Contemporary Mathematics
Volume **00**, 2003

An introduction to Triangular Witt Groups and a survey of applications

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March 2003

Abstract. These are extended notes from a survey talk on Witt groups of triangulated categories, given at the Talca–Pucon Conference, December 2002.

1. Introduction

The articles [**2**] to [**9**], presenting the theory of triangular Witt groups (TWG) and its first applications, are meant for quite a general audience, and hence contain a lot of details. Therefore, I will not repeat the whole material of TWG here, except for the basic notions of course, but I will rather try to broaden the audience. In this spirit, I give *motivations* and *preliminary explanations*. Motivating a reader for TWG is probably best achieved by reviewing applications, and this is done in the second part of the present article. The first part contains preliminary explanations, meant to fill the "triangular gap". Although basic facts about triangulated categories are recalled in [**3**] for instance, a reader not familiar with the language of triangulated categories might think it too difficult to access and find the triangular Witt groups pretty hard to use in everyday life. Consequently, I added to this survey a first part which consists of a very basic pre-introduction to TWG for non-specialists, where I sketch how complexes appear in the story, explain how triangulated categories allow us to handle these complexes with minimal pain, and show what symmetric forms and Witt groups become in this new language. The case of a ring is our running example and follows us throughout this first part.

The expository style is very slow and detailed up to the point where I consider the triangular gap sufficiently filled to be crossed with dry feet. From there on, the pace gets faster with a "guide through TWG" which is simply a list of results with references. A reader only interested in down-to-earth facts can directly skip to Part II, which is a more standard overview of applications, essentially to Witt groups of schemes. A certain number of results, by other authors as well, are collected there and references to the literature are given.

I sometimes gave priority to simple and conceptual considerations, at the price of having slightly less rigorous or less general statements than actually possible.

2000 *Mathematics Subject Classification.* Primary 11E81, 19G12, 18E30.

The author is supported by the Swiss National Science Foundation.

Acknowledgments: It is a pleasure to thank Raman Preeti and Michel Matthey for reading and commenting early versions of the paper.

Part I: Introduction to TWG for non-specialists

2. Genesis

The idea of considering complexes $P_\bullet = \quad \cdots \to P_{i+1} \to P_i \to P_{i-1} \to \cdots$ equipped with symmetric forms naturally emerged in at least two areas of mathematics: *differential topology* and *algebraic geometry.* For the moment, by *symmetric complex* we mean some complex $P_\bullet$ with some mysterious sort of "symmetric form", basically relating $P_\bullet$ and some mysterious sort of a dual of $P_\bullet$ which we shall make precise later on.

In differential topology, more precisely in *surgery theory,* symmetric complexes come from Poincaré duality and these complexes consist of $\mathbb{Z}[\pi]$-modules, where π is the fundamental group of the manifolds under the scalpel. The theory of surgery of manifolds was initiated by Milnor [**19**] and led to the development of L-theory by Wall (see for instance [**25**] and [**26**]), Mischenko, and to the full extent by Ranicki (see for instance [**23**] for a concise and purely algebraic approach to L-theory).

In algebraic geometry, the side of the story we are mainly concerned with here, symmetric complexes appear in quite a different spirit. The complexes show up for general reasons, involving direct images of modules, and then the symmetric forms follow in a second step. Let us give a simple example instead of a general theory. Let R be a commutative ring. Assume that R is noetherian and regular to fix the ideas, *i.e.* any finitely generated R-module has a finite resolution by finitely generated projective R-modules. Consider an ideal $I \subset R$ and the quotient $\pi : R \to R/I$. As an intense meditation will show, this π is different from the π considered above. Via π we can consider an R/I-module as an R-module and this is a sound activity when comparing the modules over $\mathrm{Spec}(R)$ and over the Zariski open $\mathrm{Spec}(R) \setminus V(I) = \{\mathfrak{p} \in \mathrm{Spec}(R) \,|\, I \not\subset \mathfrak{p}\}$. For instance, when $I = R \cdot f$ for some $f \in R$, comparing modules over R and over R_f involves studying modules killed by some power of f and in particular R/f-modules. The bad news is that a *projective* R/I-module N, even $N = R/I$ itself, is usually not projective anymore when considered as an R-module. The simplest example is probably something like $R = \mathbb{Z}$ or $\mathbb{Z}[\frac{1}{2}]$ and $I = 3 \cdot R$. The old solution to this old problem is then to replace N by a projective resolution. Now, assume that N was secretly carrying some good old symmetric form over R/I, then there is a good chance that the projective resolution of N over R will carry a symmetric form, for a more complicated sense of "symmetric form". One can find in [**20**] an example of how these algebro-geometrical questions of comparing forms over a scheme and an open subscheme led to symmetric complexes.

Philosophically, those two motivations for symmetric complexes are quite different. In the first framework, the meaningful invariant is indeed some class in some L-group of $\mathbb{Z}[\pi]$ (in case of doubt think "L-group:=Witt group"), depending on the manifold M. This class gives us geometric information about M. In the second framework, the invariant is the Witt group itself (in case of doubt think "Witt group:= L-group"). It is the whole group $\mathrm{W}(X)$ which might give us information about the geometry of the scheme X. The question of inverting 2 is also approached quite differently. In the first school, claiming that 2 is a unit is a sufficient condition for immediate excommunication. In the second one, if we think our schemes as being algebraic varieties over some ground field k, we basically renounce to k being of characteristic 2, which is indeed a sin, but a venial one, especially among quadratic form people. This inversion of 2, though, is of central importance in what follows.

In both frameworks we are led to study morphisms $\varphi : P_\bullet \longrightarrow (P_\bullet)^*$ from a complex $P_\bullet$ to its dual complex $(P_\bullet)^*$ whose definition is postponed to Section 4. Beware that we should not expect the "non-degeneracy" of such a "form" φ to be expressed by saying that φ is an isomorphism of complexes. This is because the considered complexes only behave well *up to homotopy.* Thus the "non-degeneracy" is expressed by saying that the form φ is a *homotopy equivalence*. Similarly, the "symmetry" of the form, which reads $\varphi^* = \varphi$ in usual terms, will become: φ^* is homotopic to φ. In L-theory, even more is needed. In fact, one would need the homotopy itself to be symmetric up to (higher) homotopy and the higher homotopy to be symmetric up to (even higher) homotopy and so on. When 2 is invertible, these higher homotopies are not necessary. Indeed, if φ is symmetric up to homotopy, then $\psi := \frac{1}{2}(\varphi + \varphi^*)$ will be strictly symmetric, not only up to homotopy, and ψ will be homotopic to φ. So, if we invert 2, we can reduce the data of a symmetric complex to a complex endowed with a symmetric homotopy equivalence to its dual. This is explained with more details below.

Once born to the importance of complexes with symmetric forms, the mathematician sometimes suffers a short postnatal depression: "My God, what am I going to do with these ugly complexes that I cannot even write completely? with those morphisms which are only up to homotopy? or worse, up to quasi-isomorphism..." and so on. The loss of the notion of exact sequence, which is not well-behaved with respect to homotopy of complexes, might finish to despair the neophyte and even drive him to relapse. This apparent hairiness of the category of complexes is also an old problem which has an old solution. Long ago, that is at the beginning of the sixties, Grothendieck sent Verdier [**24**] to teach us the answer:

3. Triangulated categories

Triangulated categories offer an axiomatization of the *derived category*, that is the category of complexes up to homotopy, or more precisely up to quasi-isomorphism. Let us fix some notations. Let $\mathcal{A} := \mathcal{P}(R)$ denote the category of finitely generated projective (left) R-modules and let

$$\mathrm{K}^{\mathrm{b}}(\mathcal{P}(R)),$$

and more generally $\mathrm{K}^{\mathrm{b}}(\mathcal{A})$ for any additive category $\mathcal{A}$, be the *homotopy category* of bounded complexes of objects in $\mathcal{A}$, with morphisms being morphisms of complexes up to homotopy.

Recall that a (bounded) complex $P_\bullet$ consists of a collection $\{P_i\}_{i\in\mathbb{Z}}$ of objects of $\mathcal{A}$ (such that $P_i = 0$ when $i \notin [-N, N]$ for some N depending on $P_\bullet$) together with a collection of morphisms $d_i : P_i \to P_{i-1}$ such that $d^2 = 0$, which means $d_{i-1} \circ d_i = 0$ for all $i \in \mathbb{Z}$. A morphism of complexes $f_\bullet : P_\bullet \longrightarrow Q_\bullet$ consists of a collection $f_i : P_i \to Q_i$ for all $i \in \mathbb{Z}$ such that $d\,f = f\,d$, which means $d_i^Q \circ f_i = f_{i-1} \circ d_i^P$ for all $i \in \mathbb{Z}$. This defines a category $\mathrm{Ch}^{\mathrm{b}}(\mathcal{A})$ of bounded chain complexes in $\mathcal{A}$.

Recall also that two morphisms of complexes $f_\bullet\,,\, g_\bullet : P_\bullet \longrightarrow Q_\bullet$ are *homotopic* if there is a collection of morphisms $\varepsilon_i : P_i \to Q_{i+1}$ for all $i \in \mathbb{Z}$ such that $f - g = d\varepsilon - \varepsilon d$, or more precisely $f_i - g_i = d_{i+1}^Q \varepsilon_i - \varepsilon_{i-1} d_i^P$ for all $i \in \mathbb{Z}$. Such a collection $\{\varepsilon_i\}_{i\in\mathbb{Z}}$ is sometimes called a homotopy between f and g. The category $\mathrm{K}^{\mathrm{b}}(\mathcal{A})$ has the same objects as $\mathrm{Ch}^{\mathrm{b}}(\mathcal{A})$, *i.e.* bounded complexes in $\mathcal{A}$, but the morphisms are the equivalence classes of morphisms up to homotopy. This category $\mathrm{K}^{\mathrm{b}}(\mathcal{A})$ is the prototype of a triangulated category.

A triangulated category $\mathcal{K}$ will first of all have to be additive (*i.e.* we have a direct sum $\oplus$ and we can add morphisms). Moreover, $\mathcal{K}$ will be equipped with a *shift*, a *translation*, a *suspension*, that is, a functor :

$$T : \mathcal{K} \longrightarrow \mathcal{K}$$

which is additive and which is an equivalence, *i.e.* there exists a shift backwards, a translation backwards, a de-suspension, $T^{-1} : \mathcal{K} \longrightarrow \mathcal{K}$ such that $T^{-1}T \cong TT^{-1} \cong \mathrm{Id}$. In fact, in the example $\mathcal{K} = \mathrm{K}^{\mathrm{b}}(\mathcal{A})$, this T is really an isomorphism of categories, so we can write and think $T^{-1}T = TT^{-1} = \mathrm{Id}$. In this example, the shift simply consists in moving a complex "to the left" and changing the sign of all the differentials :

$$T\Big(\cdots \xrightarrow{d_2} P_1 \xrightarrow{d_1} \underset{\text{(degree 0)}}{P_0} \xrightarrow{d_0} P_{-1} \xrightarrow{d_{-1}} \cdots\Big) := \quad \cdots \xrightarrow{-d_1} P_0 \xrightarrow{-d_0} \underset{\text{(degree 0)}}{P_{-1}} \xrightarrow{-d_{-1}} P_{-2} \xrightarrow{-d_{-2}} \cdots$$

Morphisms of complexes are simply shifted to the left with no sign added : $T(f_\bullet)_i := f_{i-1}$ for all $i \in \mathbb{Z}$. The shift backwards T^{-1} is obvious.

