ is simply restriction $\mathbb{N}^{\mathcal{N}(G)} \twoheadrightarrow \mathbb{N}^{\mathcal{N}(G/H)}$ along the canonical inclusion $\mathcal{N}(G/H) \hookrightarrow \mathcal{N}(G)$. By Proposition 12.9, for every twist $q \in \mathbb{N}^{\mathcal{N}(G)}$, we have a canonical isomorphism $\Psi^H(\mathbb{1}(q)) \cong \mathbb{1}(\bar{q})$. Therefore the modular fixed-points functor Ψ^H defines a ring homomorphism also denoted

$$\begin{array}{rccc} \Psi^H\colon & \mathrm{H}^{\bullet\bullet}(G) & \longrightarrow & \mathrm{H}^{\bullet\bullet}(G/H) \\ & \Big(\mathbb{1} \xrightarrow{f} \mathbb{1}(q)[s]\Big) & \longmapsto & \Big(\mathbb{1} \xrightarrow{\Psi^H(f)} \Psi^H(\mathbb{1}(q)[s]) \cong \mathbb{1}(\bar{q})[s]\Big) \end{array} \tag{14.4}$$

which is homogeneous with respect to γ_{Ψ^H} in (14.3).

Restriction is a little more subtle, as some twists pull-back to non-trivial shifts.

14.5. *Construction.* Let $\alpha\colon G' \to G$ be a group homomorphism. Restriction along α defines a tt-functor $\alpha^* = \mathrm{Infl}_{G'}^{\mathrm{Im}\,\alpha} \circ \mathrm{Res}^G_{\mathrm{Im}\,\alpha}\colon \mathcal{K}(G) \to \mathcal{K}(\mathrm{Im}\,\alpha) \to \mathcal{K}(G')$. Combining Proposition 12.10 for $\mathrm{Res}^G_{\mathrm{Im}\,\alpha}$ with the obvious behavior of the u_N under inflation (by construction), we see that $\alpha^*(u_N) \cong \mathbb{1}[2']$ if $N \geq \mathrm{Im}\,\alpha$ and $\alpha^*(u_N) \cong u_{\alpha^{-1}(N)}$ if $N \not\geq \mathrm{Im}\,\alpha$ (which is equivalent to $\alpha^{-1}(N) \in \mathcal{N}(G')$). Hence for every $(s,q) \in \mathbb{Z} \times \mathbb{N}^{\mathcal{N}(G)}$ we have a canonical isomorphism $\alpha^*(\mathbb{1}(q)[s]) \cong \mathbb{1}(q')[s']$ where $s' = s + 2' \sum_{N \geq \mathrm{Im}\,\alpha} q(N)$ and $q'\colon \mathcal{N}(G') \to \mathbb{N}$ is defined for every $N' \in \mathcal{N}(G')$ as

$$q'(N') = \sum_{N \in \mathcal{N}(G) \text{ s.t. } \alpha^{-1}(N) = N'} q(N).$$

(In particular $q'(N') = 0$ if $N' \not\geq \ker(\alpha)$.) These formulas define a homomomorphism $(s,q) \mapsto (s',q')$ of abelian monoids that we denote

$$\gamma = \gamma_{\alpha^*}\colon \mathbb{Z} \times \mathbb{N}^{\mathcal{N}(G)} \to \mathbb{Z} \times \mathbb{N}^{\mathcal{N}(G')}. \tag{14.6}$$

The restriction functor α^* defines a ring homomorphism

$$\begin{array}{rccc} \alpha^*\colon & \mathrm{H}^{\bullet\bullet}(G) & \longrightarrow & \mathrm{H}^{\bullet\bullet}(G') \\ & \big(\mathbb{1} \xrightarrow{f} \mathbb{1}(q)[s]\big) & \longmapsto & \big(\mathbb{1} \xrightarrow{\alpha^*(f)} \alpha^*(\mathbb{1}(q)[s]) \cong \mathbb{1}(q')[s']\big) \end{array} \tag{14.7}$$

which is homogeneous with respect to γ_{α^*} in (14.6).

14.8. *Remark.* For instance, $\alpha\colon G \twoheadrightarrow G/H$ can be the quotient by a normal subgroup $H \trianglelefteq G$. In that case α^* is inflation, which is a section of modular fixed-points Ψ^H. It follows that the homomorphism Ψ^H in (14.4) is split surjective. (This also means that the composed effect on gradings $\gamma_{\Psi^H} \circ \gamma_{\alpha^*} = \mathrm{id}$ is trivial.)

Without changing the group G, we can also localize the twisted cohomology ring $\mathrm{H}^{\bullet\bullet}(G)$ by restricting to an open $U(H)$ of $\mathrm{Spc}(\mathcal{K}(G))$, as defined in Proposition 13.11. Recall the elements $a_N, b_N \in \mathrm{H}^{\bullet\bullet}(G)$ from Definition 12.3.

14.9. *Definition.* Let $H \leqslant G$ be a p-subgroup. Let $S_H \subset \mathrm{H}^{\bullet\bullet}(G)$ be the multiplicative subset of the graded ring $\mathrm{H}^{\bullet\bullet}(G)$ generated by all a_N such that $H \not\leqslant N$ and all b_N such that $H \leqslant N$, for all $N \in \mathcal{N}(G)$. Recall that the a_N and b_N are central by Remark 12.20. We define a $\mathbb{Z}$-graded ring

$$\mathcal{O}^{\bullet}_G(H) := \big(\mathrm{H}^{\bullet\bullet}(G)[S_H^{-1}]\big)_{0\text{-twist}} \tag{14.10}$$

as the twist-zero part of the localization of $\mathrm{H}^{\bullet\bullet}(G)$ with respect to S_H. Explicitly, the homogeneous elements of $\mathcal{O}^{\bullet}_G(H)$ consist of fractions $\frac{f}{g}$ where $f, g \in \mathrm{H}^{\bullet\bullet}(G)$ are such that $g\colon \mathbb{1} \to \mathbb{1}(q)[t]$ is a product of the chosen a_N, b_N in S_H, meaning that $\mathbb{1}(q)[t]$ is the $\otimes$-product of the corresponding u_N for a_N and $u_N[-2']$ for b_N, whereas $f\colon \mathbb{1} \to \mathbb{1}(q)[s]$ is any morphism in $\mathcal{K}(G)$ with the same $\mathcal{N}$-twist q as the denominator. Thus $\mathcal{O}^{\bullet}_G(H)$ is $\mathbb{Z}$-graded by the shift only: The degree of $\frac{f}{g}$ is the difference $s - t$ between the shifts of f and g.

14.11. *Remark.* It follows from Lemma 12.12 (and Remark 12.24) that the $\mathbb{Z}$-graded ring $\mathcal{O}^{\bullet}_G(H)$ is generated as a k-algebra by the elements

$$\{\, \zeta_N^+, \xi_N^+ \,|\, H \leqslant N \,\} \cup \{\, \zeta_N^-, \xi_N^- \,|\, H \not\leqslant N \,\}$$

where $\zeta_N^+ = a_N/b_N$ is of degree $+2'$ and $\zeta_N^- = b_N/a_N$ of degree $-2'$ as in Remark 12.7, and where (only for p odd) the additional elements $\xi_N^{\pm}$ are $\xi_N^+ := c_N/b_N$ of degree $+1$, and $\xi_N^- := c_N/a_N$ of degree -1. (For $p = 2$, simply ignore the $\xi_N^{\pm}$.) In general, all these elements satisfy some relations; see Theorem 17.13. Beware that here ξ_N^- is never the inverse of ξ_N^+. In fact, both are nilpotent.

In fact, we can perform the central localization of the whole category $\mathcal{K}(G)$

$$\mathcal{L}(H) = \mathcal{L}_G(H) := \mathcal{K}(G)[S_H^{-1}]$$

with respect to the central multiplicative subset S_H of Definition 14.9.

14.12. *Construction.* The tt-category $\mathcal{L}(H) = \mathcal{K}(G)[S_H^{-1}]$ has the same objects as $\mathcal{K}(G)$ and morphisms $x \to y$ of the form $\frac{f}{g}$ where $g\colon \mathbb{1} \to u$ belongs to S_H, for u a tensor-product of shifts of u_N's according to g (as in Definition 14.9) and where $f\colon x \to u \otimes y$ is any morphism in $\mathcal{K}(G)$ with 'same' twist u as the denominator g. This category $\mathcal{K}(G)[S_H^{-1}]$ is also the Verdier quotient of $\mathcal{K}(G)$ by the tt-ideal $\langle\{\, \mathrm{cone}(g) \,|\, g \in S_H \,\}\rangle$ and the above fraction $\frac{f}{g}$ corresponds to the Verdier fraction $x \xrightarrow{\ f\ } u \otimes y \xleftarrow{\ g\otimes 1\ } y$. See [Bal10a, § 3] if necessary.

The $\mathbb{Z}$-graded endomorphism ring $\mathrm{End}^{\bullet}_{\mathcal{L}(H)}(\mathbb{1})$ of the unit in $\mathcal{L}(H) = \mathcal{K}(G)[S_H^{-1}]$ is thus the $\mathbb{Z}$-graded ring $(S_H^{-1}\,\mathrm{H}^{\bullet\bullet}(G))_{0\text{-twist}} = \mathcal{O}^{\bullet}_G(H)$ of Definition 14.9.

There is a general localization $\mathcal{K}_{|U}$ of a tt-category $\mathcal{K}$ over a quasi-compact open $U \subseteq \mathrm{Spc}(\mathcal{K})$ with closed complement Z. It is defined as $\mathcal{K}_{|U} = (\mathcal{K}/\mathcal{K}_Z)^\natural$. If we apply this to $U = U(H)$, we deduce from (13.12) that $U = \cap_{g \in S_H} \mathrm{open}(g)$ has closed complement $Z = \cup_{g \in S_H} \mathrm{supp}(\mathrm{cone}(g))$ whose tt-ideal $\mathcal{K}(G)_Z$ is the above $\langle\{\, \mathrm{cone}(g) \,|\, g \in S_H \,\}\rangle$. In other words, the idempotent-completion of our $\mathcal{L}_G(H) = \mathcal{K}(G)[S_H^{-1}]$ is exactly $\mathcal{K}(G)_{|U(H)}$. As with any localization, we know that $\mathrm{Spc}(\mathcal{L}_G(H))$ is a subspace of $\mathrm{Spc}(\mathcal{K}(G))$, given here by $U = \cap_{g \in S_H} \mathrm{open}(g) = U(H)$.

14.13. *Example.* For $G = E$ elementary abelian and the subgroup $H = 1$, the category $\mathcal{L}_E(1) = \mathcal{K}(E)_{|U(1)}$ in Construction 14.12 is simply the derived category $\mathcal{L}_E(1) = \mathrm{D_b}(E)$, by Proposition 13.14. In that case, $\mathcal{O}^{\bullet}_E(1) \cong \mathrm{H}^{\bullet}(E;k)$ is the actual cohomology ring of E. Since $H=1\leqslant N$ for all N, we are inverting all the b_N and no a_N. As noted in Corollary 13.16, we obtain the same ring (the cohomology of E) as soon as we invert enough $b_{N_1},\ldots,b_{N_r}$, namely, as soon as $N_1\cap\cdots\cap N_r=1$.

We again obtain an induced homomorphism of multi-graded rings.

14.14. *Construction.* Let $H\leqslant G$ be a p-subgroup and consider the above central localization $(-)_{\restriction U(H)}\colon \mathcal{K}(G)\twoheadrightarrow\mathcal{L}_G(H)$. As explained in Remark 13.10, the morphisms a_N and b_N give us explicit isomorphisms $(u_N)_{\restriction U(H)}\cong\mathbb{1}$ if $N\not\geq H$ and $(u_N)_{\restriction U(H)}\cong\mathbb{1}[2']$ if $N\geq H$. This yields a homomorphism on the grading

$$\gamma=\gamma_{U(H)}\colon \mathbb{Z}\times\mathbb{N}^{\mathcal{N}(G)}\to\mathbb{Z} \tag{14.15}$$

defined by $\gamma(s,q)=s+2'\sum_{N\geq H}q(N)$ and we obtain a ring homomorphism

$$(-)_{\restriction U(H)}\colon\ \mathrm{H}^{\bullet\bullet}(G)\longrightarrow\mathrm{End}^{\bullet}_{\mathcal{L}_G(H)}(\mathbb{1})=\mathcal{O}^{\bullet}_G(H) \tag{14.16}$$

which is homogeneous with respect to the homomorphism $\gamma_{U(H)}$ of (14.15).

14.17. *Remark.* It is easy to verify that the continuous maps induced on homogeneous spectra by the ring homomorphisms constructed above are compatible with the comparison map of Proposition 13.4. In other words, if $F\colon\mathcal{K}(G)\to\mathcal{K}(G')$ is a tt-functor and if the induced homomorphism $F\colon\ \mathrm{H}^{\bullet\bullet}(G)\to\mathrm{H}^{\bullet\bullet}(G')$ is homogenous with respect to $\gamma=\gamma_F\colon\mathbb{Z}\times\mathbb{N}^{\mathcal{N}(G)}\to\mathbb{Z}\times\mathbb{N}^{\mathcal{N}(G')}$, for instance $F=\Psi^H$ or $F=\alpha^*$ as in Constructions 14.2 and 14.5, then the following square commutes:

$$\begin{array}{ccc}\mathrm{Spc}(\mathcal{K}(G')) & \xrightarrow{\ \mathrm{Spc}(F)\ } & \mathrm{Spc}(\mathcal{K}(G))\\ \downarrow{\scriptstyle \mathrm{comp}_{G'}} & & \downarrow{\scriptstyle \mathrm{comp}_{G}}\\ \mathrm{Spec}^{\mathrm{h}}(\mathrm{H}^{\bullet\bullet}(G')) & \xrightarrow{\ \mathrm{Spec}^{\mathrm{h}}(F)\ } & \mathrm{Spec}^{\mathrm{h}}(\mathrm{H}^{\bullet\bullet}(G)).\end{array} \tag{14.18}$$

This follows from $F(\mathrm{cone}(f))\simeq\mathrm{cone}(F(f))$ in $\mathcal{K}(G')$ for any $f\in\mathrm{H}^{\bullet\bullet}(G)$.