Suppose that $\mathcal{A}$ is indeed an abelian category, like $\mathcal{A} = \mathcal{M}(R)$ the category of finitely generated (left) R-modules over a (left) noetherian ring R. Then on the category of chain complexes $\mathrm{Ch}^{\mathrm{b}}(\mathcal{A})$ with honest chain maps, not up to homotopy, we have the notion of exact sequence, which is simply an exact sequence degree-wise. Indeed $\mathrm{Ch}^{\mathrm{b}}(\mathcal{A})$ is again an abelian category. Similarly, when $\mathcal{A}$ is only additive, like for $\mathcal{A} = \mathcal{P}(R)$, or is an exact category like the category of vector bundles over a scheme, then $\mathrm{Ch}^{\mathrm{b}}(\mathcal{A})$ still has the useful notion of exact sequences degree-wise. Unfortunately, this is all ruined down by the "up to homotopy" politics. To see this, one can prove that any monomorphism in $\mathrm{K}^{\mathrm{b}}(\mathcal{A})$ must in fact be a *split* monomorphism; hence the poverty in subtle short exact sequences.

The replacement of exact sequences is the notion of *exact triangle*, or *distinguished triangle*, that we explain now.

3.1. The mapping cone construction. An important construction in categories of complexes is the following. Let $u : A_\bullet \longrightarrow B_\bullet$ be a morphism of complexes. Then the *mapping cone* of u is a new complex $\mathrm{Cone}(u)$ defined as in the following diagram (see the third line) :

$$
\begin{tikzcd}
 & & \text{(degree n)} & & \text{(degree n-1)} & \\
A_\bullet = \arrow[d, "u"] & \cdots \arrow[r] & A_n \arrow[rr, "d_n^A"] \arrow[d, "u_n"'] & & A_{n-1} \arrow[r] \arrow[d, "u_{n-1}"] & \cdots \\
B_\bullet = \arrow[d, "v:="] & \cdots \arrow[r] & B_n \arrow[rr, "d_n^B"] \arrow[d, "\begin{pmatrix}0\\1\end{pmatrix}"'] & & B_{n-1} \arrow[r] \arrow[d, "\begin{pmatrix}0\\1\end{pmatrix}"] & \cdots \\
\mathrm{Cone}(u) := \arrow[d, "w:="] & \cdots \arrow[r] & A_{n-1} \oplus B_n \arrow[rr, "\begin{pmatrix}-d_{n-1}^A & 0\\ -u_{n-1} & d_n^B\end{pmatrix}"] \arrow[d, "\begin{pmatrix}-1 & 0\end{pmatrix}"'] & & A_{n-2} \oplus B_{n-1} \arrow[r] \arrow[d, "\begin{pmatrix}-1 & 0\end{pmatrix}"] & \cdots \\
T(A_\bullet) = & \cdots \arrow[r] & A_{n-1} \arrow[rr, "-d_{n-1}^A"] & & A_{n-2} \arrow[r] & \cdots
\end{tikzcd}
$$

in which we also define morphisms $v : B \longrightarrow \mathrm{Cone}(u)$ and $w : \mathrm{Cone}(u) \longrightarrow T(A)$. In short, the differential of the mapping cone is $\begin{pmatrix} -d & 0 \\ -u & d \end{pmatrix}$ whose square is easily checked to be zero. We abstract the above data to any additive category $\mathcal{K}$ with translation T as being a triple of objects and a triple of morphisms as follows:

$$
A \xrightarrow{u} B \xrightarrow{v} C \xrightarrow{w} T(A).
$$

This is *a triangle*. Compare the last two diagrams to feel how the language of triangulated categories could simplify the handling of complexes.

The above "mapping cone construction" turns out to be very useful in $\mathrm{K}^{\mathrm{b}}(\mathcal{A})$, even for $\mathcal{A} = \mathcal{P}(R)$ which has only split exact sequences. In fact, we can prove for instance that a morphism $u : A_\bullet \longrightarrow B_\bullet$ is a homotopy equivalence (*i.e.* an isomorphism in our $\mathrm{K}^{\mathrm{b}}(\mathcal{A})$) exactly when the complex $\mathrm{Cone}(u)$ is split exact (*i.e.* isomorphic to zero in $\mathrm{K}^{\mathrm{b}}(\mathcal{A})$). One can also prove that a composition $z \circ u$ is homotopic to zero for some test morphism $z : B_\bullet \longrightarrow Z_\bullet$ if and only if there exists a morphism $\bar{z} : \mathrm{Cone}(u) \longrightarrow Z_\bullet$ such that $\bar{z} \circ v$ is homotopic to z :

$$
\begin{tikzcd}
A_\bullet \arrow[r, "u"] \arrow[dr, "\text{if } z\,u=0"'] & B_\bullet \arrow[r, "v"] \arrow[d, "z"'] & \mathrm{Cone}(u) \arrow[r, "w"] \arrow[dl, dashed, "\exists\,\bar{z}"] & T(A_\bullet) \\
 & Z_\bullet & &
\end{tikzcd}
$$

In terms of the category $\mathcal{K} = \mathrm{K}^{\mathrm{b}}(\mathcal{A})$ it says that the morphism $v : B_\bullet \longrightarrow \mathrm{Cone}(u)$ is almost a cokernel of u, except that we do not guarantee the uniqueness (even up to homotopy) of the $\bar{z}$ in the above story. One would say that $v : B_\bullet \longrightarrow \mathrm{Cone}(u)$ is a *weak cokernel* of u. Similarly, it can be shown that $T^{-1}(w) : T^{-1}(\mathrm{Cone}(u)) \longrightarrow A_\bullet$ is a *weak kernel* of u. In vaguer terms, the mapping cone $\mathrm{Cone}(u)$ contains the *homological information* about u (weak kernel, weak cokernel, detection of isomorphism).

The same will be true in "abstract" triangulated categories, but of course not for any triangle. For instance, even without knowing yet what an exact triangle will be, the normally brained reader can guess that we do not expect a triangle like

$$0 \longrightarrow B \longrightarrow 0 \longrightarrow T(0)$$

to be exact unless $B \simeq 0$. So we have to *distinguish* those triangles which contain the above kind of "homological information" among all the possible triangles. In the case of $\mathrm{K}^{\mathrm{b}}(\mathcal{A})$, those distinguished triangles are defined to be all the triangles which are isomorphic to a triangle obtained, as above, by the mapping cone construction.

In abstract triangulated categories, we assume *given* a class of *distinguished triangles*, also called *exact triangles*, and we require them to fulfil four axioms, all of them very natural. We do not reproduce the axioms here and we refer the reader to Section 1 of [**3**] to get a four-page "*baise-en-ville* of the triangulated mathematician". The original source is of course [**24**]. More important to us here is the philosophy : we need complexes but we don't like complexes, so we overcome the apparent complication of $\mathrm{K}^{\mathrm{b}}(\mathcal{A})$ by abstracting the important techniques into the concept of triangulated category.

In particular, given a morphism $u : A \to B$ in a triangulated category $\mathcal{K}$ there is an object C and a distinguished triangle $A \xrightarrow{u} B \xrightarrow{v} C \xrightarrow{w} T(A)$ in $\mathcal{K}$ (this is an axiom). One can prove from the axioms that the triple (C, v, w) is unique up to non-unique isomorphism, once $u : A \to B$ is given. One can prove that $v : B \to C$ is a weak cokernel of u, that $T^{-1}(w) : T^{-1}(C) \to A$ is a weak kernel of u, and that C is isomorphic to zero if and only if u is an isomorphism.

3.2. The cone of a morphism. Given a morphism $u : A \to B$ in a triangulated category, a triple (C, v, w) as above, or sometimes just the object C, is called *the cone* of u, out of nostalgia for the mapping cone of our good old complexes, which now start to fade away in the distance.

4. Triangulated categories with duality

We now have a feeling of what a triangulated category is. In order to talk of symmetric spaces and to define Witt groups, we need another kind of structure on our category, namely we need a *duality*. This notion of duality is of course not specific to triangulated categories and already appears for modules and for vector spaces as we all know. Here, the duality will have to be compatible with the triangulation. This is conceptually very simple but technically slightly complicated by the presence of signs. Although we try to explain both the idea and the signs simultaneously, the reader can very well ignore the signs in a first reading. We have decided to discuss that here, since a fear of signs, not cured at an early stage, might slow down a serious attempt to understand and use TWG.

As suggested in Section 2, when 2 is invertible, we can chop out the "higher homotopies" from the data of a symmetric complex. This means that a symmetric complex over a ring R will simply be an object P in $\mathrm{K}^{\mathrm{b}}(\mathcal{P}(R))$ with an isomorphism $\varphi : P \longrightarrow P^*$ in $\mathrm{K}^{\mathrm{b}}(\mathcal{P}(R))$, which is symmetric $\varphi^* = \varphi$ also in $\mathrm{K}^{\mathrm{b}}(\mathcal{P}(R))$. We now want to make this precise. Again, we stress that L-theory cannot, as far as we know, be defined over the triangulated category $\mathrm{K}^{\mathrm{b}}(\mathcal{P}(R))$. This is a great application of the invention of 0.5 : We can apply the relaxing philosophy presented in the previous

Section, in order to simplify the question of symmetric complexes by formalizing it into the triangular language. We proceed as above, that is, we explicit the example of $\mathrm{K^b}(\mathcal{P}(R))$ or $\mathrm{K^b}(\mathcal{A})$ and then abstract what is really needed.

Consider an additive contravariant functor $(-)^* : \mathcal{A}^{\mathrm{op}} \longrightarrow \mathcal{A}$ on an additive category $\mathcal{A}$, like for instance $(-)^* = \mathrm{Hom}_R(-,R) : \mathcal{P}(R)^{\mathrm{op}} \longrightarrow \mathcal{P}(R)$ the usual dual on the category of finitely generated projective left R-modules over a ring R with involution[1]. Such a functor induces a contravariant functor

$$(-)^{\#} : \mathrm{K^b}(\mathcal{A})^{\mathrm{op}} \longrightarrow \mathrm{K^b}(\mathcal{A})$$

which is defined as follows

$$\Big(\cdots \xrightarrow{d_2} P_1 \xrightarrow{d_1} \overset{\text{(degree 0)}}{P_0} \xrightarrow{d_0} P_{-1} \xrightarrow{d_{-1}} \cdots \Big)^{\#} := \quad \cdots \xrightarrow{d_{-1}^*} P_{-1}^* \xrightarrow{d_0^*} \overset{\text{(degree 0)}}{P_0^*} \xrightarrow{d_1^*} P_1^* \xrightarrow{d_2^*} \cdots$$

and on morphisms by

$$\big((f_\bullet)^{\#}\big)_i := (f_{-i})^* \qquad \text{for all } i \in \mathbb{Z}\,.$$

It is the obvious extension of an additive functor to $\mathrm{K^b}(-)$ using the fact that $\mathrm{K^b}\big(\mathcal{A}^{\mathrm{op}}\big) \cong \mathrm{K^b}(\mathcal{A})^{\mathrm{op}}$. Note that there are no signs hidden in this definition, except of course the $-i$ in indices which is not a sign in $\mathcal{A}$ but merely indicates the switched reading direction, imposed by the the contravariance of $(-)^*$.