14.19. *Remark.* Similarly, for every $H\in\mathrm{Sub}_p(G)$ the following square commutes

$$\begin{array}{ccc}U_G(H)=\mathrm{Spc}(\mathcal{L}_G(H)) & \hookrightarrow & \mathrm{Spc}(\mathcal{K}(G))\\ {\scriptstyle \mathrm{comp}_{\mathcal{L}(H)}}\downarrow & & \downarrow{\scriptstyle \mathrm{comp}_{G}}\\ \mathrm{Spec}^{\mathrm{h}}(\mathcal{O}^{\bullet}_G(H)) & \hookrightarrow & \mathrm{Spec}^{\mathrm{h}}(\mathrm{H}^{\bullet\bullet}(G))\end{array} \tag{14.20}$$

where the left-hand vertical map is the classical comparison map of [Bal10a] for the tt-category $\mathcal{L}_G(H)$ and the $\otimes$-invertible $\mathbb{1}[1]$. The horizontal inclusions are the ones corresponding to the localizations with respect to S_H, as in Constructions 14.12 and 14.14. In fact, it is easy to verify that the square (14.20) is cartesian, in view of $U_G(H)=\bigcap_{g\in S_H}\mathrm{open}(g)=\bigcap_{g\in S_H}\mathrm{comp}_G^{-1}(Z(g)^c)$ by Construction 14.12 and (13.5).

We can combine the above functors. Here is a useful example.

14.21. **Proposition.** *Let $H \trianglelefteq G$ be a normal subgroup such that G/H is elementary abelian. Then we have a commutative square*

$$\text{(14.22)} \qquad \begin{array}{ccc} V_{G/H} = \mathrm{Spc}(\mathrm{D_b}(k(G/H))) & \xrightarrow{\check{\psi}^H} & \mathrm{Spc}(\mathcal{K}(G)) \\ {\scriptstyle \mathrm{comp}_{\mathrm{D_b}(k(G/H))}}\downarrow \simeq & & \downarrow {\scriptstyle \mathrm{comp}_G} \\ \mathrm{Spec^h}(\mathrm{H}^\bullet(G/H,k)) & \hookrightarrow & \mathrm{Spec^h}(\mathrm{H}^{\bullet\bullet}(G)) \end{array}$$

and in particular, its diagonal $\mathrm{comp}_G \circ \check{\psi}^H$ *is injective.*

Proof. The functor $\check{\Psi}^H : \mathcal{K}(G) \to \mathrm{D_b}(k(G/H))$ is the modular fixed-points functor $\Psi^H : \mathcal{K}(G) \to \mathcal{K}(G/H)$ composed with $\Upsilon_{G/H} : \mathcal{K}(G/H) \twoheadrightarrow \mathrm{D_b}(k(G/H))$, which is the central localization $(-)_{\restriction U(1)}$ over the cohomological open, by Proposition 13.14; see Example 14.13. Thus we obtain two commutative squares (14.20) and (14.18):

$$\begin{array}{ccccc} \mathrm{Spc}(\mathrm{D_b}(k(G/H))) & \overset{\upsilon_{G/H}}{\hookrightarrow} & \mathrm{Spc}(\mathcal{K}(G/H)) & \xrightarrow{\psi^H} & \mathrm{Spc}(\mathcal{K}(G)) \\ {\scriptstyle \mathrm{comp}_{\mathrm{D_b}(k(G/H))}}\downarrow \simeq & & \downarrow {\scriptstyle \mathrm{comp}_{G/H}} & & \downarrow {\scriptstyle \mathrm{comp}_G} \\ \mathrm{Spec^h}(\mathrm{H}^\bullet(G/H,k)) & \hookrightarrow & \mathrm{Spec^h}(\mathrm{H}^{\bullet\bullet}(G/H)) & \hookrightarrow & \mathrm{Spec^h}(\mathrm{H}^{\bullet\bullet}(G)) \end{array}$$

the left-hand one for the central localization of $\mathcal{K}(G/H)$ over the open $U_{G/H}(1) = V_{G/H}$, and the right-hand one for the tt-functor $\Psi^H : \mathcal{K}(G) \to \mathcal{K}(G/H)$. Note that the bottom-right map is injective because the ring homomorphism in question, $\Psi^H : \mathrm{H}^{\bullet\bullet}(G) \to \mathrm{H}^{\bullet\bullet}(G/H)$ defined in (14.4), is surjective by Remark 14.8. ☐

15. The elementary abelian case

In this central section, we apply the general constructions of Sections 12 to 14 in the case of $G = E$ elementary abelian. We start with a key fact that is obviously wrong in general (*e.g.* for a non-cyclic simple group, the target space is just a point).

15.1. **Proposition.** *Let E be an elementary abelian group. The comparison map*

$$\mathrm{comp}_E : \mathrm{Spc}(\mathcal{K}(E)) \to \mathrm{Spec^h}(\mathrm{H}^{\bullet\bullet}(E))$$

of Proposition 13.4 is injective.

Proof. Let $H, N \leqslant E$ with $[E : N] = p$. Suppose first that $H \not\leqslant N$. We use the map $\check{\psi}^H = \mathrm{Spc}(\check{\Psi}^H) : V_{E/H} \to \mathrm{Spc}(\mathcal{K}(E))$ of Recollection 10.7. Then

$$\begin{array}{lll} (\check{\psi}^H)^{-1}(\mathrm{open}(b_N)) & = (\check{\psi}^H)^{-1}(\mathrm{open}(\mathrm{cone}(b_N))) & \text{by definition, see (13.8)} \\ & = \mathrm{open}(\mathrm{cone}(\check{\Psi}^H(b_N))) & \text{by general tt-geometry} \\ & = \mathrm{open}(\mathrm{cone}(0 : \mathbb{1} \to \mathbb{1})) & \text{by Proposition 12.9} \\ & = \mathrm{open}(\mathbb{1} \oplus \mathbb{1}[1]) = \varnothing. & \end{array}$$

Thus $\mathrm{Im}(\check{\psi}^H)$ does not meet $\mathrm{open}(b_N)$ when $H \not\leqslant N$. Suppose now that $H \leqslant N$. A similar computation as above shows that $(\check{\psi}^H)^{-1}(\mathrm{open}(b_N)) = \mathrm{Spc}(\mathrm{D_b}(k(E/H)))$ since in that case $\check{\Psi}^H(b_N)$ is an isomorphism in $\mathrm{D_b}(k(E/H))$. Therefore $\mathrm{Im}(\check{\psi}^H) \subseteq \mathrm{open}(b_N)$ when $H \leqslant N$. Combining both observations, we have

$$\text{(15.2)} \qquad \mathrm{Im}(\check{\psi}^H) \cap \mathrm{open}(b_N) \neq \varnothing \iff H \leqslant N.$$

Let now $\mathcal{P}, \mathcal{Q} \in \mathrm{Spc}(\mathcal{K}(E))$ be such that $\mathrm{comp}_E(\mathcal{P}) = \mathrm{comp}_E(\mathcal{Q})$ in $\mathrm{Spec^h}(\mathrm{H}^{\bullet\bullet}(E))$. Say $\mathcal{P} = \mathcal{P}_E(H, \mathfrak{p})$ and $\mathcal{Q} = \mathcal{P}_E(K, \mathfrak{q})$ for $H, K \leqslant E$ and $\mathfrak{p} \in V_{E/H}$ and $\mathfrak{q} \in V_{E/K}$.

(See Recollection 10.7.) The assumption $\mathrm{comp}_E(\mathcal{P}) = \mathrm{comp}_E(\mathcal{Q})$ implies that $\mathcal{P} \in \mathrm{open}(f)$ if and only if $\mathcal{Q} \in \mathrm{open}(f)$, for every $f \in \mathrm{H}^{\bullet\bullet}(E)$. In particular applying this to $f = b_N$, we see that for every index-p subgroup $N \lhd E$ we have $\mathcal{P} \in \mathrm{open}(b_N)$ if and only if $\mathcal{Q} \in \mathrm{open}(b_N)$. By (15.2), we have for every $N \in \mathcal{N}(E)$

$$H \leqslant N \quad \Longleftrightarrow \quad K \leqslant N.$$

Since E is elementary abelian, this forces $H = K$. So we have two points $\mathfrak{p}, \mathfrak{q} \in V_{E/H}$ that go to the same image under $V_{E/H} \xrightarrow{\check{\psi}^H} \mathrm{Spc}(\mathcal{K}(E)) \xrightarrow{\mathrm{comp}_E} \mathrm{Spec}^{\mathrm{h}}(\mathrm{H}^{\bullet\bullet}(E))$ but we know that this map in injective by Proposition 14.21 for $G = E$. ☐

In fact, we see that the open $U(H)$ of $\mathrm{Spc}(\mathcal{K}(E))$ defined in Proposition 13.11 matches perfectly the open $\mathrm{Spec}^{\mathrm{h}}(\mathcal{O}^{\bullet}_E(H))$ of $\mathrm{Spec}^{\mathrm{h}}(\mathrm{H}^{\bullet\bullet}(E))$ in Definition 14.9.

15.3. **Theorem.** *Let E be an elementary abelian p-group. Let $H \leqslant E$ be a subgroup. Then the comparison map of Proposition 13.4 restricts to a homeomorphism*

$$\mathrm{comp}_E \colon U(H) \xrightarrow{\sim} \mathrm{Spec}^{\mathrm{h}}(\mathcal{O}^{\bullet}_E(H))$$

where $\mathcal{O}^{\bullet}_E(H)$ is the $\mathbb{Z}$-graded endomorphism ring of the unit $\mathbb{1}$ in the localization $\mathcal{L}_E(H)$ of $\mathcal{K}(E)$ over the open $U(H)$.

Proof. Recall the tt-category $\mathcal{L}(H) = \mathcal{L}_E(H) := \mathcal{K}(E)[S_H^{-1}]$ of Construction 14.12, where $S_H \subset \mathrm{H}^{\bullet\bullet}(E)$ is the multiplicative subset generated by the homogeneous elements $\{ a_N \mid H \not\leqslant N \} \cup \{ b_N \mid H \leqslant N \}$ of Definition 14.9. In view of Remark 14.19, it suffices to show that the map $\mathrm{comp}_{\mathcal{L}(H)} \colon \mathrm{Spc}(\mathcal{L}(H)) \to \mathrm{Spec}^{\mathrm{h}}(\mathcal{O}^{\bullet}_E(H))$ is a homeomorphism. We have injectivity by Proposition 15.1. We also know that $\mathcal{O}^{\bullet}_E(H)$ is noetherian by Theorem 12.21. It follows from [Bal10a] that $\mathrm{comp}_{\mathcal{L}(H)}$ is surjective. Hence it is a continuous bijection and we only need to prove that it is a closed map.

We claim that $\mathcal{L}(H)$ is generated by its $\otimes$-unit $\mathbb{1}$. Namely, let $\mathcal{J} = \mathrm{thick}_{\mathcal{L}(H)}(\mathbb{1})$ be the thick subcategory of $\mathcal{L}(H)$ generated by $\mathbb{1}$ and let us see that $\mathcal{J} = \mathcal{L}(H)$. Observe that $\mathcal{J}$ is a sub-tt-category of $\mathcal{L}(H)$. Let $N \in \mathcal{N}$ be an index-p subgroup. We claim that $k(E/N)$ belongs to $\mathcal{J}$. If $N \not\geq H$, then a_N is inverted in $\mathcal{L}(H)$, so $k(E/N) = 0$ in $\mathcal{L}(H)$ by Lemma 13.2 (a). If $N \geq H$, then $b_N \colon \mathbb{1} \to u_N[-2']$ is inverted, so $u_N \in \mathcal{J}$ and we conclude again by Lemma 13.2 (a) since $a_N \colon \mathbb{1} \to u_N$ is now a morphism in $\mathcal{J}$. For a general proper subgroup $K < E$, the module $k(E/K)$ is a tensor product of $k(E/N)$ for some $N \in \mathcal{N}$. (Here we use E elementary abelian again.) Hence $k(E/K)$ also belongs to $\mathcal{J}$ as the latter is a sub-tt-category of $\mathcal{L}(H)$. In short $\mathcal{J}$ contains all generators $k(E/H)$ for $H \leqslant E$. Therefore $\mathcal{L}(H) = \mathcal{J}$ is indeed generated by its unit. It follows from this and from noetherianity of $\mathrm{End}^{\bullet}_{\mathcal{L}(H)}(\mathbb{1}) = \mathcal{O}^{\bullet}_E(H)$ that $\mathrm{Hom}^{\bullet}_{\mathcal{L}(H)}(x, y)$ is a finitely generated $\mathcal{O}^{\bullet}_E(H)$-module for every $x, y \in \mathcal{L}(H)$. We conclude from a general tt-geometric fact, observed by Lau [Lau23, Proposition 2.7], that the map comp must then be closed. ☐

15.4. **Corollary.** *Let E be an elementary abelian p-group. Let $\mathcal{O}^{\bullet}_E$ be the sheaf of $\mathbb{Z}$-graded rings on $\mathrm{Spc}(\mathcal{K}(E))$ obtained by sheafifying $U \mapsto \mathrm{End}^{\bullet}_{\mathcal{K}(E)_{|U}}(\mathbb{1})$. Then $(\mathrm{Spc}(\mathcal{K}(E)), \mathcal{O}^{\bullet}_E)$ is a Dirac scheme in the sense of* [HP23].

Proof. We identified an affine cover $\{U(H)\}_{H \leqslant E}$ in Theorem 15.3. ☐

15.5. *Remark.* This result further justifies the notation for the ring $\mathcal{O}^{\bullet}_E(H)$ in Definition 14.9. Indeed, this $\mathcal{O}^{\bullet}_E(H)$ is also the ring of sections $\mathcal{O}^{\bullet}_E(U(H))$ of the $\mathbb{Z}$-graded structure sheaf $\mathcal{O}^{\bullet}_E$ over the open $U(H)$ of Proposition 13.11.

15.6. **Corollary.** *Let E be an elementary abelian p-group. Then the comparison map of Proposition 13.4 is an open immersion. More precisely, it defines a homeomorphism between* $\mathrm{Spc}(\mathcal{K}(E))$ *and the following open subspace of* $\mathrm{Spec^h}(\mathrm{H}^{\bullet\bullet}(E))$*:*

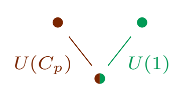
$$\big\{\, \mathfrak{p} \in \mathrm{Spec^h}(\mathrm{H}^{\bullet\bullet}(E)) \,\big|\, \text{for all } N \lhd E \text{ of index } p \text{ either } a_N \notin \mathfrak{p} \text{ or } b_N \notin \mathfrak{p} \,\big\}. \tag{15.7}$$

Proof. By Proposition 15.1, the (continuous) comparison map is injective. Therefore, it being an open immersion can be checked locally on the domain. By Proposition 13.11, the open $U(H)$ form an open cover of $\mathrm{Spc}(\mathcal{K}(E))$. Theorem 15.3 tells us that each $U(H)$ is homeomorphic to the following open of $\mathrm{Spec^h}(\mathrm{H}^{\bullet\bullet}(E))$

$$U'(H) := \bigcap_{N \not\geq H} Z(a_N)^c \cap \bigcap_{N \geq H} Z(b_N)^c$$

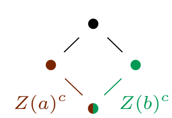
(recall that $Z(f)^c = \{\, \mathfrak{p} \,|\, f \notin \mathfrak{p} \,\}$ is our notation for a principal open). So it suffices to verify that the union $\cup_{H \leqslant E} U'(H)$, is the open subspace of the statement (15.7). Let $\mathfrak{p} \in U'(H)$ for some $H \leqslant E$ and let $N \in \mathcal{N}(E)$; then clearly either $N \not\geq H$ in which case $a_N \notin \mathfrak{p}$, or $N \geq H$ in which case $b_N \notin \mathfrak{p}$. Conversely let $\mathfrak{p}$ belong to the open (15.7) and define $H = \cap_{M \in \mathcal{N}\, \mathrm{s.t.}\, b_M \notin \mathfrak{p}} M$. We claim that $\mathfrak{p} \in U'(H)$. Let $N \in \mathcal{N}$. If $N \not\geq H$ then $b_N \in \mathfrak{p}$ by construction of H and therefore $a_N \notin \mathfrak{p}$. So the last thing we need to prove is that $N \geq H$ implies $b_N \notin \mathfrak{p}$. One should be slightly careful here, as H was defined as the intersection of the $M \in \mathcal{N}$ such that $b_M \notin \mathfrak{p}$, and certainly such M's will contain H, but we need to see why *every* $N \geq H$ satisfies $b_N \notin \mathfrak{p}$. This last fact follows from Corollary 13.16 applied to E/H. ☐

15.8. *Example.* Consider the spectrum of $\mathcal{K}(C_p)$ for the cyclic group C_p of order p. By Example 12.22, the reduced ring $\mathcal{O}^{\bullet}_{C_p}(1)_{\mathrm{red}}$ is $k[\zeta^+]$ with $\zeta^+ = a/b$ in degree $2'$ while $\mathcal{O}^{\bullet}_{C_p}(C_p)_{\mathrm{red}} = k[\zeta^-]$ with $\zeta^- = b/a$. (The former is also Example 14.13.) Each of these has homogeneous spectrum the Sierpiński space and we easily deduce that

(15.9) $\mathrm{Spc}(\mathcal{K}(C_p)) =$ 

confirming the computation of $\mathrm{Spc}(\mathcal{K}(C_{p^n}))$ in Proposition 8.3 for $n = 1$.

We can also view this as an instance of Corollary 15.6. Namely, still by Example 12.22, the reduced ring $\mathrm{H}^{\bullet\bullet}(C_p)_{\mathrm{red}}$ is $k[a, b]$ with a in degree 0 and b in degree $-2'$. Its homogeneous spectrum has one more point at the top:

$\mathrm{Spec^h}(\mathrm{H}^{\bullet\bullet}(C_p)) =$ 

and this superfluous closed point $\langle a, b \rangle$ lies outside of the open subspace (15.7).

15.10. *Remark.* Let $K \leqslant H \leqslant E$. The functor $\Psi^K \colon \mathcal{K}(E) \to \mathcal{K}(E/K)$ passes, by Proposition 12.9, to the localizations over $U_E(H)$ and $U_{E/K}(H/K)$, respectively. On the $\mathbb{Z}$-graded endomorphisms rings, we get a homomorphism $\Psi^K \colon \mathcal{O}^{\bullet}_E(H) \to \mathcal{O}^{\bullet}_{E/K}(H/K)$ that on generators a_N, b_N is given by the formulas of Proposition 12.9. By Remark 14.8 this homomorphism $\Psi^K \colon \mathcal{O}^{\bullet}_E(H) \to \mathcal{O}^{\bullet}_{E/K}(H/K)$ is surjective.

15.11. **Proposition.** *For every elementary abelian group E, the spectrum* $\mathrm{Spc}(\mathcal{K}(E))$ *admits a unique generic point η_E, namely the one of the cohomological open V_E.*

Proof. We proceed by induction on the p-rank. Let us write $\eta_E = \mathcal{P}_E(1, \sqrt{0})$ for the generic point of V_E, corresponding to the ideal $\sqrt{0}$ of nilpotent elements in $\mathrm{H}^\bullet(E;k)$. Similarly, for every $K \leqslant E$, let us write $\eta_E(K) = \mathcal{P}_E(K, \eta_{E/K})$ for the generic point of the stratum $V_E(K) \simeq V_{E/K}$. We need to prove that every point $\eta_E(K)$ belongs to the closure of $\eta_E = \eta_E(1)$ in $\mathrm{Spc}(\mathcal{K}(E))$. It suffices to show this for every cyclic subgroup $H < E$, by an easy induction argument on the rank, using the fact that $\psi^H\colon \mathrm{Spc}(\mathcal{K}(E/H)) \hookrightarrow \mathrm{Spc}(\mathcal{K}(E))$ is closed. So let $H \leqslant E$ be cyclic.

Note that inflation $\mathrm{Infl}_E^{E/H}\colon \mathcal{K}(E/H) \to \mathcal{K}(E)$ passes to the localization of the former with respect to all $b_{N/H}$ for all $N \in \mathcal{N}(E)$ containing H (which is just the derived category of E/H) and of the latter with respect to the corresponding b_N:

$$\mathrm{Infl}_E^{E/H}\colon \mathrm{D_b}(k(E/H)) \longrightarrow \mathcal{K}(E)\big[\{\, b_N \,\big|\, N \geq H \,\}^{-1}\big]. \tag{15.12}$$

This being a central localization of a fully-faithful functor with respect to a multiplicative subset in the source, it remains fully-faithful. One can further localize both categories with respect to all non-nilpotent $f \in \mathrm{H}^\bullet(E/H;k)$ in the source, to obtain a fully-faithful

$$\mathrm{Infl}_E^{E/H}\colon \mathrm{D_b}(k(E/H))[\{\, f \,\big|\, f \notin \sqrt{0}\,\}^{-1}] \longrightarrow \mathcal{L} \tag{15.13}$$

where $\mathcal{L}$ is obtained from $\mathcal{K}(E)$ by first inverting all b_N for $N \geq H$ as in (15.12) and then inverting all $\mathrm{Infl}_E^{E/H}(f)$ for $f \in \mathrm{H}^\bullet(E/H;k) \smallsetminus \sqrt{0}$.

At the level of spectra, $\mathrm{Spc}(\mathcal{L})$ is a subspace of $\mathrm{Spc}(\mathcal{K}(E))$. By construction, it meets the closed subset $\mathrm{Im}(\psi^H) \cong \mathrm{Spc}(\mathcal{K}(E/H))$ of $\mathrm{Spc}(\mathcal{K}(E))$ only at the image of the generic point $\eta_E(H)$. Indeed, inverting all b_N for $N \geq H$ on $\mathrm{Im}(\psi^H)$ corresponds to inverting all $b_{N/H}$ in $\mathcal{K}(E/H)$, hence shows that $\mathrm{Spc}(\mathcal{L}) \cap \mathrm{Im}(\psi^H)$ is in the image under ψ^H of the cohomological open $V_{E/H}$. Similarly, inverting all $f \notin \sqrt{0}$ removes all non-generic points of $V_{E/H}$. In particular, the generic point $\eta_E(H)$ of $V_E(H)$ is now a closed point of the subspace $\mathrm{Spc}(\mathcal{L})$ of $\mathrm{Spc}(\mathcal{K}(E))$.

Using that (15.13) is fully-faithful and that the endomorphism ring of the source is the cohomology of E/H localized at its generic point (in particular not a product of two rings), we see that $\mathcal{L}$ is not a product of two tt-categories and therefore $\mathrm{Spc}(\mathcal{L})$ is not disconnected. Also η_E belongs to $\mathrm{Spc}(\mathcal{L})$ and is distinct from $\eta_E(H)$. Hence the closed point $\eta_E(H) \in \mathrm{Spc}(\mathcal{L})$ cannot be isolated. Thus $\eta_E(H)$ belongs to the closure of some other point in $\mathrm{Spc}(\mathcal{L})$.

Let then $\mathcal{Q} \in \mathrm{Spc}(\mathcal{K}(E))$ be a point in the subspace $\mathrm{Spc}(\mathcal{L})$, such that $\mathcal{Q} \neq \eta_E(H)$ and $\eta_E(H) \in \overline{\{\mathcal{Q}\}}$, which reads $\mathcal{Q} \subsetneq \eta_E(H)$. We know by Corollary 7.13 that this can only occur for $\mathcal{Q} = \mathcal{P}(H', \mathfrak{p})$ with $H' \leqslant H$, that is, either $H' = H$ or $H' = 1$ since here H was taken cyclic. The case $H' = H$ is excluded, as in the subspace $\mathrm{Spc}(\mathcal{L})$ the only prime of the form $\mathcal{P}(H, \mathfrak{q})$ that remained was $\eta_E(H)$ itself, and $\mathcal{Q}$ is different from $\eta_E(H)$. Thus $H' = 1$, which means that $\mathcal{Q} \in V_E = \overline{\{\eta_E(1)\}}$ and we therefore have $\eta_E(H) \in \overline{\{\mathcal{Q}\}} \subseteq \overline{\{\eta_E(1)\}}$ as claimed. □

We can now determine the Krull dimension of the spectrum of $\mathcal{K}(E)$.

15.14. **Proposition.** *Let E be a elementary abelian p-group. Then the Krull dimension of* $\mathrm{Spc}(\mathcal{K}(E))$ *is the p-rank of E.*

Proof. By Proposition 13.11, the dimension of $\mathrm{Spc}(\mathcal{K}(E))$ is the maximum of the dimensions of the open subsets $U(H)$, for $H \leqslant E$. Each of these spaces has the same generic point η_E (by Proposition 15.11) and a unique closed point $\mathcal{M}(H)$ by

Proposition 13.11 (and the fact that $\mathcal{M}(K) \in U(H)$ forces K and H to be contained in the same subgroups $N \in \mathcal{N}(G)$ by Proposition 12.9, which in turn forces $K = H$ because E is elementary abelian). Using Theorem 15.3 we translate the problem into one about the graded ring $\mathcal{O}^{\bullet}_{E}(H)$. Let $\eta_E = \mathfrak{p}_0 \subsetneq \mathfrak{p}_1 \subsetneq \cdots \subsetneq \mathfrak{p}_n = \mathcal{M}(H)$ be a chain of homogeneous prime ideals in $\mathcal{O}^{\bullet}_{E}(H)$. Note that $\mathfrak{p}_{n-1}$ belongs to the open $Z(f)^c$ of $\mathrm{Spec}^{\mathrm{h}}(\mathcal{O}^{\bullet}_{E}(H))$ for some $f = \zeta^{+}_{N}$, $H \leqslant N$, or some $f = \zeta^{-}_{N}$, $H \not\leqslant N$. Each of these has non-zero degree so the graded ring $\mathcal{O}^{\bullet}_{E}(H)[f^{-1}]$ is periodic. We deduce that $\dim(\mathrm{Spec}^{\mathrm{h}}(\mathcal{O}^{\bullet}_{E}(H)))$ is the maximum of $1 + \dim(R)$ where R ranges over the ungraded rings $R = \mathcal{O}^{\bullet}_{E}(H)[f^{-1}]_{(0)}$ for f as above. The reduced ring R_{red} is a finitely generated k-algebra with irreducible spectrum, hence a domain. Therefore $\dim(R) = \dim(R_{\mathrm{red}})$ is the transcendence degree of the residue field at the unique generic point. As observed above, this generic point is the same for all $H \leqslant E$, namely the generic point of $U(1) = \mathrm{Spc}(\mathrm{D_b}(kE))$. We conclude that $\dim(\mathrm{Spc}(\mathcal{K}(E))) = \dim(\mathrm{Spc}(\mathrm{D_b}(kE)))$ which is indeed the p-rank of E. □

15.15. *Remark.* In fact, the proof shows that all closed points $\mathcal{M}(H) \in \mathrm{Spc}(\mathcal{K}(E))$ have the same codimension (height), namely the p-rank of E.

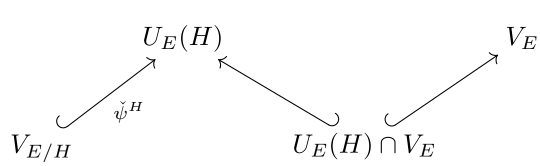
15.16. *Remark.* Thus for E elementary abelian, the Krull dimension of $\mathrm{Spc}(\mathcal{K}(E))$ is the same as the Krull dimension of the classical cohomological open $\mathrm{Spc}(\mathrm{D_b}(kE)) \cong \mathrm{Spec}^{\mathrm{h}}(\mathrm{H}^{\bullet}(E,k))$. In other words, the spectrum of $\mathcal{K}(E)$ is not monstrously different from that of $\mathrm{D_b}(kE)$, at least in terms of dimension, or 'vertical complexity'. There is however 'horizontal complexity' in $\mathrm{Spc}(\mathcal{K}(E))$: each $U(H)$ has its own shape and form, and there are as many $U(H)$ as there are subgroups $H \leqslant E$. We give a finite presentation of the corresponding k-algebras $\mathcal{O}^{\bullet}_{E}(H)$ in Section 17.

16. Closure in elementary abelian case

In this section, E is still an elementary abelian p-group. Following up on Remark 11.11, we can now use Theorem 15.3 to analyze inclusion of tt-primes $\mathcal{P}, \mathcal{Q}$ in $\mathcal{K}(E)$, which amounts to asking when $\mathcal{Q}$ belongs to $\overline{\{\mathcal{P}\}}$ in $\mathrm{Spc}(\mathcal{K}(E))$.

16.1. *Remark.* Using again that every $\psi^H\colon \mathrm{Spc}(\mathcal{K}(E/H)) \hookrightarrow \mathrm{Spc}(\mathcal{K}(E))$ is a closed immersion, induction on the p-rank easily reduces the above type of questions to the case where the 'lower' point $\mathcal{P}$ belongs to $U_E(1) = V_E$. More generally, given a closed piece Z of the cohomological open V_E, we consider its closure $\bar{Z}$ in $\mathrm{Spc}(\mathcal{K}(E)) = \sqcup_{H \leqslant E} V_E(H)$ and we want to describe the part $\bar{Z} \cap V_E(H)$ in each stratum $V_E(H) \cong V_{E/H}$ for $H \leqslant E$.