4.1. The opposite triangulation, or the first appearance of signs. The above isomorphism $\mathrm{K^b}(\mathcal{A})^{\mathrm{op}} \cong \mathrm{K^b}\big(\mathcal{A}^{\mathrm{op}}\big)$ forces a triangular structure on $\mathrm{K^b}(\mathcal{A})^{\mathrm{op}}$. More generally, the opposite $\mathcal{K}^{\mathrm{op}}$ of a triangulated category $\mathcal{K}$ also has a triangulation, which should agree with this example, and therefore involves some signs as we explain now. Let $\mathcal{K}$ be a triangulated category. Then the translation $T : \mathcal{K} \longrightarrow \mathcal{K}$ induces a translation $T^{\mathrm{op}} : \mathcal{K}^{\mathrm{op}} \longrightarrow \mathcal{K}^{\mathrm{op}}$ defined by

$$T^{\mathrm{op}}(A^\circ) := \big(T^{-1}(A)\big)^\circ \qquad \text{and} \qquad T^{\mathrm{op}}(f^\circ) := \big(T^{-1}(f)\big)^\circ$$

where A° stands for the object A seen in $\mathcal{K}^{\mathrm{op}}$ and where $f^\circ : A^\circ \to B^\circ$ in $\mathcal{K}^{\mathrm{op}}$ corresponds to $f : B \to A$ in $\mathcal{K}$. In short, T^{op} is T^{-1}. One has to "reverse" the translation in order to *at least make sense* of a triangle in $\mathcal{K}^{\mathrm{op}}$. Indeed, a triangle

$$A^\circ \xrightarrow{f^\circ} B^\circ \xrightarrow{g^\circ} C^\circ \xrightarrow{h^\circ} T^{\mathrm{op}}(A^\circ)$$

is *distinguished* in $\mathcal{K}^{\mathrm{op}}$ if the following triangle is distinguished in $\mathcal{K}$:

$$T^{-1}(A) \xrightarrow{-h} C \xrightarrow{-g} B \xrightarrow{-f} A$$

obtained by displaying the morphisms in reversed order: h, g, f, but with signs. This is what we meant above: just because the fourth object must be the translation of the first, imposes the definition of T^{op} as being T^{-1}. (This can be carefully checked as a familiarizing exercise.)

[1]Exercise: When M is a left R-module, its dual M^* is *a priori* rather a right R-module in a natural way. One requires the ring R to have an involution $\sigma : R \to R$, $\sigma(rs) = \sigma(s)\sigma(r)$, in order to restore a left module structure on M^* and more generally on any right R-module.

We take this opportunity to recall that in a triangulated category $\mathcal{K}$ we can change any two signs in a triangle but usually not the three of them. Rotating a triangle involves inserting a sign (this is an axiom), and so if the reader is rather object-oriented than morphism-oriented (may Alexander forgive him!), we can equivalently require the following triangle to be distinguished:

$$C \xrightarrow{g} B \xrightarrow{f} A \xrightarrow{T(h)} T(C).$$

Here the objects are in the reversed order: C, B, A, and the morphisms are slightly mixed up, but no sign appears.

4.2. Lemma. *Let $\mathcal{A}$ be an additive category. Then with the above triangulation, $\mathrm{K}^{\mathrm{b}}(\mathcal{A})^{\mathrm{op}}$ is isomorphic as a triangulated category to $\mathrm{K}^{\mathrm{b}}\big(\mathcal{A}^{\mathrm{op}}\big)$.*

In particular, for any additive functor $(-)^* : \mathcal{A}^{\mathrm{op}} \longrightarrow \mathcal{A}$ the induced functor $(-)^{\#} : \mathrm{K}^{\mathrm{b}}(\mathcal{A})^{\mathrm{op}} \longrightarrow \mathrm{K}^{\mathrm{b}}(\mathcal{A})$ described above sends distinguished triangles to distinguished triangles.

Proving this Lemma amounts to understanding the connection between the mapping cone construction of Section 3 performed in $\mathrm{K}^{\mathrm{b}}(\mathcal{A}^{\mathrm{op}})$ and the opposite of the mapping cone construction performed in $\mathrm{K}^{\mathrm{b}}(\mathcal{A})$. For an arbitrary morphism $u : A_\bullet \longrightarrow B_\bullet$ there is an isomorphism between the complex $\mathrm{Cone}(u^\circ)$ and something like $\big(\mathrm{Cone}(u)\big)^\circ$. In fact, this isomorphism is of degree one:

$$\mathrm{Cone}(u^\circ) \simeq \big(T^{-1}\mathrm{Cone}(u)\big)^\circ.$$

One has to check that this isomorphism is compatible with the morphisms v and w defined in the mapping cone constructions. This is a long but straightforward exercise.

Let us summarize the order of the sign choices. The above Lemma justifies the choice of the triangulation on $\mathrm{K}^{\mathrm{b}}(\mathcal{A})^{\mathrm{op}}$. In turn this triangulation on $\mathrm{K}^{\mathrm{b}}(\mathcal{A})^{\mathrm{op}}$ justifies the choice of signs in defining the triangulation on $\mathcal{K}^{\mathrm{op}}$ for any triangulated category $\mathcal{K}$.

4.3. Definition. A *duality on a triangulated category* $\mathcal{K}$ can be remembered as a contravariant functor

$$(-)^{\#} : \mathcal{K}^{\mathrm{op}} \longrightarrow \mathcal{K}$$

which is *exact*, *i.e.* sends a distinguished triangle to a distinguished triangle, and such that

$$\#^2 = \mathrm{Id}\,.$$

A precise definition is to be found in [**3**, § 2], where the notion of δ-duality for $\delta = \pm 1$ is given. A duality is simply a $(+1)$-duality. We will explain below how one ends up studying δ-dualities, but first, let us see how the $\#$ of our example $\mathrm{K}^{\mathrm{b}}(\mathcal{A})$ is a duality in the above sense.

If we compute the functor $\#^2$ on $\mathrm{K}^{\mathrm{b}}(\mathcal{A})$ we simply get a functor which sends a complex $P_\bullet$ to the same complex with a decoration ** everywhere. So, if assume that we have a natural isomorphism $can : \mathrm{Id}_P \xrightarrow{\sim} P^{**}$ for $P \in \mathcal{A}$, then by applying it degree-wise, we immediately get a natural isomorphism:

$$\varpi : \mathrm{Id}_{\mathrm{K}^{\mathrm{b}}(\mathcal{A})} \xrightarrow{\sim} (-)^{\#} \circ (-)^{\#}\,.$$

In the special case $\mathcal{A} = \mathcal{P}(R)$, this can_P is defined as the usual identification between a projective module and its double dual[2]. In the general case, we suppose this isomorphism ϖ given on $\mathcal{A}$ from start.

4.4. Infinitely many dualities for the price of one. Assume you have bought a triangulated category with duality $(\mathcal{K}, \#)$ then in the definition of $\#$ being exact you find that $\#$ sends distinguished triangles to distinguished triangles. For this to make sense, you get in particular (see the definition of the opposite triangulation) :

$$\# \circ T^{-1} \;\cong\; T \circ \# .$$

In the example of $\mathrm{K}^{\mathrm{b}}(\mathcal{A})$ this is even an equality : shift a complex to the right and dualize it, you get the dual of the complex but shifted to the left. But now compute like a beast :

$$(T \circ \#)^2 \;\cong\; (\# \circ T^{-1}) \circ (T \circ \#) \;\cong\; \# \circ \# \xleftarrow[\varpi]{\sim} \mathrm{Id}\,.$$

So you have discovered that $T \circ \#$ is also a duality ! And you conclude inductively and enthusiastically that:

$$T^i \circ \# \;\text{ is again a duality for any } i \in \mathbb{Z}\,!$$

After applying the second fundamental principle of pedagogy : "Teach something possibly wrong but understandable", we have to say that unfortunately, life is more complicated than we expect (this being the first principle). In fact, it is true that $T^i \circ \#$ squares to the identity for any $i \in \mathbb{Z}$ but $T^i \circ \#$ is exact only when i is even. This comes from the fact that the translation $T : \mathcal{K} \longrightarrow \mathcal{K}$ is not exact. Indeed, applying three times the Rotation Axiom, one sees that T is *skew-exact* in the sense that given a distinguished triangle $A \xrightarrow{u} B \xrightarrow{v} C \xrightarrow{w} T(A)$ the triangle $T(A) \xrightarrow{T(u)} T(B) \xrightarrow{T(v)} T(C) \xrightarrow{T(w)} T^2(A)$ is not distinguished but becomes distinguished when changing all signs : $T(A) \xrightarrow{-T(u)} T(B) \xrightarrow{-T(v)} T(C) \xrightarrow{-T(w)} T^2(A)$ or equivalently only one of the signs.

In short, the slogan becomes : Given a triangulated category with duality $(\mathcal{K}, \#)$, we automatically inherit infinitely many triangulated categories with δ-duality $(\mathcal{K}, T^i \circ \#)$ for $\delta = (-1)^i$, where a δ-*duality* for $\delta = \pm 1$ means a duality when $\delta = 1$ and a "skew-exact duality" when $\delta = -1$. Still, in a first reading, one can consider that we have infinitely many dualities on $\mathcal{K}$ and ignore the precise definition [**3**, 2.2].

We will think of these ± 1-dualities in the following order :

$$\ldots \qquad T^{-2}\# \qquad T^{-1}\# \qquad \# \qquad T^{1}\# \qquad T^{2}\# \qquad \ldots$$

and might call $T \circ \#$ the *next* duality *after* $\#$, or $T^{-1} \circ \#$ the *previous* duality.

[2] For a ring R with involution σ, the canonical identification $can_P : P \to P^{**}$ sends $p \in P$ to the homomorphism $\mathrm{ev}_p : P^* \to R$ defined by $\mathrm{ev}_p(f) = \sigma(f(p))$. The outside $\sigma(\ldots)$ is sometimes forgotten in the literature, but is needed for R-linearity.

5. Triangular Witt groups

5.1. Symmetric forms. The notion of symmetric form has nothing to do with the triangulation. As soon as you have a category with duality, you can consider *symmetric spaces*, that are pairs (E,φ) formed of an object E of your category and a symmetric isomorphism $\varphi=\varphi^{\#}: E \overset{\sim}{\to} E^{\#}$ from E to its dual. More carefully written, $\varphi=\varphi^{\#}\circ\varpi_E$ where ϖ_E is the (given) identification between E and its double dual $E^{\#\#}$. The isomorphism φ is called *the form* of the symmetric space (E,φ). There is also immediately a notion of *isometry*, that is, an isomorphism on the E-part which transports the form.