16.2. *Construction.* Let $H \leqslant E$ be a subgroup of our elementary abelian group E. Consider the open subsets $U_E(H)$ of Proposition 13.11, the cohomological open $U_E(1) = V_E$ and their intersection $U_E(H) \cap V_E$. Consider also the stratum $V_E(H) = \check{\psi}^H(V_{E/H})$, that is a closed subset of $U_E(H)$ homeomorphic to $V_{E/H}$ via $\check{\psi}^H$:



On graded endomorphism rings of the unit (Definition 14.9) this corresponds to

$$\begin{array}{ccccc} & & \mathcal{O}^{\bullet}_{E}(H) & & \mathrm{H}^{\bullet}(E;k) \\ & \swarrow^{\Psi^H} & & \searrow^{Q} \quad \swarrow^{Q'} & \\ \mathrm{H}^{\bullet}(E/H;k) & & & \mathcal{O}^{\bullet}_{E}(U_E(H)\cap V_E) & \end{array} \tag{16.3}$$

where Q is the localization of $\mathcal{O}^{\bullet}_{E}(H)$ with respect to $\zeta_N^- = \frac{b_N}{a_N}$ for all $N \not\geq H$, where Q' is the localization of $\mathcal{O}^{\bullet}_{E}(1) = \mathrm{H}^{\bullet}(E,k)$ with respect to $\zeta_N^+ = \frac{a_N}{b_N}$ for all $N \not\geq H$ and where Ψ^H is the epimorphism of Remark 15.10 for $K = H$.

16.4. **Lemma.** *With above notation, let $I \subseteq \mathrm{H}^{\bullet}(E,k)$ be a homogeneous ideal of the cohomology of E. Define the homogeneous ideal $J = \Psi^H(Q^{-1}(\langle Q'(I)\rangle))$ in the cohomology $\mathrm{H}^{\bullet}(E/H;k)$ of E/H by 'carrying around' the ideal I along* (16.3)*:*

$$\begin{array}{ccccc} & & Q^{-1}(\langle Q'(I)\rangle) & & I \\ & \swarrow & & \nwarrow \quad \swarrow & \\ J := \Psi^H(Q^{-1}(\langle Q'(I)\rangle)) & & & \langle Q'(I)\rangle & \end{array}$$

Let Z be the closed subset of V_E defined by the ideal I. Then the closed subset of $V_{E/H}$ defined by J is exactly the intersection $\bar{Z} \cap V_E(H)$ of the closure $\bar{Z}$ of Z in $\mathrm{Spc}(\mathcal{K}(E))$ with the subspace $V_{E/H}$, embedded via $\check{\psi}^H$.

Proof. Once translated by Theorem 15.3, it is a general result about the multi-graded ring $A = \mathrm{H}^{\bullet\bullet}(E)$. We have two open subsets, $U(H) = \cap_{s\in S_H} Z(s)^c$ and $V_E = U(1) = \cap_{s\in S_1} Z(s)^c$ for the multiplicative subsets S_H and S_1 of Definition 14.9. These open subsets are 'Dirac-affine', meaning they correspond to the homogenous spectra of the $\mathbb{Z}$-graded localizations $S_H^{-1}(A)_{\text{0-twist}} = \mathcal{O}^{\bullet}_{E}(H)$ and $S_1^{-1}(A)_{\text{0-twist}} = \mathcal{O}^{\bullet}_{E}(1) = \mathrm{H}^{\bullet}(E;k)$, where $(-)_{\text{0-twist}}$ refers to 'zero-twist', as before. The intersection of those two affine opens corresponds to inverting both S_H and S_1, that is, inverting $\left\{ \frac{b_N}{a_N} \,\middle|\, N \not\geq H \right\}$ from $\mathcal{O}^{\bullet}_{E}(H)$ and $\left\{ \frac{a_N}{b_N} \,\middle|\, N \not\geq H \right\}$ from $\mathrm{H}^{\bullet}(E;k)$. This explains the two localizations Q and Q' and why their targets coincide.

The intersection $U(H) \cap \bar{Z}$ coincides with the closure in $U(H)$ of $U(H) \cap Z$. The latter is a closed subset of $U(H) \cap V_E$ defined by the ideal $\langle Q'(I)\rangle$. The preimage ideal $Q^{-1}(\langle Q'(I)\rangle)$ then defines that closure $U(H) \cap \bar{Z}$ in $U(H)$. Finally, to further intersect this closed subset of $U(H)$ with the closed subset $V_{E/H} = \mathrm{Im}(\mathrm{Spec}^{\mathrm{h}}(\Psi^H))$, it suffices to project the defining ideal along the corresponding epimorphism $\Psi^H : \mathcal{O}^{\bullet}_{E}(H) \twoheadrightarrow \mathrm{H}^{\bullet}(E/H;k)$. □

Before illustrating this method, we need a technical detour via polynomials.

16.5. **Lemma.** *Let I be a homogeneous ideal of the cohomology $\mathrm{H}^{\bullet}(E,k)$ and let $1 \neq H \lneqq E$ be a fixed non-trivial subgroup. Suppose that the only homogeneous prime containing I and all the ζ_N for $N \geq H$ (Remark 12.7) is the maximal ideal $\mathrm{H}^{+}(E,k)$. Then there exists in I a homogeneous*[6] *polynomial f of the form*

$$f = \prod_{M \not\geq H} \zeta_M^d + \sum_m \lambda_m \cdot \prod_{N\in\mathcal{N}} \zeta_N^{m(N)}$$

[6] The grading is the usual $\mathbb{N}$-grading in which all the ζ_N have the same degree $2'$. In particular, the first term $\prod_{M\not\geq H} \zeta_M^d$ in f has degree $2' \cdot d \cdot \left|\left\{ M \in \mathcal{N} \,\middle|\, M \not\geq H \right\}\right|$.

for some integer $d \geq 1$ and scalars $\lambda_m \in k$ and finitely many exponents $m \in \mathbb{N}^{\mathcal{N}}$ that satisfy the following properties:

$$m(N) \geq 1 \text{ for at least one } N \geq H \quad \text{and} \quad m(N') < d \text{ for all } N' \not\geq H. \tag{16.6}$$

Proof. For simplicity, we work in the subring $\mathrm{H}^* \subseteq \mathrm{H}^\bullet(E,k)$ generated by the ζ_N. (For $p = 2$, this is the whole cohomology anyway and for p odd we only miss nilpotent elements, which are mostly irrelevant for the problem, as we can always raise everything in sight to a large p-th power.) Let us denote the maximal ideal by $\mathfrak{m} = \langle \zeta_N \mid N \in \mathcal{N} \rangle$. It is also convenient to work in the quotient $\mathbb{N}$-graded ring

$$A^* := \mathrm{H}^*(E,k)/I$$

which is generated, as a k-algebra, by the classes $\bar{\zeta}_N$ of all ζ_N modulo I.

The assumption about $Z(I + \{\, \zeta_N \,|\, N \geq H \,\}) = \{\mathfrak{m}\}$ implies that $\mathfrak{m}$ has some power contained in $I + \langle \{\, \zeta_N \,|\, N \geq H \,\} \rangle$. In other words when $N' \not\geq H$ we have

$$(\bar{\zeta}_{N'})^d \in \langle \bar{\zeta}_N \mid N \geq H \rangle_{A^*} \tag{16.7}$$

for $d \gg 1$ that we take large enough to work for all the (finitely many) $N' \not\geq H$.

Consider this ideal $J = \langle\, \bar{\zeta}_N \mid N \geq H \,\rangle$ of A^* more carefully. It is a k-linear subspace generated by the classes $\bar{\theta}_m$ of the following products in H^*

$$\theta_m := \prod_{N \in \mathcal{N}} (\zeta_N)^{m(N)} \tag{16.8}$$

with $m \in \mathbb{N}^{\mathcal{N}}$ such that $m(N) \geq 1$ for at least one $N \geq H$. We claim that J is in fact generated over k by the subset of the $\bar{\theta}_m$ for the special $m \in \mathbb{N}^{\mathcal{N}}$ satisfying (16.6). Indeed, let $J' \subseteq J$ be the k-subspace generated by the $\bar{\theta}_m$ for the special m. Then we can prove that the class $\bar{\theta}_m$ of each product (16.8) belongs to J', by using (16.7) and (descending) induction on the number $\sum_{N \geq H} m(N)$, for a fixed total degree $\sum_N m(N)$. We conclude that $J = J'$.

By (16.7), the monomial $\prod_{M \not\geq H} (\bar{\zeta}_M)^d$ belongs to J and therefore to J': It is a k-linear combination of monomials of the form $\bar{\theta}_m$ for $m \in \mathbb{N}^{\mathcal{N}}$ satisfying (16.6). Returning from $A^* = \mathrm{H}^*(E,k)/I$ to $\mathrm{H}^*(E,k)$, the difference between $\prod_{M \not\geq H} (\zeta_M)^d$ and the same k-linear combination of the lifts θ_m in $\mathrm{H}^*(E,k)$ is an element of I, that we call f and that fulfills the statement. □

16.9. **Proposition.** *Let $Z \subset V_E$ be a non-empty closed subset of the cohomological open and let $1 \neq H \lneqq E$ be a non-trivial subgroup. Suppose that in V_E, the subset Z intersects the image of the cohomological open of H in the smallest possible way:*

$$Z \cap \rho_H(V_H) = \{\mathcal{M}(1)\}.$$

Consider the closure $\bar{Z}$ of Z in the whole spectrum $\mathrm{Spc}(\mathcal{K}(E))$. *Then $\bar{Z}$ does not intersect the stratum $V_E(H) = \psi^H(V_{E/H})$. Hence $\mathcal{M}(H)$ does not belong to $\bar{Z}$.*

Proof. Let $I \subset \mathrm{H}^\bullet(E,k)$ be the homogeneous ideal that defines Z. The closed image $\rho_H(V_H)$ is given by the (partly redundant) equations $\zeta_N = 0$ for all $N \geq H$. It follows that the intersection $Z \cap \rho_H(V_H)$ is defined by the ideal $I + \langle\, \zeta_N \mid N \geq H \,\rangle$. So our hypothesis translates exactly in saying that I satisfies the hypothesis of Lemma 16.5. Hence there exists a homogeneous element of I

$$f = \prod_{M \not\geq H} \zeta_M^d + \sum_m \lambda_m \prod_{N \in \mathcal{N}} \zeta_N^{m(N)}$$

for scalars $\lambda_m \in k$ and finitely many exponents $m \in \mathbb{N}^{\mathcal{N}}$ satisfying (16.6). We can now use Lemma 16.4 and follow Diagram (16.3) with the ideal I and particularly with its element f. The element $Q'(f)$ is just f seen in $\mathcal{O}_E^\bullet(U_E(H) \cap V_E)$. But it does not belong to the image of $\mathcal{O}_E^\bullet(H)$ under Q because f contains some b_M with $M \ngeq H$ in denominators in the ζ_M's. Still, we can multiply $Q'(f)$ by $\prod_{M \ngeq H}(\frac{b_M}{a_M})^d = \prod_{M \ngeq H}(\zeta_M)^{-d}$ to get a degree-zero homogeneous element

$$\tilde{f} = 1 + \sum_m \lambda_m \prod_{N \in \mathcal{N}} \zeta_N^{m'(N)} \tag{16.10}$$

in the ideal $\langle Q'(f) \rangle$, where we set the exponent $m'(N) := m(N) - d$ if $N \ngeq H$ and $m'(N) := m(N)$ if $N \geq H$. Note that by (16.6) the exponent $m'(N)$ is negative if $N \ngeq H$ and is non-negative if $N \geq H$ and strictly positive for at least one $N \geq H$. Both types of exponents of ζ_N are allowed in $\mathcal{O}_E^\bullet(H)$, namely, when $N \ngeq H$, the element $\zeta_N^- = \frac{b_N}{a_N}$ exists in $\mathcal{O}_E^\bullet(H)$. In other words, the element $\tilde{f} \in \langle Q'(f) \rangle$ satisfies

$$\tilde{f} = Q(1 + \tilde{g})$$

where $\tilde{g} \in \mathcal{O}_E^\bullet(H)$ belongs to the ideal $\langle \zeta_N \mid N \geq H \rangle$ in $\mathcal{O}_E^\bullet(H)$ and must be of degree zero by homogeneity. Now, for $N \geq H$, we have $\Psi^H(\zeta_N) = \zeta_{N/H}$ by Proposition 12.9. It follows that $\Psi^H(\tilde{g})$ belongs to the maximal ideal $\langle \zeta_{\bar{N}} \mid \bar{N} \in \mathcal{N}(E/H) \rangle \subseteq \mathrm{H}^+(E/H, k)$ of $\mathrm{H}^\bullet(E/H, k)$ and still has degree zero. This forces $\Psi^H(\tilde{g}) = 0$ and therefore $\Psi^H(1 + \tilde{g}) = 1$ in $\mathrm{H}^\bullet(E/H, k)$. In the notation of Lemma 16.4, we have shown that J contains 1, which implies that $\bar{Z} \cap V_E(H) = \varnothing$. □

16.11. **Corollary.** *Let $Z \subset V_E$ be a closed subset of the cohomological open, strictly larger than the unique closed point $\mathcal{M}(1)$ of V_E. Suppose that in V_E, the subset Z intersects the images of all proper subgroups trivially, i.e. $Z \cap \big(\bigcup_{H \lneqq E} \rho_H(V_H)\big) = \{\mathcal{M}(1)\}$. Then the closure $\bar{Z}$ of Z in the whole spectrum* $\mathrm{Spc}(\mathcal{K}(E))$ *has only one more point, namely $\bar{Z} = Z \cup \{\mathcal{M}(E)\}$.*

Proof. By Proposition 16.9, we see that $\bar{Z}$ does not meet any stratum $V_E(H)$ for $H \neq E$. Thus the only point of $\mathrm{Spc}(\mathcal{K}(E))$ outside of Z itself, hence outside of V_E, that remains candidate to belong to $\bar{Z}$ must belong to $\mathrm{supp}(\mathcal{K}_{\mathrm{ac}}(E)) \smallsetminus \cup_{H \lneqq E} V_E(H) = V_E(E) = \{\mathcal{M}(E)\}$. We know that $\mathcal{M}(E) = \langle k(E/H) \mid H \lneqq E \rangle$ in $\mathcal{K}(E)$, by Example 7.30. Take $\mathcal{P} \in Z$ different from $\mathcal{M}(1)$. Since $\mathcal{P}$ does not belong to any $\mathrm{Im}(\rho_H) = \mathrm{supp}(k(E/H))$ by assumption, it must contain $k(E/H)$. Consequently, $\mathcal{M}(E) \subseteq \mathcal{P}$, meaning that $\mathcal{M}(E) \in \overline{\{\mathcal{P}\}} \subseteq \bar{Z}$. □

16.12. **Corollary.** *Let E be an elementary abelian group of rank r. Let $\mathcal{P}$ be a point of height $r-1$ in the cohomological open V_E, that is, a closed point of the classical projective support variety $\mathcal{V}_E(k) := V_E \smallsetminus \{\mathcal{M}(1)\} \cong \mathrm{Proj}(\mathrm{H}^\bullet(E,k)) \cong \mathbb{P}_k^{r-1}$. Suppose that $\mathcal{P}$ does not belong to the image $\rho_H(V_H)$ of the support variety of any proper subgroups $H \lneqq E$. Then the closure of $\mathcal{P}$ in* $\mathrm{Spc}(\mathcal{K}(E))$ *is exactly the following*

$$\overline{\{\mathcal{P}\}} = \{\mathcal{M}(E), \mathcal{P}, \mathcal{M}(1)\}.$$

Proof. Apply Corollary 16.11 to $Z = \{\mathcal{P}, \mathcal{M}(1)\}$, the closure of $\mathcal{P}$ in V_E. □

16.13. *Example.* We can review the proof of Proposition 16.9 in the special case of Corollary 16.12, to see how elements like $f \in \mathcal{O}_E^\bullet(1)$ and $\tilde{f} \in \mathcal{O}_E^\bullet(H)$ come into play. We do it in the special case where $\mathcal{P}$ is a k-rational point (e.g. if k is algebraically closed). Let $1 \neq H \lneqq E$ be a non-trivial subgroup (the case $r = 1$ being trivial). Choose $N_0, N_1 \lhd E$ index-p subgroups with $H \leqslant N_0$ and $H \nleqslant N_1$. They define coordinates ζ_0, ζ_1 in $\mathbb{P}^{r-1}$ (where $\zeta_i = \zeta_{N_i}$ as in Remark 12.7). There exists a hyperplane of $\mathbb{P}^{r-1}$

$$\lambda_0\zeta_0 + \lambda_1\zeta_1 = 0, \qquad [\lambda_0 : \lambda_1] \in \mathbb{P}^1(k), \tag{16.14}$$

going through the rational point $\mathcal{P}$. Note that $\lambda_1 \neq 0$ as $\mathcal{P} \notin Z(\zeta_0) = \rho_{N_0}(V_{N_0})$, by assumption. As in Lemma 16.4, the following two localizations agree

$$\mathcal{O}^\bullet_E(H)\big[(\zeta_N^-)^{-1} \mid H \nleqslant N\big] = \mathcal{O}^\bullet_E(1)\big[(\zeta_N^+)^{-1} \mid H \nleqslant N\big] \tag{16.15}$$

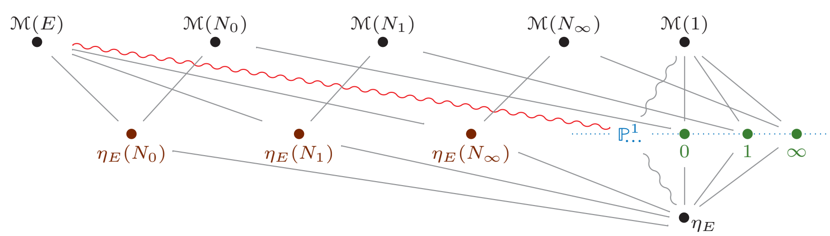
where $N \lhd E$ ranges over the index-p subgroups as usual. We find a lift

$$\tilde{f} := \lambda_0\zeta_0\zeta_{N_1}^- + \lambda_1 \in \mathcal{O}^\bullet_E(H)$$

of the element $f = \lambda_0\zeta_0 + \lambda_1\zeta_1 \in \mathcal{O}^\bullet_E(1)$ of (16.14) suitably multiplied by $\zeta_1^{-1} = \zeta_{N_1}^-$ in the localization (16.15). Then we have $\Psi^H(\zeta_{N_1}^-) = 0$ since $H \nleqslant N_1$, by Proposition 12.9, so $\Psi^H(\tilde{f}) = \lambda_1 \in k^\times$ is an isomorphism. We deduce that $\operatorname{cone}(\tilde{f})$ belongs to $\mathcal{M}(H) \smallsetminus \mathcal{P}$, which shows that $\mathcal{P}$ does not specialize to $\mathcal{M}(H)$.

16.16. *Example.* Let $E = C_2 \times C_2$ be the Klein-four group. Let us justify the description of $\operatorname{Spc}(\mathcal{K}(E))$ announced in Example 8.10 in some detail:

(16.17)



By Recollection 10.7, we have a partition of the spectrum as a set

$$\operatorname{Spc}(\mathcal{K}(E)) = V_E(E) \sqcup V_E(N_0) \sqcup V_E(N_1) \sqcup V_E(N_\infty) \sqcup V_E,$$

where we write N_0, N_1, N_∞ for the three cyclic subgroups C_2 and where $V_E = V_E(1)$ is the cohomological open as usual. Let us review those five parts $V_E(H) = \psi^H(V_{E/H})$ separately, in growing order of complexity, *i.e.* from left to right in (16.17).

For $H = E$, the stratum $V_E(E) = \psi^E(V_{E/E}) = \{\mathcal{M}(E)\}$ is just a closed point.

For each cyclic subgroup $N_i < E$, the quotient $E/N_i \simeq C_2$ is cyclic, so $\operatorname{Spc}(E/N_i)$ is the space of Example 15.8. Its image under ψ^{N_i} is $\{\mathcal{M}_E(E), \eta_E(N_i), \mathcal{M}_E(N_i)\}$, defining the (brown) point $\eta_E(N_i) := \psi^{N_i}(\eta_{E/N_i})$, as in the proof of Proposition 15.11. The stratum $V_E(N_i)$ is the image of the cohomological open V_{E/N_i} only, that is, the Sierpiński space $\{\eta_E(N_i), \mathcal{M}(N_i)\}$, whose non-closed point $\eta_E(N_i)$ is the generic point of the irreducible $\{\mathcal{M}_E(E), \eta_E(N_i), \mathcal{M}_E(N_i)\}$ in $\operatorname{Spc}(\mathcal{K}(E))$.

Finally, for $H = 1$, the cohomological open $V_E = \operatorname{Spc}(\mathrm{D_b}(kE)) \cong \operatorname{Spec^h}(k[\zeta_0, \zeta_1])$ is a $\mathbb{P}^1_k$ with a closed point $\mathcal{M}(1)$ on top. We denote by η_E the generic point of $\operatorname{Spc}(\mathcal{K}(E))$ as in Proposition 15.11 and by $0, 1, \infty$ the three $\mathbb{F}_2$-rational points of $\mathbb{P}^1_k$ (in green). The notation $\mathbb{P}^1_{\because}$ refers to all remaining points of $\mathbb{P}^1_k$. The undulated lines indicate that *all* points of $\mathbb{P}^1_{\because}$ have the same behavior. Namely, η_E specializes to all points of $\mathbb{P}^1_{\because}$ and every point of $\mathbb{P}^1_{\because}$ specializes to $\mathcal{M}(1)$ and the (red) undulated line towards $\mathcal{M}(E)$ indicates that all points of $\mathbb{P}^1_{\because}$ specialize to $\mathcal{M}(E)$, as follows from Corollary 16.12. (Note that the latter was rather involved: Its proof occupies most of this section, and relies on technical Lemma 16.5.)

We have described the closure of every point in $\operatorname{Spc}(\mathcal{K}(E))$, except for the $\mathbb{F}_2$-rational points $0, 1, \infty$. For this, we use the closed immersion $\rho_{N_i}\colon \operatorname{Spc}(\mathcal{K}(N_i)) \hookrightarrow \operatorname{Spc}(\mathcal{K}(E))$ induced by restriction Res_{N_i}. The point i is the image of the generic point η_{N_i} of the V-shaped space $\operatorname{Spc}(\mathcal{K}(N_i))$ of Example 15.8. Hence its closure is $\operatorname{Im}(\rho_{N_i}) = \{\mathcal{M}(E_i), i, \mathcal{M}(1)\}$. So specializations are exactly those of (16.17).

We revisit this picture in more geometric terms in Example 18.2.

16.18. *Remark.* It is possible to extend Corollary 16.12 to a general finite group G by means of the Colimit Theorem 11.10. Let $Z \subseteq \operatorname{Spc}(\mathcal{K}(G))$ be a one-dimensional irreducible closed subset. Write its generic point as $\mathcal{P} = \mathcal{P}(K, \mathfrak{p})$ for (unique) $K \in \operatorname{Sub}_p(G)_{/G}$ and $\mathfrak{p} \in V_{G/\!\!/K}$. By Quillen applied to $\bar{G} = G/\!\!/K$, there exists a minimal elementary abelian subgroup $E \leqslant \bar{G}$ such that $\mathfrak{p} \in \operatorname{Im}(\rho_E\colon V_E \to V_{\bar{G}})$, also unique up to $\bar{G}$-conjugation. This $E \leqslant \bar{G} = (N_G K)/K$ is given by $E = H/K$ for $H \leqslant N_G K$ containing K. Then $\mathcal{P} = \varphi_{(H,K)}(\mathcal{Q})$ where $\mathcal{Q} \in \operatorname{Spc}(\mathcal{K}(E))$ is given by $\mathcal{Q} = \mathcal{P}_E(1, \mathfrak{q})$ for some $\mathfrak{q} \in V_E$. By Lemma 11.9, the map $\varphi_{(H,K)}\colon \operatorname{Spc}(\mathcal{K}(E)) \to \operatorname{Spc}(\mathcal{K}(G))$ is closed and preserves the dimension of points. It follows that $\mathcal{Q}$ is also the generic point of a one-dimensional irreducible in $\operatorname{Spc}(\mathcal{K}(E))$. By minimality of E, the point $\mathcal{Q} \in V_E$ does not belong to $V_{H'}$ for any proper subgroup $H' < E$. By Corollary 16.12, we have $\overline{\{\mathcal{Q}\}} = \{\mathcal{M}_E(E), \mathcal{Q}, \mathcal{M}_E(1)\}$ in $\operatorname{Spc}(\mathcal{K}(E))$. The map $\varphi_{(H,K)}$ sends this subset to $\{\mathcal{M}_G(H), \mathcal{P}, \mathcal{M}_G(K)\}$. In summary, every one-dimensional irreducible subset of $\operatorname{Spc}(\mathcal{K}(G))$ is of the form $Z = \{\mathcal{M}(H), \mathcal{P}, \mathcal{M}(K)\}$, where H and K are uniquely determined by the generic point $\mathcal{P}$ via the above method.

17. Presentation of twisted cohomology

We remain in the case of an elementary abelian group E. In this section we want to better understand the local $\mathbb{Z}$-graded rings $\mathcal{O}^{\bullet}_E(H)$ that played such an important role in Section 15. Thankfully they are reasonable k-algebras.

17.1. *Terminology.* Recall that we write $C_p = \langle\, \sigma \mid \sigma^p = 1 \,\rangle$ for the cyclic group of order p with a chosen generator σ. For brevity we call an $\mathbb{F}_p$-linear surjection $\pi\colon E \twoheadrightarrow C_p$ a *coordinate*. For two coordinates π, π' we write $\pi \sim \pi'$ if $\ker(\pi) = \ker(\pi')$. Finally, for a subgroup H, we often abbreviate $H \mid \pi$ to mean $H \leqslant \ker(\pi)$.

Recall from Definition 12.3 and Remark 12.5 that each coordinate π yields an invertible object $u_\pi = \pi^* u_p$ in $\mathcal{K}(E)$. It comes with maps $a_\pi, b_\pi, c_\pi\colon k \to u_\pi[*]$.

17.2. *Remark.* If $\pi \sim \pi'$ then there exists a unique $\lambda \in \mathbb{F}_p^\times$ such that $\pi' = \pi^\lambda$. Hence, if $p = 2$ then necessarily $\pi = \pi'$ and $u_\pi = u_{\pi'}$. On the other hand, if $p > 2$ is odd then we still have $u_\pi \cong u_{\pi'}$ as already mentioned. Explicitly, consider the automorphism $\lambda\colon C_p \to C_p$ that sends σ to σ^λ. The isomorphism $u_\pi = \pi^* u_p \overset{\sim}{\to} \pi^* \lambda^* u_p = (\pi^\lambda)^* u_p = u_{\pi'}$ will be the pullback $\pi^*\Lambda$ along π of an isomorphism of complexes $\Lambda\colon u_p \overset{\sim}{\to} \lambda^* u_p$. This isomorphism Λ can be given explicitly by the identity in degree 0 and by the kC_p-linear maps $kC_p \to \lambda^* kC_p$ in degree 1 (resp. 2) determined by $1 \mapsto 1$ (resp. $1 \mapsto 1 + \sigma + \cdots \sigma^{\lambda-1}$). One checks directly that $\Lambda \circ a_p = a_p$ and $\Lambda \circ b_p = \lambda \cdot b_p$. By applying π^* we obtain

$$(\pi^*\Lambda) \circ a_\pi = a_{\lambda\pi} \qquad \text{and} \qquad (\pi^*\Lambda) \circ b_\pi = \lambda \cdot b_{\lambda\pi}. \tag{17.3}$$

17.4. **Lemma.** *Given coordinates* $\pi_1 \not\sim \pi_2$ *set* $\pi_3 = \pi_1^{-1}\pi_2^{-1}$. *Write* u_i, a_i *and* b_i *for* u_{π_i}, a_{π_i} *and* b_{π_i} *in* $\mathcal{K}(E)$. *Then we have the relation*

$$a_1b_2b_3 + b_1a_2b_3 + b_1b_2a_3 = 0$$

as a map from $\mathbb{1}$ *to* $(u_1 \otimes u_2 \otimes u_3)[-2' \cdot 2]$ *in* $\mathcal{K}(E)$. *(See Notation 12.2 for* $2'$*.)*

Proof. Let $N_i = \ker(\pi_i)$ for $i = 1, 2, 3$, which are all distinct. Let $N = N_1 \cap N_2 \cap N_3$ be the common kernel, which is of index p^2 in E. By inflation along $E \twoheadrightarrow E/N$, it suffices to prove the lemma for $E = C_p \times C_p$ and π_1 and π_2 the two projections on the factors. We abbreviate u for the complex of permutation kE-modules $u := u_1 \otimes u_2 \otimes u_3$. Consider the permutation module $M := kC_p \otimes kC_p \otimes kC_p \cong k(E/N_1) \otimes k(E/N_2) \otimes k(E/N_3)$ which appears as a summand in various degrees of the complex u. One element in M is of particular interest:

$$m := \sum_{i_1, i_2 = 0}^{p-1} \sigma^{i_1} \otimes \sigma^{i_2} \otimes \sigma^{-i_1 - i_2}.$$