When the category is additive – *e.g.* triangulated – and when the duality is additive as well, we can perform the *orthogonal sum* of two spaces, exactly as usual. In cash, we have

$$(E,\varphi)\perp(F,\psi) := \big(E\oplus F, \begin{pmatrix}\varphi & 0\\ 0 & \psi\end{pmatrix}\big)\,.$$

The question is now to understand the collection of symmetric spaces, at least up to isometry. To do that, we start by producing some cheap symmetric spaces, which will be called *neutral* or *metabolic*. The Witt group will measure the rest: how much information about symmetric spaces remains, apart from these cheap ones.

5.2. The symmetric cone of a symmetric morphism. That is where the triangulation comes into the game. Let $(\mathcal{K},\#)$ be a triangulated category with duality, that is, a $(+1)$-duality to fix the ideas. We have to assume here that 2 is invertible, in the obvious sense, see 6.1 if necessary. Suppose we are given a morphism $u: L \longrightarrow T^{-1}(L^{\#})$ which is symmetric for the duality $T^{-1}\circ\#$:

$$T^{-1}(u^{\#}) = u\,.$$

Note that we use here another duality than the one we start with, namely the "previous one". Then, for any choice of a distinguished triangle starting with u

$$L \xrightarrow{u} T^{-1}(L^{\#}) \xrightarrow{v} C \xrightarrow{w} T(L)$$

the "cone" C will carry a symmetric form for $\#$. That is, you start with a mere symmetric morphism, preferably not an isomorphism (even zero $u=0: L\longrightarrow T^{-1}(L^{\#})$ will do), but *for the previous duality* $T^{-1}\#$, then you can create a symmetric *space* on the cone of u with respect to the considered duality $\#$. This form on the cone of u is unique up to isometry. How is this form defined then? Just from the axioms! In fact this symmetric space (C,φ) is characterized by the following commutative diagram, in which the lines are distinguished triangles:

$$\begin{tikzcd} L \arrow[r, "u"] \arrow[d, "1"'] & T^{-1}(L^{\#}) \arrow[r, "v"] \arrow[d, "1"'] & C \arrow[r, "w"] \arrow[d, "\varphi=\varphi^{\#}"'] & T(L) \arrow[d, "1"'] \\ L \arrow[r, "T^{-1}(u^{\#})"'] & T^{-1}(L^{\#}) \arrow[r, "w^{\#}"'] & C^{\#} \arrow[r, "v^{\#}"'] & T(L). \end{tikzcd}$$

The first line is the distinguished triangle over u defining the cone of u, and the second line is the dual of the first, slightly turned around by using the Rotation Axiom. The left-hand square commutes because u is symmetric. So a fill-in map $\varphi : C \to C^{\#}$ must exist (axiom again). Using $\frac{1}{2}$ to replace φ by $\frac{1}{2}(\varphi + \varphi^{\#})$, we can assume that $\varphi = \varphi^{\#}$. All this is explained in Theorem 2.6 of [**3**]. The term "Theorem" might seem excessive here. It was awarded because this is the starting point of Witt groups over triangulated categories, and not because of the difficulty of the proof, which is in fact totally elementary. Moreover this form φ on C is also proven there to be unique up to isometry. So it sounds reasonable to call it *the symmetric cone* of the symmetric morphism u over L and to write it

$$\mathrm{Cone}(L, u) := (C, \varphi)\,.$$

5.3. A first glimpse at triangular Witt groups. The above symmetric cone construction can be generalized as follows. Consider for any $i \in \mathbb{Z}$ the monoid S^i of isometry classes of pairs (L, u) where L is an object and u is a symmetric morphism $u : L \longrightarrow T^i(L^{\#})$, with u not necessarily an isomorphism. This S^i is an abelian monoid with orthogonal sum as operation. Then the above "symmetric cone construction" gives us a series of maps $c^i : S^i \longrightarrow S^{i+1}$

$$\cdots \longrightarrow S^{i-1} \xrightarrow{c^{i-1}} S^i \xrightarrow{c^i} S^{i+1} \longrightarrow \cdots$$

Note that $c^i(u) = 0$ means precisely that $\mathrm{Cone}(u) \simeq 0$, which, as we know, is equivalent to ask u to be an isomorphism. So, the "kernel" (be careful: S^i is only a monoid not an abelian group) of c^i is exactly the monoid of symmetric *spaces* we are interested in, for the duality $T^i\#$. Since the cone of a symmetric morphism is really a symmetric space, we get

$$c \circ c = 0\,.$$

The i-th Witt group will be in some sense the i-th homology group of the above complex, except that these S^i are monoids, not groups, and thus it is not so clear what the right notion of "homology" should be. We give details below.

5.4. Watch the signs! In fact, the above is again essentially true but slightly too optimistic. The symmetric cone construction $\mathrm{Cone}(L, u)$ we have defined exists because the 2x4 diagram of 5.2 was as it was. Had we started with a *skew*-duality #, the story would of course involve a sign. This is because the duality was used to deduce the exactness of the second line from the exactness of the first line. If the duality is skew-exact instead of exact as assumed above, there will be a sign in this second line. The details are presented in the reference already mentioned: Section 2 of [**3**] and the bottom line is the following.

If # was a skew-duality instead of a $(+1)$-duality (and this generality has to be considered to define the above c^i for all $i \in \mathbb{Z}$), then the cone $\mathrm{Cone}(L, u) = (C, \varphi)$ of a symmetric $u : L \to T^{-1}(L^{\#})$ will be *skew-symmetric* for # ! This means $\varphi^{\#} = -\varphi$. In other words, our $c^i : S^i \longrightarrow S^{i+1}$ described above only works for i odd ($i+1$ even) and actually goes $c^i : S^i \longrightarrow S^{i+1}_{-}$ for i even, where the little sign "$-$" down there means *skew*-symmetric forms.

This symmetric cone construction is responsible for the connection between the *a priori* independent notions of δ-exactness of the duality ($\delta = \pm 1$) and ϵ-symmetry of forms ($\epsilon = \pm 1$). One can live with this and keep signs everywhere. This was indeed the naive notation used in the original [**2**]. There is a way though of hiding the signs under the carpet, which was suggested to me by Charles Walter, and has been consistently used ever since [**3**] as well as in the other authors' articles. We come to this in 5.7, after strengthening in 5.6 the definition of triangulated category with duality.

5.5. No distinction between symmetric and skew-symmetric forms. There is an old trick to avoid speaking of ± 1-symmetric forms: hide the sign ± 1 in the identification $\varpi : \mathrm{Id} \xrightarrow{\sim} \# \circ \#$. If one replaces ϖ with $-\varpi$ then a skew-symmetric form becomes symmetric. Using this, there is a way of making the above symmetric cone construction really going from the i-th duality to the $(i+1)$-st duality and it is to *define* the $(i+1)$-st duality in a different way depending on the parity of i. To do that, we need to include the identification ϖ in what we call a triangulated category with duality. So:

5.6. The full definition. If $\delta = \pm 1$, *a triangulated category with δ-duality* is not only a pair $(\mathcal{K}, \#)$ but a *triple* $(\mathcal{K}, \#, \varpi)$ formed of a triangulated category $\mathcal{K}$, a δ-exact duality $\# : \mathcal{K}^{\mathrm{op}} \longrightarrow \mathcal{K}$ and a *chosen* identification $\varpi : \mathrm{Id} \xrightarrow{\sim} \#^2$. See details in [**3**, Def. 2.2].

5.7. Shifting the dualities. Let $(\mathcal{K}, \#, \varpi)$ be a triangulated category with δ-duality as above. The *shifted* triangulated category with $(-\delta)$-duality is the triple

$$T(\mathcal{K}, \#, \varpi) \quad := \quad \big(\mathcal{K} \ , \ T \circ \# \ , \ (-\delta) \cdot \varpi \,\big).$$

Note that this duality $T \circ \#$ is now $(-\delta)$-exact, not because of the $-\delta$ in front of ϖ but because of the skew-exactness of the translation T. This is also called the *next duality*, or the *first shifted duality*.

This definition is made so that the above "symmetric cone construction" really associates to any symmetric morphism $(L, u : L \longrightarrow L^{\#})$ for the duality $(\mathcal{K}, \#, \varpi)$ a symmetric *space* for the next duality $T(\mathcal{K}, \#, \varpi)$.

The above process is clearly invertible in a unique way and we set

$$T^{-1}(\mathcal{K}, \#, \varpi) \quad := \quad \big(\mathcal{K} \ , \ T^{-1} \circ \# \ , \ \delta \cdot \varpi\big)$$

the *previous* duality or the *duality shifted backwards*, which is also $(-\delta)$-exact when $\#$ is δ-exact. So the exactness does not come from the sign in front of ϖ! Let us repeat slowly: shifting the dualities from $\#$ to $T\#$, $T^2\#$, and so on, changes exactness at each step simply because T is skew-exact. So much for exactness, and nothing more is needed on that side. On the side of ϖ, we put signs in a strange way to make the above "symmetric cone construction" smoother. Indeed, when going from $\#$ to $T\#$, we change the sign of ϖ when $\#$ is exact and we do not change that sign when $\#$ is skew-exact. The rest is a little game with signs and we spend time explaining that not because of any mathematical complexity but to avoid the reader getting lost because of that.

We obtain a series of *shifted dualities* $T^i(\mathcal{K}, \#, \varpi)$ for $i \in \mathbb{Z}$ and we clearly have $T^i\big(T^j(\mathcal{K}, \#, \varpi)\big) = T^{i+j}(\mathcal{K}, \#, \varpi)$ for any $i, j \in \mathbb{Z}$. We leave it as an exercise to check that

$$T^i(\mathcal{K}, \#, \varpi) = (\mathcal{K} \ , \ T^i\# \ , \ \epsilon^i \cdot \varpi) \,,$$

with $\epsilon^i = (-1)^{\frac{i\cdot(i+1)}{2}} \cdot \delta^i$, is a triangulated category with $((-1)^i \cdot \delta)$-exact duality. Since these two signs are 4-periodic in i, we have in other words:

$$T^i(\mathcal{K},\#,\varpi) = \begin{cases} (\mathcal{K}\ ,\ T^i\#\ ,\ \varpi\ ) & \text{if } i \equiv 0 \text{ modulo } 4 \\ (\mathcal{K}\ ,\ T^i\#\ ,\ (-\delta)\cdot\varpi) & \text{if } i \equiv 1 \text{ modulo } 4 \\ (\mathcal{K}\ ,\ T^i\#\ ,\ -\varpi\ ) & \text{if } i \equiv 2 \text{ modulo } 4 \\ (\mathcal{K}\ ,\ T^i\#\ ,\ \delta\cdot\varpi\ ) & \text{if } i \equiv 3 \text{ modulo } 4 \end{cases}$$

which is a triangulated category with $\begin{cases} \delta\text{-exact duality} & \text{if } i \equiv 0 \text{ modulo } 4 \\ (-\delta)\text{-exact duality} & \text{if } i \equiv 1 \text{ modulo } 4 \\ \delta\text{-exact duality} & \text{if } i \equiv 2 \text{ modulo } 4 \\ (-\delta)\text{-exact duality} & \text{if } i \equiv 3 \text{ modulo } 4. \end{cases}$

Note finally that $T^i(\mathcal{K},\#,\varpi)$ and $T^{i+2}(\mathcal{K},\#,\varpi)$ have the same exactness (both exact or both skew-exact) but opposite symmetry (one with $\epsilon^i\varpi$ and the other one with $-\epsilon^i\varpi$). In particular, $T^i(\mathcal{K},\#,\varpi)$ and $T^{i+4}(\mathcal{K},\#,\varpi)$ have same exactness and same symmetry. We will see that the whole theory is indeed 4-periodic.