It is easy to check that m is E-invariant, thus defines a kE-linear map $\tilde{m}\colon k \to M$, that can be used to define the required homotopies. This depends on p. If $p = 2$, the homotopy is given by $\tilde{m}$ when viewed from $\mathbb{1}$ to the only M-entry of $u[-2]$ in degree one. If $p > 2$, the homotopy is given by $(\tilde{m}, \tilde{m}, \tilde{m})$ as a map from $\mathbb{1}$ to the three M-entries of $u[-4]$ in degree one. Verifications are left to the reader. □

17.5. *Construction.* We construct a commutative k-algebra $\underline{\mathcal{O}}^\bullet_E(H)$ by generators and relations. Its generators are indexed by coordinates $\pi\colon E \twoheadrightarrow C_p$ (Terminology 17.1)

$$\{\, \zeta^+_\pi \mid \pi \text{ s.t. } H \leqslant \ker(\pi) \,\} \;\cup\; \{\, \zeta^-_\pi \mid \pi \text{ s.t. } H \not\leqslant \ker(\pi) \,\}.$$

These generators come equipped with a degree in $\mathbb{Z}$: If $H \mid \pi$ the generator ζ^+_π is set to have degree $2'$, whereas if $H \nmid \pi$ the generator ζ^-_π is set to have degree $-2'$. We impose the following four families of homogeneous relations. First for every coordinate π and every $\lambda \in \mathbb{F}_p^\times$ (for p odd), we have a rescaling relation

(a) $\zeta^+_{\pi^\lambda} = \lambda\zeta^+_\pi$ if $H \mid \pi$ and $\zeta^-_{\pi^\lambda} = \lambda^{-1}\zeta^-_\pi$ if $H \nmid \pi'$

and whenever $\pi_3 = \pi_1^{-1}\pi_2^{-1}$ and $\pi_1 \not\sim \pi_2$, writing $\zeta^\pm_i := \zeta^\pm_{\pi_i}$, we impose one of the following relations, inspired by Lemma 17.4:

(b) $\zeta^+_1 + \zeta^+_2 + \zeta^+_3 = 0$, if $H \mid \pi_1$ and $H \mid \pi_2$ (and therefore $H \mid \pi_3$)

(c) $\zeta^-_1 + \zeta^-_2 + \zeta^-_1\zeta^-_2\zeta^+_3 = 0$, if $H \nmid \pi_1$ and $H \nmid \pi_2$ but $H \mid \pi_3$

(d) $\zeta^-_1\zeta^-_2 + \zeta^-_2\zeta^-_3 + \zeta^-_3\zeta^-_1 = 0$ if $H \nmid \pi_i$ for all $i = 1, 2, 3$.

Since these relations are homogeneous, the ring $\underline{\mathcal{O}}^\bullet_E(H)$ is a $\mathbb{Z}$-graded ring.

17.6. *Remark.* We could also define a multi-graded commutative k-algebra $\underline{\mathrm{H}}^{\bullet\bullet}(E)$ generated by all a_π, b_π subject to the relations in (17.3) and Lemma 17.4. This algebra $\underline{\mathrm{H}}^{\bullet\bullet}(E)$ would be $\mathbb{Z} \times \mathbb{N}^{\mathcal{N}}$-graded with a_π in degree $(0, 1_{\ker(\pi)})$ and b_π in degree $(-2', 1_{\ker(\pi)})$. Then $\underline{\mathcal{O}}^\bullet_E(H)$ is simply the 'zero-twist' part of the localization of $\underline{\mathrm{H}}^{\bullet\bullet}(E)$ with respect to the a_π, b_π that become invertible in $U(H)$, that is, those a_π such that $H \nmid \pi$ and those b_π such that $H \mid \pi$, as in Definition 14.9.

17.7. *Remark.* By (17.3) and Lemma 17.4, there exists a canonical homomorphism

$$\underline{\mathcal{O}}^\bullet_E(H) \to \mathcal{O}^\bullet_E(H) \tag{17.8}$$

mapping ζ^+_π to $\frac{a_\pi}{b_\pi}$ and ζ^-_π to $\frac{b_\pi}{a_\pi}$.

17.9. *Example.* Let $H = 1$. Recall from Example 14.13 that $\mathcal{O}^\bullet_E(1)$ is the cohomology ring. Then the homomorphism (17.8) is the standard one $\underline{\mathcal{O}}^\bullet_E(1) \to \mathrm{H}^\bullet(E;k)$, that maps ζ^+_π to the usual generator $\zeta_\pi = \pi^*(\zeta_{C_p})$. Note that here $H = 1 \mid \pi$ for all π, so there is no ζ^-_π. For E elementary abelian, it is well-known that this homomorphism $\underline{\mathcal{O}}^\bullet_E(1) \to \mathrm{H}^\bullet(E;k)$ is an isomorphism modulo nilpotents. See for instance [Car96].

For two subgroups $H, K \leqslant E$, the open subsets $U_E(H)$ and $U_E(K)$ can intersect in $\mathrm{Spc}(\mathcal{K}(E))$. Similarly, we can discuss what happens with the rings $\underline{\mathcal{O}}^\bullet_E(H)$.

17.10. **Proposition.** *Let $H, K \leqslant E$ be two subgroups. Define $S = S(H,K) \subset \underline{\mathcal{O}}^\bullet_E(H)$ to be the multiplicative subset generated by the finite set*

$$\left\{ \zeta^+_\pi \;\middle|\; \text{for } \pi \text{ with } H \mid \pi \text{ and } K \nmid \pi \right\} \cup \left\{ \zeta^-_\pi \;\middle|\; \text{for } \pi \text{ with } H \nmid \pi \text{ and } K \mid \pi \right\}$$

and similarly, swapping H and K, let $T = S(K,H) \subset \underline{\mathcal{O}}^\bullet_E(K)$ be the multiplicative subset generated by $\left\{ \zeta^+_\pi \,\middle|\, H \nmid \pi \text{ and } K \mid \pi \right\} \cup \left\{ \zeta^-_\pi \,\middle|\, H \mid \pi \text{ and } K \nmid \pi \right\}$. Then we have a canonical isomorphism of (periodic) $\mathbb{Z}$-graded rings

$$S^{-1}\underline{\mathcal{O}}^\bullet_E(H) \cong T^{-1}\underline{\mathcal{O}}^\bullet_E(K)$$

and in particular of their degree-zero parts. Thus the open of $\mathrm{Spec}^{\mathrm{h}}(\underline{\mathcal{O}}^\bullet_E(H))$ defined by S is canonically homeomorphic to the open of $\mathrm{Spec}^{\mathrm{h}}(\underline{\mathcal{O}}^\bullet_E(K))$ defined by T.

Proof. The left-hand side $S^{-1}\underline{\mathcal{O}}^\bullet_E(H)$ is the ('zero-twist' part of the) localization of the multi-graded ring $\underline{\mathrm{H}}^{\bullet\bullet}(E)$ of Remark 17.6 with respect to

$$\begin{aligned}\left\{ a_\pi \,\middle|\, H \nmid \pi \right\} \cup \left\{ b_\pi \,\middle|\, H \mid \pi \right\} \cup \left\{ a_\pi \,\middle|\, H \mid \pi, K \nmid \pi \right\} \cup \left\{ b_\pi \,\middle|\, H \nmid \pi, K \mid \pi \right\} \\ = \left\{ a_\pi \,\middle|\, H \nmid \pi \text{ or } K \nmid \pi \right\} \cup \left\{ b_\pi \,\middle|\, H \mid \pi \text{ or } K \mid \pi \right\}\end{aligned}$$

which is symmetric in H and K. This completes the proof. □

17.11. *Remark.* The above isomorphism is compatible with the homomorphism (17.8), namely the obvious diagram commutes when we perform the corresponding localizations on $\mathcal{O}^\bullet_E(H)$ and $\mathcal{O}^\bullet_E(K)$.

17.12. **Proposition.** *Let $K \leqslant H \leqslant E$. There is a canonical split epimorphism ${}^{`}\Psi^K \colon \underline{\mathcal{O}}^\bullet_E(H) \twoheadrightarrow \underline{\mathcal{O}}^\bullet_{E/K}(H/K)$ whose kernel is $\langle \zeta^-_\pi \mid K \nmid \pi \rangle$. It is compatible with the homomorphism Ψ^K of Remark 15.10, in that the following diagram commutes*

$$\begin{array}{ccc}
\underline{\mathcal{O}}^\bullet_E(H) & \xrightarrow{(17.8)} & \mathcal{O}^\bullet_E(H) \\
{\scriptstyle {}^{`}\Psi^K} \big\downarrow & & \big\downarrow {\scriptstyle \Psi^K} \\
\underline{\mathcal{O}}^\bullet_{E/K}(H/K) & \xrightarrow{(17.8)} & \mathcal{O}^\bullet_{E/K}(H/K).
\end{array}$$

Proof. Set $\bar{H} := H/K \leqslant \bar{E} := E/K$. Similarly, for every coordinate $\pi \colon E \twoheadrightarrow C_p$ such that $K \mid \pi$, let us write $\bar{\pi} \colon E/K \twoheadrightarrow C_p$ for the induced coordinate. The morphism ${}^{`}\Psi^K$ will come from a morphism ${}^{``}\Psi^K \colon \underline{\mathrm{H}}^{\bullet\bullet}(E) \to \underline{\mathrm{H}}^{\bullet\bullet}(E/K)$, with respect to the homomorphism of gradings (14.3). As these algebras are constructed by generators and relations (Remark 17.6), we need to give the image of generators. In view of Proposition 12.9 we define ${}^{``}\Psi^K \colon \underline{\mathrm{H}}^{\bullet\bullet}(E) \to \underline{\mathrm{H}}^{\bullet\bullet}(E/K)$ on generators by

$$a_\pi \mapsto \begin{cases} a_{\bar{\pi}} & \text{if } K \mid \pi \\ 1 & \text{if } K \nmid \pi \end{cases} \qquad\qquad b_\pi \mapsto \begin{cases} b_{\bar{\pi}} & \text{if } K \mid \pi \\ 0 & \text{if } K \nmid \pi. \end{cases}$$

It is easy to see that the relations in $\underline{\mathrm{H}}^{\bullet\bullet}(E)$ are preserved; thus the map “Ψ^K is well-defined. Let $\varpi\colon E \twoheadrightarrow E/K$ and for every $\bar\pi\colon \bar E \twoheadrightarrow C_p$ consider the coordinate $\pi = \bar\pi \circ \varpi\colon E \twoheadrightarrow C_p$. Then $\bar H \mid \bar\pi$ if and only if $H \mid \pi$. It follows that the morphism passes to the localizations ‘$\Psi^K\colon \underline{\mathcal{O}}^\bullet_E(H) \twoheadrightarrow \underline{\mathcal{O}}^\bullet_{E/K}(H/K)$ as announced. The statement about its kernel is easy and commutativity of the square follows from the fact (Remark 15.10) that Ψ^K treats the a_π and b_π according to the same formulas.

The section of ‘Ψ^K is inspired by inflation. Namely, $a_{\bar\pi} \mapsto a_\pi$ and $b_{\bar\pi} \mapsto b_\pi$ defines a map of graded k-algebras $\underline{\mathrm{H}}^{\bullet\bullet}(\bar E) \to \underline{\mathrm{H}}^{\bullet\bullet}(E)$ that is already a section to “Ψ^K and passes to the localizations. □

17.13. **Theorem.** *The canonical homomorphism* (17.8) *induces an isomorphism*

$$\underline{\mathcal{O}}^\bullet_E(H)_{\mathrm{red}} \xrightarrow{\sim} \mathcal{O}^\bullet_E(H)_{\mathrm{red}}$$

of reduced $\mathbb{Z}$-graded k-algebras.

Proof. It follows from Remark 14.11 that the map is surjective. We will now show that the closed immersion $\mathrm{Spec}^{\mathrm{h}}(\mathcal{O}^\bullet_E(H)) \hookrightarrow \mathrm{Spec}^{\mathrm{h}}(\underline{\mathcal{O}}^\bullet_E(H))$ is surjective—this will complete the proof, by the usual commutative algebra argument, which can be found in [HP23, Lemma 2.22] for the graded case. By Theorem 15.3, this is equivalent to showing the surjectivity of the composite with comp_E, that we baptize β^H

$$\beta^H\colon\qquad U_E(H) \xrightarrow[\mathrm{comp}_E]{\simeq} \mathrm{Spec}^{\mathrm{h}}(\mathcal{O}^\bullet_E(H)) \hookrightarrow \mathrm{Spec}^{\mathrm{h}}(\underline{\mathcal{O}}^\bullet_E(H)). \tag{17.14}$$

We proceed by induction on the order of the subgroup H. If $H = 1$ the result follows from Example 17.9. So suppose that $H \neq 1$ and pick a homogeneous prime $\mathfrak{p} \in \mathrm{Spec}^{\mathrm{h}}(\underline{\mathcal{O}}^\bullet_E(H))$. We distinguish two cases.

If for every coordinate $\pi\colon E \twoheadrightarrow C_p$ such that $H \nmid \pi$ we have $\zeta^-_\pi \in \mathfrak{p}$ then $\mathfrak{p}$ belongs to $V(\{\, \zeta^-_\pi \mid H \nmid \pi \,\})$, which we identify with the image of $\mathrm{Spec}^{\mathrm{h}}(\underline{\mathcal{O}}^\bullet_{E/H}(1))$ by Proposition 17.12 applied to $K = H$. Namely, we have a commutative square

$$\begin{array}{ccc}
U_E(H) & \xleftarrow{\quad\psi^H\quad} & U_{E/H}(1) \\
{\scriptstyle\beta^H}\downarrow & & \downarrow{\scriptstyle\beta^1} \\
\mathrm{Spec}^{\mathrm{h}}(\underline{\mathcal{O}}^\bullet_E(H)) & \xleftarrow{\mathrm{Spec}^{\mathrm{h}}(‘\Psi^H)} & \mathrm{Spec}^{\mathrm{h}}(\underline{\mathcal{O}}^\bullet_{E/H}(1))
\end{array}$$

and since the right-hand vertical arrow is surjective by the case already discussed, we conclude that $\mathfrak{p}$ belongs to the image of β^H in (17.14) as well.

Otherwise, there exists a coordinate π_1 such that $H \nmid \pi_1$ and $\zeta^-_{\pi_1} \notin \mathfrak{p}$. Let $K := H \cap \ker(\pi_1)$ and let $S = S(H, K)$ be defined as in Proposition 17.10:

$$S = \{\, \zeta^-_\pi \mid \text{for } \pi \text{ with } H \nmid \pi \text{ and } K \mid \pi \,\}.$$

We claim that $\mathfrak{p}$ belongs to the open of $\mathrm{Spec}^{\mathrm{h}}(\underline{\mathcal{O}}^\bullet_E(H))$ defined by S. Indeed, let $\zeta^-_{\pi_2} \in S$, that is for π_2 with $H \nmid \pi_2$ and $K \mid \pi_2$, and let us show that $\zeta^-_{\pi_2} \notin \mathfrak{p}$. If $\pi_2 \sim \pi_1$ this is clear from $\zeta^-_{\pi_1} \notin \mathfrak{p}$ and the relation (a) in $\underline{\mathcal{O}}^\bullet_E(H)$. If $\pi_2 \not\sim \pi_1$, let $h \in H \smallsetminus K$ (so that h generates the cyclic group $H/K \cong C_p$). As $\pi_1(h) \neq 1$ and $\pi_2(h) \neq 1$ we may replace π_1 by an equivalent coordinate $\tilde\pi_1 := \pi_1^\lambda$ such that $\tilde\pi_1(h) = \pi_2(h)^{-1}$ and therefore $H \mid \pi_3 := \tilde\pi_1^{-1}\pi_2^{-1}$. Then relation (c) exhibits $\zeta^-_{\tilde\pi_1}$ as a multiple of $\zeta^-_{\pi_2}$. As the former does not belong to $\mathfrak{p}$ (by the previous case), neither does $\zeta^-_{\pi_2}$. At this point we may apply Proposition 17.10 for our subgroups H and K. By Remark 17.11, we have a commutative triangle:

$$\begin{array}{ccc} & U_E(H)\cap U_E(K) & \\ {\scriptstyle\beta^H}\swarrow & & \searrow{\scriptstyle\beta^K} \\ \mathrm{Spec}^{\mathrm{h}}(\underline{\mathcal{O}}^\bullet_E(H)[S^{-1}]) & \xleftrightarrow{\approx} & \mathrm{Spec}^{\mathrm{h}}(\underline{\mathcal{O}}^\bullet_E(K)[T^{-1}]) \end{array}$$

We just proved that $\mathfrak{p}$ belongs to the open subset in the bottom left corner. As K is a proper subgroup of H, we know that β^K is surjective by induction hypothesis and we conclude that $\mathfrak{p}$ belongs to the image of β^H as well. □

17.15. *Remark.* In Theorem 17.13 we have proved something slightly more precise, namely that the map

$$\underline{\mathcal{O}}^\bullet_E(H) \to \mathcal{O}^\bullet_E(H)/\langle\xi^\pm_\pi\rangle \tag{17.16}$$

(where π ranges over all coordinates) is surjective with nilpotent kernel. We expect that $\underline{\mathcal{O}}^\bullet_E(H)$ is already reduced, which would imply that (17.16) is in fact an isomorphism of graded rings. In particular, for $p = 2$ we expect that $\underline{\mathcal{O}}^\bullet_E(H) \xrightarrow{\sim} \mathcal{O}^\bullet_E(H)$.

18. Applications and examples

In this final section, we push our techniques further and compute more examples.

18.1. *Remark.* For E elementary abelian, Corollary 15.4 and Theorem 17.13 allow us to think of the *geometry* of $\mathrm{Spc}(\mathcal{K}(E))$, beyond its mere topology, by viewing $\mathrm{Spc}(\mathcal{K}(E))$ as a Dirac scheme. Consider further the 'periodic' locus of $\mathrm{Spc}(\mathcal{K}(E))$, which is the open complement of the closed points $\{\,\mathcal{M}(H)\,|\,H \leqslant E\,\}$; see Recollection 10.7. This is analogous to considering the projective support variety $\mathrm{Proj}(\mathrm{H}^\bullet(E,k)) \cong \mathbb{P}^{r-1}_k$ by removing the 'irrelevant ideal' $\mathcal{M}(1) = \mathrm{H}^+(E,k)$ from $\mathrm{Spec}^{\mathrm{h}}(\mathrm{H}^\bullet(E,k))$. To avoid confusion with the phrase 'closed points', we now refer to the $\mathcal{M}(H)$ as *very closed points*, allowing us to speak of closed points of $\mathbb{P}^{r-1}_k$ in the usual sense (as we did in Corollary 16.12). Removing those finitely many 'irrelevant' points allows us to draw more geometric pictures by depicting the (usual) closed points of the periodic locus, as in classical algebraic geometry.

In fact, for any finite group G, we can speak of the *periodic locus* of $\mathrm{Spc}(\mathcal{K}(G))$ to mean the open $\mathrm{Spc}'(\mathcal{K}(G)) := \mathrm{Spc}(\mathcal{K}(G)) \smallsetminus \{\,\mathcal{M}(H)\,|\,H \in \mathrm{Sub}_p(G)\,\}$ obtained by removing the 'irrelevant' very closed points. However, we do not endow these spectra with a scheme-theoretic structure beyond the elementary abelian case, since we do not have Corollary 15.4 in general. We postpone a systematic treatment of the periodic locus to later work. For now we focus on examples.

18.2. *Example.* Let us revisit Klein-four, with the notation of Example 16.16. From the picture in (16.17) we see that the union of the open subsets $U_E(1)$ and $U_E(E)$ only misses (three) very closed points hence covers the periodic locus. We have

$$\begin{aligned} \underline{\mathcal{O}}^\bullet_E(1) &= \frac{k[\zeta^+_{N_0},\zeta^+_{N_1},\zeta^+_{N_\infty}]}{\langle\zeta^+_{N_0}+\zeta^+_{N_1}+\zeta^+_{N_\infty}\rangle} \qquad (= \mathrm{H}^*(E;k))\,, \\ \underline{\mathcal{O}}^\bullet_E(E) &= \frac{k[\zeta^-_{N_0},\zeta^-_{N_1},\zeta^-_{N_\infty}]}{\langle\zeta^-_{N_0}\zeta^-_{N_1}+\zeta^-_{N_1}\zeta^-_{N_\infty}+\zeta^-_{N_\infty}\zeta^-_{N_0}\rangle} \end{aligned} \tag{18.3}$$

and their homogeneous spectra are both a projective line with a unique closed point added. (For $\underline{\mathcal{O}}^{\bullet}_{E}(E)$, the coordinate transformation for $i = 0, 1$, $\zeta^{-}_{N_i} \mapsto \tilde{\zeta}^{-}_{i} := \zeta^{-}_{N_i} + \zeta^{-}_{N_\infty}$, identifies the ring with $k[\tilde{\zeta}^{-}_{0}, \tilde{\zeta}^{-}_{1}, \zeta^{-}_{N_\infty}]/\langle \tilde{\zeta}^{-}_{0}\tilde{\zeta}^{-}_{1} + (\zeta^{-}_{N_\infty})^2\rangle$, which corresponds to the image of a degree-two Veronese embedding of $\mathbb{P}^1$ in $\mathbb{P}^2$.) Removing the very closed points (Remark 18.1), it is a straightforward exercise to check that the two lines are glued along the open complement of the $\mathbb{F}_2$-rational points, according to the rule $(\zeta^{+}_{N_i})^{-1} = \zeta^{-}_{N_i}$. In other words, we obtain the following picture of $\mathrm{Spc}'(\mathcal{K}(E))$:

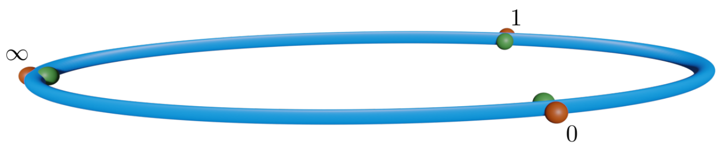


Figure 3. A $\mathbb{P}^1_k$ with three doubled points.

To translate between this picture and the one in (16.17), think of the blue part as $\mathbb{P}^1_{\cdot\cdot}$, the three green points as the $\mathbb{F}_2$-rational points $i = 0, 1, \infty$ in $U_E(1) = V_E$ and the brown points as the $\eta_E(N_i)$ in $U_E(E)$.

18.4. *Example.* In view of later applications let us consider the action induced on spectra by the involution on $E = C_2 \times C_2$ that interchanges the two C_2-factors. Let us say that the two factors correspond to the subgroups N_0 and N_1. On the generators $\zeta^{\pm}_{N_i}$ of $\underline{\mathcal{O}}^{\bullet}_{E}(1)$ and $\underline{\mathcal{O}}^{\bullet}_{E}(E)$ in (18.3), the effect of the involution is

$$\zeta^{\pm}_{N_0} \mapsto \zeta^{\pm}_{N_1} \qquad \zeta^{\pm}_{N_1} \mapsto \zeta^{\pm}_{N_0} \qquad \zeta^{\pm}_{N_\infty} \mapsto \zeta^{\pm}_{N_\infty}.$$

The subrings of invariants in $\underline{\mathcal{O}}^{\bullet}_{E}(1)$ and $\underline{\mathcal{O}}^{\bullet}_{E}(E)$ are, respectively,

$$\frac{k[e^{+}_{1}, e^{+}_{2}, \zeta^{+}_{N_\infty}]}{\langle e^{+}_{1} + \zeta^{+}_{N_\infty}\rangle} \cong k[e^{+}_{2}, \zeta^{+}_{N_\infty}] \qquad \text{and} \qquad \frac{k[e^{-}_{1}, e^{-}_{2}, \zeta^{-}_{N_\infty}]}{\langle e^{-}_{1}\zeta^{-}_{N_\infty} + e^{-}_{2}\rangle} \cong k[e^{-}_{1}, \zeta^{-}_{N_\infty}]$$

where $e^{\pm}_{1} = \zeta^{\pm}_{N_0} + \zeta^{\pm}_{N_1}$ and $e^{\pm}_{2} = \zeta^{\pm}_{N_0}\zeta^{\pm}_{N_1}$ are the first and second symmetric polynomials in $\zeta^{\pm}_{N_0}$ and $\zeta^{\pm}_{N_1}$. Thus $e^{\pm}_{i}$ has degree $\pm i$. The homogeneous spectra of these rings (with unique very closed point removed) are again two projective lines ([7]) and they are glued together along the complement of two points. In other words, the quotient of $\mathrm{Spc}'(\mathcal{K}(E))$ by the involution is a $\mathbb{P}^1_k$ with two doubled points:

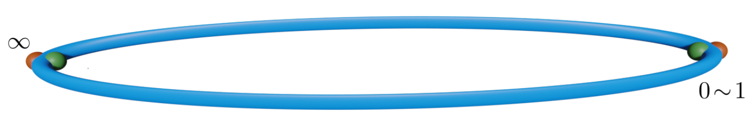


Figure 4. A $\mathbb{P}^1_k$ with two doubled points.

Alternatively, the topological space underlying this quotient may be obtained more directly at the level of Figure 3. Indeed, this involution fixes the two colored points corresponding to ∞, fixes no other points, and swaps the points corresponding to 0 with the points corresponding to 1, respecting the color. So, again, the quotient can be pictured as a $\mathbb{P}^1_k$ with only two doubled points as in Figure 4.

[7] More precisely, as already in Example 18.2, we are dealing with weighted projective spaces which happen to be isomorphic to projective lines.

Let us return to general finite groups. We want to optimize the Colimit Theorem 11.10 by revisiting the category of elementary abelian p-sections $\mathcal{E}_p(G)$.

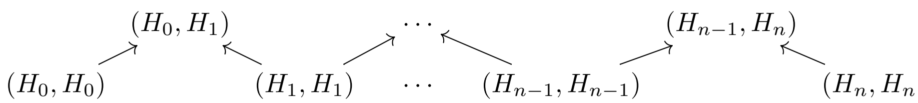
18.5. *Remark.* In Construction 11.1, we gave a 'raw' version of the morphisms in the indexing category $\mathcal{E}_p(G)$, which could be fine-tuned without changing the colimit (11.8). As with any colimit, we can quotient-out the indexing category $\mathcal{E}_p(G) \twoheadrightarrow \bar{\mathcal{E}}_p(G)$ by identifying any two morphisms that induce the same map by the functor under consideration, here $\mathrm{Spc}(\mathcal{K}(-))$. We then still have

$$\operatorname*{colim}_{(H,K)\in\bar{\mathcal{E}}_p(G)} \mathrm{Spc}(\mathcal{K}(H/K)) \xrightarrow{\sim} \mathrm{Spc}(\mathcal{K}(G)). \tag{18.6}$$

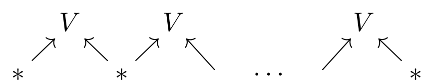
The same holds for any intermediate quotient $\mathcal{E}_p(G) \twoheadrightarrow \tilde{\mathcal{E}}_p(G) \twoheadrightarrow \bar{\mathcal{E}}_p(G)$. For instance if $Z(G)$ denotes the center of G, we can consider the category $\tilde{\mathcal{E}}_p(G)$ obtained from $\mathcal{E}_p(G)$ by modding out the obvious right action of the group $Z(G)\cdot H'$ on each hom set $\hom_{\mathcal{E}_p(G)}((H,K),(H',K'))$.

Let us illustrate how such reductions can be used in practice.

18.7. *Example.* Let $G = C_{p^n}$ be the cyclic group of order p^n. As with any abelian group, using $Z(G) = G$, the reduced category $\bar{\mathcal{E}}_p(G)$ discussed in Remark 18.5 just becomes a poset. Here, if we denote by $1 = H_n < H_{n-1} < \cdots < H_1 < H_0 = G$ the tower of subgroups of G then the poset $\bar{\mathcal{E}}_p(G)$ looks as follows:

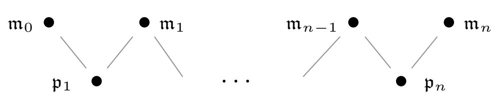


From Theorem 11.10 we deduce that $\mathrm{Spc}(\mathcal{K}(G))$ is the colimit of the diagram



with $* = \mathrm{Spc}(\mathcal{K}(1))$ and $V = \mathrm{Spc}(\mathcal{K}(C_p))$ the V-shaped space in (15.9). In the above diagram, the arrow to the right (resp. left) captures the left-most (resp. right-most) point of V. We conclude that the spectrum of $\mathcal{K}(C_{p^n})$ is equal to

(18.8)



This example reproves Proposition 8.3. It will provide the starting point for our upcoming work on the tt-geometry of Artin motives over finite fields.

18.9. *Remark.* The category of elementary abelian p-sections $\mathcal{E}_p(G)$ is a finite *EI-category*, meaning that all endomorphisms are invertible. The same is true of its reduced versions $\tilde{\mathcal{E}}_p(G)$ and $\bar{\mathcal{E}}_p(G)$ in Remark 18.5. Theorem 11.10 then implies formally that $\mathrm{Spc}(\mathcal{K}(G))$ is the quotient of the spectra for the maximal elementary abelian p-sections by the maximal relations. Let us spell this out.