5.8. Neutral spaces and Witt groups. Given a triangulated category with δ-duality $(\mathcal{K},\#,\varpi)$ the cone $\mathrm{Cone}(L,u)$ of any symmetric morphism $u : L \longrightarrow T^{-1}L^{\#}$ for the previous duality is a symmetric space for $\#$. Such a symmetric space is called *neutral* or *metabolic*. The *Witt group* of $(\mathcal{K},\#,\varpi)$

$$\mathrm{W}(\mathcal{K},\#,\varpi)$$

is defined to be the quotient of the abelian monoid of symmetric spaces by the submonoid of neutral spaces. See details in [**3**, § 2]. The quotient of an abelian monoid M by a submonoid $N \subset M$ is simply the monoid of equivalence classes of M modulo the relation $m \sim m'$ when there exist $n, n' \in N$ such that $m + n = m' + n'$. We shall denote by $[E,\varphi]$ the class of a symmetric space (E,φ) modulo this equivalence relation.

In other words, $\mathrm{W}(\mathcal{K},\#,\varpi)$ is an abelian group (see below why it is a group and not only a monoid) given by generators and relations as follows. Take a generator $[E,\varphi]$ for any symmetric space (E,φ) and impose the two relations:

1. $[E,\varphi] = [E',\varphi']$ if the spaces (E,φ) and (E',φ') are isometric,
2. $[(E,\varphi)\perp(E',\varphi')] = [E,\varphi] + [E',\varphi']$,
3. $\big[\mathrm{Cone}(L,u)\big] = 0$ for any morphism $u : L \longrightarrow T^{-1}L^{\#}$ which is symmetric for the previous duality $T^{-1}(\mathcal{K},\#,\varpi)$, or equivalently: $[E,\varphi] = 0$ for any neutral space (E,φ).

5.9. Hyperbolic spaces. Why is this quotient of monoids a *group* in the first place? This is because of the freedom to choose arbitrarily "degenerate" symmetric morphisms $u : L \to T^{-1}L^{\#}$ to define neutral spaces. In fact, if we take $u = 0$, the symmetric cone is just the *hyperbolic space* $L \oplus L^{\#}$ with the usual form. Since 2 is assumed to be invertible, for any symmetric space (E,φ), the orthogonal sum $(E,\varphi)\perp(E,-\varphi)$ is hyperbolic. This last fact is indeed true in any additive category in which 2 is invertible. (This would still work without $\frac{1}{2}$ though: one can prove that $(E,\varphi)\perp(E,-\varphi)$ is *one* symmetric cone of $(E,0)$, the latter not being unique up to isometry anymore. See 5.12.)

5.10. Lagrangians. If we unfold the definition of a symmetric space (E,φ) being the symmetric cone of a symmetric morphism (L,u), we see an obvious connection with the classical definition of a metabolic space. A metabolic space (E,φ) in an exact category (which is the typical space to kill when constructing the "classical" Witt group *à la* Knebusch), is a symmetric space such that there exists an exact sequence:

$$M \rightarrowtail^{\alpha} E \xrightarrow{\alpha^*\varphi} M^*.$$

This pair (M,α) is called a *Lagrangian* of (E,φ). Similarly here, the right part of the 2x4 diagram of 5.2 can be re-written as a piece of exact triangle:

$$M \xrightarrow{v} C \xrightarrow{v^\#\varphi} M^\#$$

where we put $M := T^{-1}(L^\#)$ and used $w = v^\#\varphi$. This means that the space (C,φ) has a Lagrangian, in a triangular sense of the term: the (symmetric) exact sequence is now replaced by a (symmetric) exact triangle. See more on this in [**3**, Def. 2.12] or unfold the details to familiarize yourself with triangular dualities.

5.11. The shifted Witt groups. When you have a triangulated category with δ-duality $(\mathcal{K},\#,\varpi)$, you immediately obtain infinitely many Witt groups:

$$\mathrm{W}^i(\mathcal{K},\#,\varpi) := \mathrm{W}\big(T^i(\mathcal{K},\#,\varpi)\big)$$

called the *i-th shifted Witt group* of $(\mathcal{K},\#,\varpi)$ for any $i\in\mathbb{Z}$.

5.12. What about 2 not invertible? There is no guaranteed existence nor uniqueness of the "symmetric cone" anymore, but we can still define a neutral space as being one which fits in a 2x4 diagram as in 5.2. The Witt group will still be a group, even if $(E,\varphi)\perp(E,-\varphi)$ is not really hyperbolic, but only split metabolic. The real problem is that I have no idea if a group defined in this way coincides with "usual" Witt groups. I have no idea either if the cohomological behaviour discussed below can be established without $\frac{1}{2}$.

5.13. Symmetric forms in categories of complexes. Let us consider our running example $(\mathcal{K},\#,\varpi)$ where $\mathcal{K}=\mathrm{K}^{\mathrm{b}}(\mathcal{A})$ is the homotopy category of $(\mathcal{A},*,can)$ an additive category with duality $*:\mathcal{A}^{\mathrm{op}}\longrightarrow\mathcal{A}$, where $\#$ is derived from $*$, and where ϖ is *can* in each degree. For instance, $\mathcal{A}$ can be the category of finitely generated projective R-modules $\mathcal{P}(R)$. All this was explained in Sections 3 and 4 above.

Fix an $i\in\mathbb{Z}$. A symmetric space (E,φ) in $T^i(\mathcal{K},\#,\varpi)$ consists of two things:

1. A bounded complex $E=P_\bullet\in\mathrm{Ch}^{\mathrm{b}}(\mathcal{A})$. Note that we do not impose *a priori* a global restriction on the size of the support of $P_\bullet$, except that it must be finite, of course.
2. A symmetric form, that is, a homotopy equivalence $\varphi: P_\bullet\longrightarrow T^i(P_\bullet{}^\#)$ which is symmetric with respect to the i-shifted ϖ, which is $\varpi^i := (-1)^{\frac{i\cdot(i+1)}{2}}\cdot\varpi$.

Set $\epsilon^i = (-1)^{\frac{i\cdot(i+1)}{2}}$. For $i \equiv 0,1,2,3$ modulo 4, we have $\epsilon^i = 1,-1,-1,1$ respectively. Explicitly, our symmetric form φ is a morphism of complexes as follows:

$$\begin{tikzcd}
 & & \text{degree } n & & \\
P = \arrow[d, "\varphi"', "\varphi^{\#}\circ\varpi^i_P"] & \cdots \arrow[r] & P_{n+1} \arrow[r, "d"] \arrow[d, "\varphi_{n+1}"', "\epsilon^i(\varphi_{-n+i-1})^*"] & P_n \arrow[r, "d"] \arrow[d, "\varphi_n"', "\epsilon^i(\varphi_{-n+i})^*"] & P_{n-1} \arrow[r] \arrow[d, "\varphi_{n-1}"', "\epsilon^i(\varphi_{-n+i+1})^*"] & \cdots \\
T^i(P^{\#}) = & \cdots \arrow[r] & (P_{-n+i-1})^* \arrow[r, "(-1)^i d^*"'] & (P_{-n+i})^* \arrow[r, "(-1)^i d^*"'] & (P_{-n+i+1})^* \arrow[r] & \cdots
\end{tikzcd}$$

where we represent φ on the left of the vertical arrows and $\varphi^{\#}\circ\varpi^i$ on the right. We require the morphism φ to be a homotopy equivalence and the morphisms φ and $\varphi^{\#}\varpi^i$ to be homotopic to each other. Using $\frac{1}{2}$, we can take the mean between φ and $\varphi^{\#}\,\varpi^i$ and assume that φ is *strongly symmetric*, *i.e.*

$$\varphi_n = \epsilon^i\cdot(\varphi_{-n+i})^* \qquad \text{for all } n\in\mathbb{Z}$$

or equivalently that φ is already symmetric in $\mathrm{Ch}^{\mathrm{b}}(\mathcal{A})$ and not only in $\mathrm{K}^{\mathrm{b}}(\mathcal{A})$. In our triangulated category $\mathrm{K}^{\mathrm{b}}(\mathcal{A})$, this new form $\frac{1}{2}(\varphi+\varphi^{\#}\varpi^i)$ is indeed *equal* to φ. Both are the *same* morphism from P to $T^i(P^{\#})$ in $\mathrm{K}^{\mathrm{b}}(\mathcal{A})$!

Let us give a special example for $i=1$ for instance. In that case $\epsilon^i=-1$. Suppose moreover that our complex is bounded in degrees 1 and 0. Then a form which has been replaced by the mean with its dual looks as follows:

$$\begin{tikzcd}
 & & & & \text{degree } 0 & & & \\
P = \arrow[d, "\varphi"] & \cdots \arrow[r] & 0 \arrow[r] \arrow[d] & P_1 \arrow[r, "d"] \arrow[d, "\alpha"'] & P_0 \arrow[r] \arrow[d, "-\alpha^*"] & 0 \arrow[r] \arrow[d] & 0 \arrow[r] \arrow[d] & \cdots \\
P^{\#} = & \cdots \arrow[r] & 0 \arrow[r] & (P_0)^* \arrow[r, "-d^*"'] & (P_1)^* \arrow[r] & 0 \arrow[r] & 0 \arrow[r] & \cdots
\end{tikzcd}$$

where $\alpha := \varphi_1 = -\varphi_0^*$. Requiring that the form φ is non-degenerate, *i.e.* requiring that φ is a homotopy equivalence, exactly means that its cone

$$\cdots \longrightarrow 0 \longrightarrow P_1 \xrightarrow{\begin{pmatrix}-d\\-\alpha\end{pmatrix}} P_0\oplus P_0^* \xrightarrow{(\alpha^* \quad -d^*)} P_1^* \longrightarrow 0 \longrightarrow \cdots$$

is a (split) exact complex. The reader can do the exercise and will see that P_1 turns out to be a second Lagrangian in the skew-symmetric hyperbolic form over $P_0\oplus P_0^*$, which of course already has P_0 as a Lagrangian. This produces then a space with two Lagrangians, sometimes called a *formation*. It was indicated in [**5**, Lem 3.2] that any class in $\mathrm{W}^1(\mathcal{A})$ is the class of a form on such a short complex of length 1. This is known to L-theorists. More generally, the connection between triangular Witt groups and L-groups, when both are defined, that is, over additive $\mathbb{Z}[\frac{1}{2}]$-categories, is going to appear in Walter [**27**].