18.10. *Construction.* Let I be a finite EI-category. The (isomorphism classes of) objects in I inherit a poset structure with $x \le y$ if $\mathrm{Hom}_I(x,y) \neq \emptyset$. Maximal objects $\mathrm{Max}(I) \subseteq I$ are by definition the maximal ones in this poset. Now, let x_1, x_2 be two objects in I, possibly equal. The category $\mathrm{Rel}(x_1, x_2)$ of spans $x_1 \leftarrow y \rightarrow x_2$ (or 'relations') between x_1 and x_2, with obvious morphisms (on the y part, compatible with the spans), is also a finite EI-category and we may consider its maximal objects.

18.11. *Notation.* We denote by $\mathrm{Max}^{\mathrm{elab}}_{p\text{-sec}}(G)$ the set of maximal objects in $\mathcal{E}_p(G)$. A word of warning: In general, there can be more maximal elementary abelian p-sections than just the elementary abelian p-sections of maximal rank.

18.12. **Corollary.** *Let G be a finite group. The components $\varphi_{(H,K)}$ of* (11.7) *induce a homeomorphism between the following coequalizer in topological spaces*

$$\mathrm{coeq}\left(\coprod_{\substack{E_1 \xleftarrow{g_1} L \xrightarrow{g_2} E_2 \\ \text{maximal relations}}} \mathrm{Spc}(\mathcal{K}(L)) \underset{\mathrm{Spc}(\mathcal{K}(g_2))}{\overset{\mathrm{Spc}(\mathcal{K}(g_1))}{\rightrightarrows}} \coprod_{E \in \mathrm{Max}^{\mathrm{elab}}_{p\text{-sec}}(G)} \mathrm{Spc}(\mathcal{K}(E)) \right) \tag{18.13}$$

and $\mathrm{Spc}(\mathcal{K}(G))$*, for 'maximal relations' in $\mathcal{E}_p(G)$ or any variant of Remark 18.5.*

Proof. Applying Theorem 11.10 we obtain

$$\mathrm{Spc}(\mathcal{K}(G)) \simeq \mathrm{coeq}\left(\coprod_{E_1 \xleftarrow{g_1} L \xrightarrow{g_2} E_2} \mathrm{Spc}(\mathcal{K}(L)) \underset{\mathrm{Spc}(\mathcal{K}(g_2))}{\overset{\mathrm{Spc}(\mathcal{K}(g_1))}{\rightrightarrows}} \coprod_{E \in \mathcal{E}_p(G)} \mathrm{Spc}(\mathcal{K}(E)) \right)$$

where E ranges over *all* elementary abelian p-sections and (g_1, g_2) over *all* relations. There is a canonical map from the coequalizer in the statement to this one and it is straightforward to produce an inverse, as with any finite EI-category. □

We can apply Corollary 18.12 to find the irreducible components of $\mathrm{Spc}(\mathcal{K}(G))$.

18.14. **Proposition.** *The set of irreducible components of* $\mathrm{Spc}(\mathcal{K}(G))$ *is in bijection with the set* $\mathrm{Max}^{\mathrm{elab}}_{p\text{-sec}}(G)$ *of maximal elementary abelian p-sections of G up to conjugation, via the following bijection with generic points:*

$$\begin{aligned} \mathrm{Max}^{\mathrm{elab}}_{p\text{-sec}}(G)_{/G} &\overset{\sim}{\longleftrightarrow} \mathrm{Spc}(\mathcal{K}(G))^0 \\ (H,K) &\longmapsto \varphi_{(H,K)}(\eta_{H/K}). \end{aligned}$$

In particular, $\dim(\mathrm{Spc}(\mathcal{K}(G))) = p\text{-rank}_{\mathrm{sec}}(G)$ *is the sectional p-rank of G.*

Proof. We use coequalizer (18.13). Recall from Proposition 15.11 that $\mathrm{Spc}(\mathcal{K}(E))$ for an elementary abelian p-group E is always irreducible. We get immediately that the map $\mathrm{Max}^{\mathrm{elab}}_{p\text{-sec}}(G)_{/G} \twoheadrightarrow \mathrm{Spc}(\mathcal{K}(G))^0$ is a surjection. Assume now that $\varphi_E(\eta_E) = \varphi_{E'}(\eta_{E'})$ for $E, E' \in \mathrm{Max}^{\mathrm{elab}}_{p\text{-sec}}(G)$ and let us show that E and E' are conjugate p-sections. By Corollary 18.12, there exists a finite sequence of maximal relations responsible for the identity $\varphi_E(\eta_E) = \varphi_{E'}(\eta_{E'})$ and we will treat one relation at a time. More precisely, assuming that the generic point in $\mathrm{Spc}(\mathcal{K}(E_1))$ is in the image of (the map on spectra induced by) some relation $E_1 \xleftarrow{g_1} L \xrightarrow{g_2} E_2$, with $E_1, E_2 \in \mathrm{Max}^{\mathrm{elab}}_{p\text{-sec}}(G)$, we will show below that both g_i are conjugation isomorphisms (type (a) in Examples 11.2). In particular, E_1, E_2 are conjugate. And as conjugation identifies the unique generic points in the spectra for E_1 and E_2 one can apply induction on the number of relations to conclude.

As the map induced by g_1 is a closed immersion (Lemma 11.9) it must be a homeomorphism once its image contains the generic point. From this, we deduce that g_1 itself must be an isomorphism. (Indeed, the map induced by restriction to a proper subgroup of E_1 is not surjective, already on the cohomological open. And similarly, the map induced by modular fixed-points with respect to a nontrivial subgroup of E_1 does not even meet the cohomological open.) Hence $L \simeq E_1$

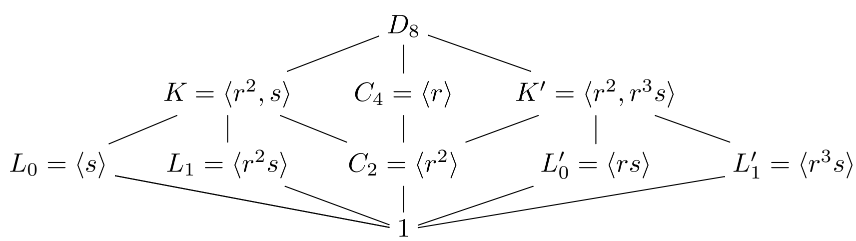
is maximal too and therefore g_2 is also an isomorphism. The only isomorphisms in $\mathcal{E}_p(G)$ are conjugations (Remark 11.3) and we conclude.

The second statement follows from this together with Proposition 15.14. $\square$

18.15. *Remark.* For G not elementary abelian, we already saw with $G = Q_8$ in Example 8.12 that $\mathrm{Spc}(\mathcal{K}(G))$ can have larger Krull dimension than $\mathrm{Spc}(\mathrm{D_b}(kG))$. And indeed, Q_8 has sectional p-rank two and p-rank one.

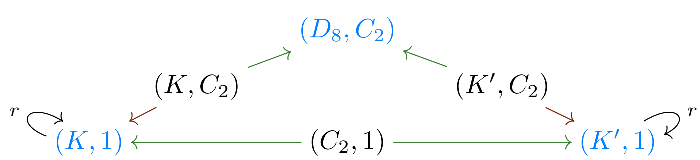
18.16. *Remark.* For every maximal $(H,K) \in \mathrm{Max}^{\mathrm{elab}}_{p\text{-sec}}(G)$, since $\varphi_{(H,K)}$ is a closed map, it yields a surjection $\varphi_{(H,K)}\colon \mathrm{Spc}(\mathcal{K}(E)) \twoheadrightarrow \overline{\{\varphi_{H,K}(\eta_{H/K})\}}$ from the spectrum of the elementary abelian $E = H/K$ onto the corresponding irreducible component of $\mathrm{Spc}(\mathcal{K}(G))$. We illustrate this with $G = D_8$ in Example 18.17 below, where said surjection coincides with the folding of Example 18.4.

18.17. *Example.* Let us compute $\mathrm{Spc}(\mathcal{K}(D_8))$ for $G = D_8 = \langle\, r, s \mid r^4 = s^2 = 1, rs = sr^{-1}\,\rangle$ the dihedral group of order 8. We label its subgroups as follows ([8]):



Since L_0 and L_1 (resp. L'_0 and L'_1) are G-conjugate, by the element r, we have exactly eight very closed points $\mathcal{M}(H)$ for $H \in \mathrm{Sub}_p(G)_{/G}$. We shall focus on the open complement of these very closed points, *i.e.* the periodic locus $\mathrm{Spc}'(\mathcal{K}(D_8))$ of Remark 18.1, which is of Krull dimension one. Since all maps in the coequalizer diagram (18.13) preserve the dimension of points (Lemma 11.9) we may first remove these very closed points and *then* compute the coequalizer.

Let us describe $\mathrm{Max}^{\mathrm{elab}}_{p\text{-sec}}(D_8)$ and the maximal relations. In addition to the maximal elementary abelian subgroups K and K' there is one maximal elementary abelian subquotient D_8/C_2. So we have three maximal sections: $\mathrm{Max}^{\mathrm{elab}}_{p\text{-sec}}(D_8) = \{(K,1),(K',1),(D_8,C_2)\}$. We compute the relations in the category $\tilde{\mathcal{E}}_2(D_8)$ which is obtained from $\mathcal{E}_2(D_8)$ by quotienting each hom-set $\hom((H,M),(H',M'))$ by the action of H', as in Remark 18.5. One then easily finds by inspection five non-degenerate ([9]) maximal relations up to isomorphism, pictured as follows:

(18.18)


[8] The two Klein-four subgroups are called K and K'. The names L_0 and L_1 for the cyclic subgroups of K (resp. L'_0 and L'_1 in K') are chosen to evoke N_0 and N_1 in Example 16.16. The third cyclic subgroup, N_∞, corresponds to $C_2 = Z(D_8)$ and is common to K and K'.

[9] that is, not of the form $x \overset{\mathrm{id}}{\longleftarrow} x \overset{\mathrm{id}}{\longrightarrow} x$ (which would not affect the coequalizer (18.13) anyway)

Here, the loops labeled r represent the relations $(K,1) \xleftarrow{1} (K,1) \xrightarrow{r} (K,1)$, and similarly for K'. All unlabeled arrows are given by $1 \in D_8$, as in Examples 11.2 (b)-(c). We explain below the brown/green color-coding in the other three relations.

Hence the space $\mathrm{Spc}'(\mathcal{K}(D_8))$ is a quotient of three copies of the space $\mathrm{Spc}'(\mathcal{K}(E))$ for E the Klein-four group, equal to $\mathbb{P}^1_k$ with three doubled points as in Figure 3.

Let us discuss the relations. We start with the self-relation corresponding to the loop r on $(K,1)$. As the conjugation by r on K simply swaps the subgroups L_0 and L_1, we deduce from Example 18.4 that the quotient of $\mathrm{Spc}'(\mathcal{K}(K))$ by this relation is a $\mathbb{P}^1_k$ with two doubled points, as in Figure 4. The same is true for K'.

At this stage we have identified the three irreducible components (see Figure 5) and the three remaining relations will tell us how to glue these components.

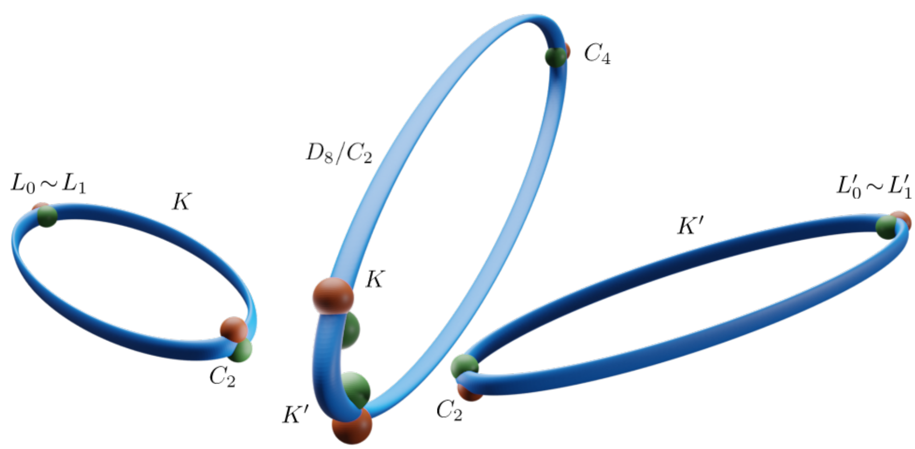


FIGURE 5. Three $\mathbb{P}^1_k$ with several points doubled.

The three sides of the 'triangle' (18.18) display maximal relations that identify a single point of one irreducible component with a single point of another. Indeed, each of the middle sections K/C_2, K'/C_2 and $C_2/1$ is a C_2, whose periodic locus is a single point η_{C_2} (Example 15.8). Each edge in (18.18) identifies the image of that single point η_{C_2} in the two corresponding irreducible components in Figure 5. The color in (18.18) records the color of that image: A brown point or a green point in the $\mathbb{P}^1_k$ with doubled points. Let us do all three. First, the relation between the two Klein-fours, K and K', at the bottom of (18.18), identifies the two green points corresponding to C_2, as we are used to with projective support varieties. Then, the last two relations in (18.18), on the sides, identify a brown point in the K- or K'-component with the green point in the D_8/C_2-component corresponding to K/C_2 and K'/C_2, respectively. This is a direct verification, for instance using that $(\psi^{C_2})^{-1}(\mathrm{Im}(\rho_K^{D_8})) = (\psi^{C_2})^{-1}(\mathrm{supp}_{D_8}(k(D_8/K))) = \mathrm{supp}_{D_8/C_2}(\Psi^{C_2}(k(D_8/K))) = \mathrm{supp}_{D_8/C_2}(k(D_8/K)) = \mathrm{Im}(\rho_{K/C_2}^{D_8/C_2})$ in $\mathrm{Spc}(\mathcal{K}(D_8/C_2))$.

Thus we obtain $\mathrm{Spc}'(\mathcal{K}(D_8))$ from these three identifications on the space of Figure 5. The result is the space that appeared in Figure 1:

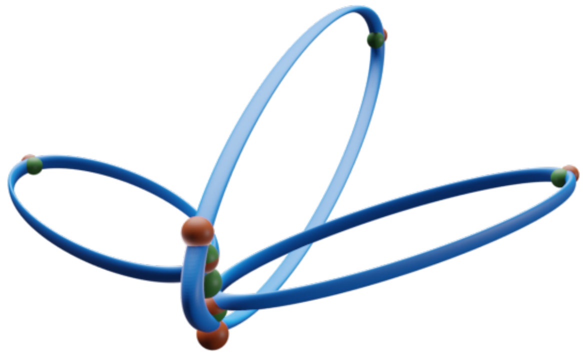


FIGURE 6. Three $\mathbb{P}^1_k$ with several points doubled and some identified.

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