The explicit construction of the symmetric cone can also be unfolded in this example of $\mathrm{K}^{\mathrm{b}}(\mathcal{A})$. This is described with some details in [**4**, 2.10].

5.14. Example. Here is a well-known example of a complex which has to do with symmetric forms. Let $(R, \mathfrak{m})$ be a regular local ring of dimension d and let $\mathfrak{m} =< x_1, \ldots, x_d >$ be a regular sequence of parameters.

Assume that $d \equiv 1$ modulo 4 for instance and write $d = 2e + 1$ with e even. Consider the usual Koszul complex but written in a zig-zag, *i.e.* half of it on the first line and half of it on the second line, with the differential "in the middle" written vertically in degree 0:

$$\begin{tikzcd}
 & & & & \text{degree } 0 \arrow[d, dotted, no head] & & & & & \\
\cdots 0 \arrow[r] & R^{\binom{d}{d}} \arrow[r] \arrow[d] & \cdots \arrow[r] & R^{\binom{d}{e+2}} \arrow[r] \arrow[d] & R^{\binom{d}{e+1}} \arrow[r] \arrow[d] & 0 \arrow[r] \arrow[d] & \cdots \arrow[r] & 0 \arrow[r] \arrow[d] & 0 \arrow[r] \arrow[d] & 0 \cdots \\
\cdots 0 \arrow[r] & 0 \arrow[r] & \cdots \arrow[r] & 0 \arrow[r] & R^{\binom{d}{e}} \arrow[r] & R^{\binom{d}{e-1}} \arrow[r] & \cdots \arrow[r] & R^{d} \arrow[r] & R \arrow[r] & 0 \cdots
\end{tikzcd}$$

Now think of the first line, completed with zeros as above, as a complex $E \in \mathrm{K}^{\mathrm{b}}(\mathcal{P}(R))$, and think of the vertical maps as a morphism $\varphi : E \to E^{\#}$. This morphism is not a homotopy equivalence. Why? Just compute the cone! This is nothing else than the Koszul complex itself, which is a resolution of the residue field $\kappa := R/\mathfrak{m}$, and in particular has non-zero homology.

Nevertheless, there is an open subscheme $U := \mathrm{Spec}(R) \setminus \{\mathfrak{m}\}$, the so-called *punctured spectrum* of R, on which the module κ is zero, *i.e.* on which φ becomes a quasi-isomorphism. We haven't talked about quasi-isomorphisms so far, but there is a nice triangulated category $\mathrm{D}^{\mathrm{b}}(\mathrm{VB}_U)$ of bounded complexes of vector bundles over U, with quasi-isomorphisms inverted (see details in the next Remark). No need for a new story: such categories are triangulated categories with duality and the above definitions make sense. This means that by restricting the above (E, φ) to the open U, we find a symmetric space in the triangulated category $\mathrm{D}^{\mathrm{b}}(\mathrm{VB}_U)$. It is a general result of TWG, which we recall below, that this will then define an element in the usual Witt group $\mathrm{W}(U)$. The localization theorem will show that this class is not in the image of $\mathrm{W}(R) \longrightarrow \mathrm{W}(U)$, which was a rather unpredicted fact: even if an isomorphism in dimensions 2, 3 and 4, the restriction $\mathrm{W}(R) \longrightarrow \mathrm{W}(U)$ is only injective with non-zero cokernel in dimensions $1, 5, 9, \ldots$ in which cases one can show that the cokernel of $\mathrm{W}(R) \longrightarrow \mathrm{W}(U)$ is isomorphic to $\mathrm{W}(\kappa)$. See [**8**, Thm 9.1] or [**6**].

Let us finally say that the assumption that $d \equiv 1$ modulo 4 is indeed hidden in the fact that we really need φ to have the right symmetry. This is not visible above because we haven't made the Koszul complex explicit, or in other words, we haven't explicitly explained how the second line was the dual of the first. This requires some care and is the reason for requiring e to be even in $d = 2e+1$. By the way, the case $d \equiv 3$ modulo 4 also provides us with some class in $\mathrm{W}^2(U) = \mathrm{W}_-(U)$, which is non-zero and hence does not extend to $\mathrm{W}_-(R)$ since the latter is zero (the minus sign indicates Witt groups of skew-symmetric forms).

5.15. Derived categories. In the last example appeared another type of triangulated category, slightly more involved than $\mathrm{K}^{\mathrm{b}}(\mathcal{A})$, namely $\mathrm{D}^{\mathrm{b}}(\mathcal{A})$, the *derived category* of $\mathcal{A}$. This category is in fact needed when $\mathcal{A}$ is homologically more subtle than an additive category, that is, when there are in $\mathcal{A}$ some kind of "exact sequences" which do not split. The simplest example is $\mathcal{A}$ an abelian category, like categories of modules over a ring. More generally $\mathcal{A}$ could be an exact category in the sense of Quillen [**22**], like the category VB_X of vector bundles over a scheme X. In these cases, something painful happens with $\mathrm{K^b}(\mathcal{A})$, namely to the natural functor $\mathcal{A} \to \mathrm{K^b}(\mathcal{A})$ which sends everything to complexes concentrated in degree zero. In fact, this functor does not necessarily send an exact sequence in $\mathcal{A}$ to a distinguished triangle in $\mathrm{K^b}(\mathcal{A})$. Pretty bad for our motto: *distinguished triangles replace exact sequences*, no? This functor would send exact sequences to distinguished triangles if *quasi-isomorphisms* were isomorphisms in $\mathrm{K^b}(\mathcal{A})$. (A quasi-isomorphism is a morphism of complexes which is an isomorphism in homology.) This is the conceptual reason for *inverting* quasi-isomorphisms and creating out of $\mathrm{K^b}(\mathcal{A})$ a new triangulated category $\mathrm{D^b}(\mathcal{A}) := \mathrm{K^b}(\mathcal{A})[\text{quasi-isos}^{-1}]$. Apart from this conceptual reason, there is a practical reason for considering $\mathrm{D^b}(\mathcal{A})$ and it is the fact that these derived categories are the right framework for derived functors, but this is another story.

When $\mathcal{A}$ has a duality $(-)^* : \mathcal{A}^{\mathrm{op}} \longrightarrow \mathcal{A}$ which is moreover *exact* in the sense that it preserves the (now possibly more subtle) exact sequences of $\mathcal{A}$, then the functor $\# : \mathrm{K^b}(\mathcal{A})^{\mathrm{op}} \longrightarrow \mathrm{K^b}(\mathcal{A})$ preserves quasi-isomorphisms. Hence it localizes into a unique functor still written $\# : \mathrm{D^b}(\mathcal{A})^{\mathrm{op}} \longrightarrow \mathrm{D^b}(\mathcal{A})$, which is easily seen to be again a duality. Since we have a general theory of Witt groups for triangulated categories with duality, it will apply to this one too. This is the source of *derived Witt groups* of schemes for instance, and more generally of derived Witt groups of exact categories with duality, as defined in [**4**, § 2].

6. Guide through TWG

Here is a list of general facts with references to [**3**] and [**4**] where a detailed treatment is already available. These are central facts but can be considered as a "black box" by a customer only interested in applications.

6.1. The $\frac{1}{2}$-assumption. We have to assume below that the additive categories under consideration "contain $\frac{1}{2}$" which means they are $\mathbb{Z}[\frac{1}{2}]$-categories, *i.e.* that we can divide any morphism by 2 in a unique way. For categories obtained from a ring R for instance, this means that $1+1$ is unit in R.

6.2. Periodicity. First of all, for any triangulated category with duality $(\mathcal{K}, \#, \varpi)$ the triangular Witt groups are 4-periodic:

$$\mathrm{W}^i \cong \mathrm{W}^{i+4} \qquad \text{for all } i \in \mathbb{Z}$$

with the isomorphism being given by the double shift $T^2 : \mathcal{K} \longrightarrow \mathcal{K}$. See if necessary [**3**, Prop 2.14]. One can if fact prove that $\mathrm{W}^{i+2}(\mathcal{K})$ is isomorphic to the i-th Witt group but of skew-symmetric forms: $\mathrm{W}^i(\mathcal{K}, -\varpi)$. In short, W^i and W^{i+2} classify forms for the same duality but for opposite symmetries. There is more information about shifted dualities and signs in [**4**, 5.1].

6.3. Agreement with usual Witt groups. This is where we connect triangular results to classical results. The most general "classical" framework is the one of an exact category with duality. Such a category has a derived category with duality as mentioned in Remark 5.15 and if it contains $\frac{1}{2}$ (see 6.1), then the 0-th Witt group of its derived category (triangular world) is naturally isomorphic to the usual Witt group (classical world). Of course this means that triangular Witt groups generalize even more classical Witt groups, like Witt groups of schemes, Witt groups of rings with involution, or simply Witt groups of fields.

6.4. The example of local rings and fields. If R is a *local* commutative ring containing $\frac{1}{2}$ then $\mathrm{W}^i(R) = 0$ unless $i \equiv 0$ modulo 4, see [**4**, Thm 5.6]. This holds in particular for R a field of characteristic different from 2. This result can be extended to semi-local commutative rings, as will appear in [**7**].

6.5. Weak Witt cancellation. It is not true "classically" that a stably metabolic space is metabolic. It is true for triangulated categories. More precisely, if (E, φ) is a space such that its class $[E, \varphi] \in \mathrm{W}(\mathcal{K})$ is zero then (E, φ) is neutral. This is [**3**, Thm 3.5]. In particular, if two spaces (E_1, φ_1) and (E_2, φ_2) are stably isometric then they are not necessarily isometric but $(E_1, \varphi_1) \perp (E_2, -\varphi_2)$ is neutral.

♡ ♡ ♡

6.6. The 12-term periodic long exact sequence. There is a *localization theorem* in TWG, which is the heart of the whole story. It requires the notion of "localization of triangulated categories with duality" and is in fact the main goal of [**3**], see in particular [**3**, Thm 6.2 and 6.8]. This produces long exact sequences of localization analogous to what is known in K-theory for instance. The input for such a localization exact sequence is a *short exact sequence of triangulated categories with duality*

$$\mathcal{J} \rightarrowtail \mathcal{K} \twoheadrightarrow \mathcal{L}$$

by which we mean the following: $\mathcal{K}$ is a triangulated category with duality, $\mathcal{J}$ is a triangulated thick subcategory which is stable under the duality and $\mathcal{L}$ is the quotient $\mathcal{K}/\mathcal{J}$, *i.e.* the localization $S^{-1}\mathcal{K}$ of $\mathcal{K}$ with respect to the class S of those morphisms whose cone belongs to $\mathcal{J}$. Requiring $\mathcal{J}$ to be thick in $\mathcal{K}$ simply means that for $A, B \in \mathcal{K}$ if $A \oplus B \in \mathcal{J}$ then $A, B \in \mathcal{J}$. Associated to this short exact sequence we get a long exact sequence of Witt groups:

$$\cdots \longrightarrow \mathrm{W}^{i-1}(\mathcal{L}) \longrightarrow \mathrm{W}^i(\mathcal{J}) \longrightarrow \mathrm{W}^i(\mathcal{K}) \longrightarrow \mathrm{W}^i(\mathcal{L}) \longrightarrow \mathrm{W}^{i+1}(\mathcal{J}) \longrightarrow \cdots$$

which is 12-term periodic because of the 4-periodicity of W^i in i. The connecting homomorphisms $\mathrm{W}^i(\mathcal{L}) \longrightarrow \mathrm{W}^{i+1}(\mathcal{J})$ are defined explicitly and heavily rely on the above "symmetric cone construction", see [**3**, § 5]. Even before checking all the details in *loc. cit.*, the reader can get the slogan of the above result: *to a short exact sequence of triangulated categories is associated a long exact sequence of Witt groups.*

♡ ♡ ♡

6.7. The spectral sequence. Once we have a long exact sequence associated to any thick inclusion $\mathcal{J} \subset \mathcal{K}$ as above, we immediately and quite formally get a spectral sequence for any *filtration* of $\mathcal{K}$, for instance for a finite filtration like $0 = \mathcal{J}_m \subset \cdots \subset \mathcal{J}_1 \subset \mathcal{J}_0 = \mathcal{K}$, in which case the spectral sequence starts with the Witt groups of the quotients $\mathcal{J}_p/\mathcal{J}_{p+1}$ and converges to the Witt groups of $\mathcal{K}$. This is explained in [**8**, § 3].

6.8. Sublagrangians. In a triangulated category, a *sublagragian* of a space (E, φ) is a morphism $\alpha : L \to E$ such that $\alpha^{\#}\,\varphi\,\alpha = 0$. This means that the form φ "restricted to L" via α is zero. Note that we use here *any* morphism $L \to E$, unlike what happens in the classical frameworks of projective modules or of vector bundles where we would only consider morphisms $L \rightarrowtail E$ which are admissible monomorphisms. Then, mimicking the classical construction, one could try to "remove" the sublagrangian L from the form (E, φ), up to Witt-equivalence. Slightly more precisely, one tries to define a symmetric form over an object like $L^{\perp}/L$, in such a way that this new symmetric space defines the same class as (E, φ) in the Witt group. This is indeed a slightly more complicated question in triangulated categories, due to the non-uniqueness of the "fill-in" map, making the definition of the form on $L^{\perp}/L$ rather technical. For an introduction to these questions, see [**3**, § 4].

6.9. Witt groups are K_0-like. We end Part I by a non-mathematical comment. By their very definition, triangular Witt groups are "K_0-like" : they are defined by generators and relations, in a way which is reminiscent of the analogous definition of K_0. Unlike the higher K-groups K_i, the shifted Witt groups W^i have a purely algebraic definition which does not require some space whose i-th homotopy group would be W^i. On the other hand, there is a localization long exact sequence, which sometimes led us to call these groups *higher and lower* Witt groups. In the present state of the author's understanding, this might be misleading and we should stick to the terminology of *shifted* Witt groups.

Part II : Survey of applications

7. Derived Witt groups of schemes

7.1. General assumptions on our schemes. In this second part, we consider schemes X and assume once and for all that $\frac{1}{2} \in \Gamma(X, \mathcal{O}_X)$. We also assume without further mention that all the considered schemes are noetherian.

A reader who is not familiar with the language of schemes should have two examples in mind : X an algebraic variety over a ground field k of characteristic different from 2, or $X = \mathrm{Spec}(R)$ the spectrum of a commutative noetherian ring R containing $\frac{1}{2}$.

We shall abbreviate by *regular* scheme, a scheme which is noetherian, separated and has regular local rings. A noetherian (local) ring is regular if any finitely generated module has a finite resolution by finitely generated projective modules.

7.2. Various definitions of Witt groups via derived categories. Given a scheme X, we can consider at least two kinds of derived categories associated to it. One is obtained from coherent $\mathcal{O}_X$-modules, which are the finitely generated modules in the affine case. The other one is obtained from vector bundles over X, or locally free coherent $\mathcal{O}_X$-modules, which are the finitely generated projective modules in the affine case. These two different possibilities should remind us of what happens with G- and K-theory respectively.

Let us fix some notations. Let $\mathrm{D^b}(\mathrm{Coh}_X)$ be the derived category of coherent $\mathcal{O}_X$-modules and $\mathrm{D^b}(\mathrm{VB}_X)$ be the derived category of vector bundles over X. There is a natural functor $\mathrm{D^b}(\mathrm{VB}_X) \longrightarrow \mathrm{D^b}(\mathrm{Coh}_X)$ which is an equivalence when X is regular.

For defining triangular Witt groups, a triangulated category is not enough: we also need a duality. The duality on $\mathrm{D^b}(\mathrm{VB}_X)$ is obvious: it is just derived from the duality $\mathrm{Hom}_{\mathcal{O}_X}(-,\mathcal{O}_X)$ on VB_X, as we saw above in the case of projective modules. The Witt groups obtained this way are called the *derived Witt groups* of the scheme:

$$\mathrm{W}^i(X) := \mathrm{W}^i\big(\mathrm{D^b}(\mathrm{VB}_X)\big)$$

and were first introduced in [**2**]. The W^i with $i \equiv 0 \bmod 4$ coincide with Knebusch's classical Witt group of schemes [**18**] by 6.3. The W^i with $i \equiv 2 \bmod 4$ coincide with classical Witt groups of skew-symmetric forms. The $\mathrm{W}^1(X)$ and $\mathrm{W}^3(X)$ are new. Note however that Charles Walter has a formation-style description of those groups by generators and relations, see [**27**].

On the other side, the side with "G-theoretic" flavour, the triangulated category $\mathrm{D^b}(\mathrm{Coh}_X)$ does not always carry a duality. There is a duality when the $\mathcal{O}_X$-module $\mathcal{O}_X$ has a finite resolution by injective $\mathcal{O}_X$-modules. This is the case when X is Gorenstein of finite Krull dimension. See more on this in Gille [**12**]. In fact, since these injective modules are not coherent in general, the right framework is the category $\mathrm{D^b_{Coh_X}}(\mathrm{Qcoh}_X)$ of bounded complexes of quasi-coherent $\mathcal{O}_X$-modules having coherent homology. Since our schemes X are assumed to be noetherian, the natural functor $\mathrm{D^b}(\mathrm{Coh}_X) \longrightarrow \mathrm{D^b_{Coh_X}}(\mathrm{Qcoh}_X)$ is an equivalence of categories and we will stick to $\mathrm{D^b}(\mathrm{Coh}_X)$ to avoid heavy notations. The Witt groups obtained in this second way are called the *coherent (derived) Witt groups* of the scheme:

$$\widetilde{\mathrm{W}}^{\,i}(X) := \mathrm{W}^i\big(\mathrm{D^b}(\mathrm{Coh}_X)\big).$$

Note that $\mathrm{Hom}_{\mathcal{O}_X}(-,\mathcal{O}_X)$ does *not* define a duality on coherent or quasi-coherent module, since the double dual is not the identity (think of R-modules already with $R = \mathbb{Z}[\frac{1}{2}]$). This new duality on the derived category is an important idea of Alexander Grothendieck.

7.3. Functoriality. Of course, given a morphism of schemes $f: Y \to X$ we have an exact functor $f^*: \mathrm{D^b}(\mathrm{VB}_X) \longrightarrow \mathrm{D^b}(\mathrm{VB}_Y)$ which is induced from $\mathrm{VB}_X \longrightarrow \mathrm{VB}_Y$, $E \mapsto f^*(E) = \mathcal{O}_Y \otimes_{\mathcal{O}_X} E$. These functors are compatible with the dualities and hence induce Witt group homomorphisms: $\mathrm{W}^i(X) \longrightarrow \mathrm{W}^i(Y)$. In particular, if $U \hookrightarrow X$ is an open subscheme, we have a restriction $\mathrm{W}^i(X) \longrightarrow \mathrm{W}^i(U)$.

The functoriality of $\widetilde{\mathrm{W}}^{\,i}(X)$ is more tricky and works essentially only for flat morphisms. The case of an inclusion $U \hookrightarrow X$ still qualifies though.

7.4. Coherent Witt groups, regular schemes, or perfect complexes?
As in G- and K-theory, the coherent Witt groups behave well with localization and homotopy invariance. To be sure that they have something to do with the classical group of Knebusch, we need derived Witt groups of vector bundles. The latter are also better behaved with respect to functoriality. When both coincide, everything is nice and therefore, in the sequel, we assume most of the time that X is regular. We will try to indicate when we can drop the regularity assumption at the price of working with coherent Witt groups.

In K-theory, Thomason used Grothendieck's perfect complexes to have a construction which is close enough to K-theory but behaves better with respect to localization, like G-theory does. There is a duality on perfect complexes. The problem is that we need idempotent completion to express localization and Witt groups do not behave well with respect to idempotent completion (another instance of their K_0-attitude).

8. A cohomology theory

8.1. Comment on localization. Why do we introduce and study triangular Witt groups if all of them have a "classical" analogue? First, we have the example of coherent Witt groups, which are really triangular and not classical, as explained above. But in fact, the real advantage is that the triangular framework is more flexible than the classical framework. The localization theorem is the best example of this flexibility: if $U \subset X$ is an open subscheme of our scheme X, there is no good description of the exact category of vector bundles over U as a "quotient" or a "localization" of the exact category of vector bundles over X, even if X is regular. The biggest problem is the absence of an interesting "kernel" category of vector bundles over X which would vanish over U, since this kernel is often zero, at least when X is connected. This rigidity of exact categories explains why the Waldhausen categories of complexes or even the associated derived categories $\mathrm{D}^{\mathrm{b}}(\mathrm{VB}_X)$ and $\mathrm{D}^{\mathrm{b}}(\mathrm{VB}_U)$ are better suited for localization purposes. To avoid speaking of perfect complexes, we now assume for simplicity that X is regular. Then the functor

$$\mathrm{D}^{\mathrm{b}}(\mathrm{VB}_X) \longrightarrow \mathrm{D}^{\mathrm{b}}(\mathrm{VB}_U)$$

is a "quotient" of triangulated categories. That is, there is a "kernel" (defined below)

$$\mathrm{D}^{\mathrm{b}}_Z(\mathrm{VB}_X) \subset \mathrm{D}^{\mathrm{b}}(\mathrm{VB}_X),$$

such that the composition $\mathrm{D}^{\mathrm{b}}_Z(\mathrm{VB}_X) \longrightarrow \mathrm{D}^{\mathrm{b}}(\mathrm{VB}_X) \longrightarrow \mathrm{D}^{\mathrm{b}}(\mathrm{VB}_U)$ is zero and such that the induced functor

$$\mathrm{D}^{\mathrm{b}}(\mathrm{VB}_X) \ / \ \mathrm{D}^{\mathrm{b}}_Z(\mathrm{VB}_X) \longrightarrow \mathrm{D}^{\mathrm{b}}(\mathrm{VB}_U)$$

is an equivalence of categories.

Here $Z := X - U$ is the closed complement of U, and $\mathrm{D}^{\mathrm{b}}_Z(\mathrm{VB}_X)$ is defined precisely as a kernel, *i.e.* it is the full subcategory of $\mathrm{D}^{\mathrm{b}}(\mathrm{VB}_X)$ on those complexes which become isomorphic to zero in $\mathrm{D}^{\mathrm{b}}(\mathrm{VB}_U)$. In other words, $\mathrm{D}^{\mathrm{b}}_Z(\mathrm{VB}_X)$ is the full subcategory of $\mathrm{D}^{\mathrm{b}}(\mathrm{VB}_X)$ on those complexes whose homology is concentrated on the closed subset Z. The above quotient $\mathrm{D}^{\mathrm{b}}(\mathrm{VB}_X) \,/\, \mathrm{D}^{\mathrm{b}}_Z(\mathrm{VB}_X)$ is a special case of the localization of triangulated categories with duality mentioned in 6.6. The duality actually restricts to $\mathrm{D}^{\mathrm{b}}_Z(\mathrm{VB}_X)$ because, for a complex $P \in \mathrm{D}^{\mathrm{b}}(\mathrm{VB}_X)$, the condition $P \in \mathrm{D}^{\mathrm{b}}_Z(\mathrm{VB}_X)$ amounts to check $(P)_x \simeq 0$ in $\mathrm{D}^{\mathrm{b}}(\mathrm{VB}_{\mathcal{O}_x})$ for all $x \in X - Z$ and the latter condition is stable by passage to the dual since $(P^{\#})_x \simeq ((P)_x)^{\#}$.

8.2. Witt groups with support. These groups were introduced in [**5**, § 1] and are defined as follows. Let X be a scheme and $Z \subset X$ be a closed subset. (Recall the assumptions of 7.1.) Consider the triangulated category $\mathrm{D}^{\mathrm{b}}_Z(\mathrm{VB}_X)$ with the duality restricted from $\mathrm{D}^{\mathrm{b}}(\mathrm{VB}_X)$. We define the i-th *derived Witt group of X with supports in Z* by

$$\mathrm{W}^i_Z(X) := \mathrm{W}^i\big(\mathrm{D}^{\mathrm{b}}_Z(\mathrm{VB}_X)\big).$$

Similarly, the coherent Witt groups with supports are defined for X Gorenstein of finite Krull dimension [**12**] as

$$\widetilde{\mathrm{W}}{}^i_Z(X) := \mathrm{W}^i\big(\mathrm{D}^{\mathrm{b}}_Z(\mathrm{Coh}_X)\big)$$

with a similar definition of $\mathrm{D}^{\mathrm{b}}_Z(\mathrm{Coh}_X) := \mathrm{Ker}\big(\mathrm{D}^{\mathrm{b}}(\mathrm{Coh}_X) \longrightarrow \mathrm{D}^{\mathrm{b}}(\mathrm{Coh}_U)\big)$.

In these examples, we see how triangulated categories which are not derived categories of vector bundles can be used to define new Witt groups, which have no classical interpretation *a priori*. The following long exact sequence is then a direct consequence of the localization long exact sequence 6.6.

8.3. Theorem. *Let X be a regular scheme and let $U \subset X$ be an open subscheme with closed complement $Z = X - U$. Then there is a natural long exact sequence*

$$\cdots \longrightarrow \mathrm{W}^{i-1}(U) \longrightarrow \mathrm{W}^i_Z(X) \longrightarrow \mathrm{W}^i(X) \longrightarrow \mathrm{W}^i(U) \longrightarrow \mathrm{W}^{i+1}_Z(X) \longrightarrow \cdots$$

See [**5**, Thm 1.6]. This can be thought of as a result for coherent Witt groups as in [**12**, Thm 2.19], regularity being used here to replace $\widetilde{\mathrm{W}}{}^i$ by W^i.

8.4. Flat excision. Let $Z \subset X$ be a closed subset of a regular scheme X and let $f : Y \to X$ be a flat morphism from another regular scheme Y, such that f induces an isomorphism of schemes $f^{-1}(Z) \overset{\sim}{\to} Z$, where those closed subsets are endowed with the reduced scheme structure. An example of this situation is an inclusion $V \hookrightarrow X$ of an open subscheme containing Z, which means that X is covered in the Zariski topology by V and $U := X - Z$. Another example appears in the Nisnevich topology as explained in [**4**, Rem 2.6].

For such a flat morphism $f : Y \to X$ such that $f^{-1}(Z) \overset{\sim}{\to} Z$, the functor $f^* : \mathrm{D}^{\mathrm{b}}(\mathrm{VB}_X) \longrightarrow \mathrm{D}^{\mathrm{b}}(\mathrm{VB}_Y)$ induces an equivalence of categories:

$$f^* : \mathrm{D}^{\mathrm{b}}_Z(\mathrm{VB}_X) \longrightarrow \mathrm{D}^{\mathrm{b}}_{f^{-1}(Z)}(\mathrm{VB}_Y)\,.$$

So, *simply because they are defined on the triangular level*, the Witt groups with supports are isomorphic:

$$f^* : \mathrm{W}^i_Z(X) \overset{\sim}{\longrightarrow} \mathrm{W}^i_{f^{-1}(Z)}(Y) \qquad \text{for all } i \in \mathbb{Z}\,.$$

Let us stress this little miracle: simply because the theory "factors" via triangulated categories, it satisfies flat excision. Why? Because, so to speak, the derived categories themselves satisfy flat excision. This is explained in [**5**, § 2], where the following immediate corollary is also to be found.

8.5. Theorem. *Let $X = U \cup V$ be an open cover of a regular scheme. Then there is a Mayer-Vietoris long exact sequence:*

$$\cdots \mathrm{W}^{i-1}(U \cap V) \longrightarrow \mathrm{W}^i(X) \longrightarrow \mathrm{W}^i(U) \oplus \mathrm{W}^i(V) \longrightarrow \mathrm{W}^i(U \cap V) \longrightarrow \mathrm{W}^{i+1}(X) \cdots$$

8.6. Dévissage and transfer. The above result is an example of a useful theorem which can be obtained from the localization long exact sequence 8.3, but without explicitly computing the "relative Witt groups" $\mathrm{W}^i_Z(X)$. Nevertheless, it is interesting to ask if those groups with supports in Z have something to do with some Witt groups of Z. This question is not answered in complete generality yet, but there are answers in the affine case at least, which can be found in [**12**], whose main result is the following.

8.7. Theorem. (S. Gille) *Let R be a regular $\mathbb{Z}[\frac{1}{2}]$-algebra of finite Krull dimension and J an ideal generated by a regular sequence of length r. Assume that R/J is also regular. Then for all $i \in \mathbb{Z}$ there is an isomorphism:*

$$\mathrm{W}^i(R/J) \xrightarrow{\sim} \mathrm{W}^{i+r}_J(R)$$

where of course $\mathrm{W}^i_J(R)$ means $\mathrm{W}^i_Z(X)$ for $Z := \mathrm{Spec}(R/J) \hookrightarrow \mathrm{Spec}(R) =: X$.

8.8. Twisted dualities. Let X be a scheme and let $\mathcal{L}$ be a line bundle over X. Then there is a so-called *twisted* duality on vector bundles which is defined by $(-)^* \otimes \mathcal{L}$ (using $\mathcal{L}^* \otimes \mathcal{L} \simeq \mathcal{O}_X$). This induces a twisted duality on $\mathrm{D^b}(\mathrm{VB}_X)$ as well, and hence produces derived Witt groups, with supports and with possibly twisted dualities. So at this point, we can play with the following decorations:

$$\mathrm{W}^i_Z(X, \mathcal{L})$$

where $i \in \mathbb{Z}$ or $\mathbb{Z}/4$, where Z is a closed subset of X and where $\mathcal{L}$ is a line bundle over X. Since the main ideas are better understood in the special case $\mathcal{L} = \mathcal{O}_X$ and easily generalized if needed, we shall not use twisted dualities here, except in Walter's projective bundle Theorem 10.4.

8.9. Homotopy invariance. It is a result of Karoubi that $\mathrm{W}(R) \cong \mathrm{W}(R[T])$ for any ring R containing $\frac{1}{2}$, where $R[T]$ is the polynomial ring in one variable. It is possible to extend this result to shifted Witt groups over regular rings: $\mathrm{W}^i(R) \cong \mathrm{W}^i(R[T])$ for all $i \in \mathbb{Z}$. This is [**5**, Thm 3.1]. By Mayer-Vietoris, we immediately get [**5**, Thm 3.4], which says:

8.10. Theorem. *Let X be a regular scheme; then $\mathrm{W}^i(\mathbb{A}^1_X) = \mathrm{W}^i(X)$ for all $i \in \mathbb{Z}$.*

8.11. Commercials. Note that the above statement has a "classical" meaning for $i = 0, 2$: the usual Witt groups of symmetric and skew-symmetric forms are globally homotopy invariant. This was not known before triangular Witt groups. Similarly, in the Mayer-Vietoris Theorem 8.5, exactness at $\mathrm{W}^i(U) \oplus \mathrm{W}^i(V)$ for $i = 0, 2$, can be explained to someone who does not know TWG: it is a classical statement with triangular proof. We will see further examples of this below.

8.12. Generalization of homotopy invariance. Stefan Gille has generalized the above Theorem in [**13**] as follows: *Let X and Y be separated Gorenstein schemes of finite Krull dimension and let $f : Y \longrightarrow X$ be an affine and flat morphism. (For example, Y could be a vector bundle over X with X regular.) Then $f^* : \widetilde{\mathrm{W}}^{\,i}(X) \longrightarrow \widetilde{\mathrm{W}}^{\,i}(Y)$ is an isomorphism of coherent Witt groups. (In the regular example, this reads $\mathrm{W}^i(X) \simeq \mathrm{W}^i(Y)$.)* Using this, he can compute some Witt groups. Here is an example inspired by Joanoulou's work. This illustrates again how K-theoretic results can be extended to Witt groups via TWG.

8.13. Proposition. (S. Gille) *Let R be a regular local ring (a field e.g.) and let $n \geq 1$. Consider $\Sigma_R^{2n-1} := \mathrm{Spec}\Big(R\,[X_1, \ldots, X_n, T_1, \ldots, T_n] \;/\; (1 - \sum\limits_{j=1}^{n} X_j T_j)\Big)$, the hyperbolic sphere. Then the usual Witt group of Σ_R^{2n-1} is*

$$\mathrm{W}(\Sigma_R^{2n-1}) = \begin{cases} \mathrm{W}(R) & \text{if } n \not\equiv 1 \text{ modulo } 4 \\ \mathrm{W}(R) \oplus \mathrm{W}(R) & \text{if } n \equiv 1 \text{ modulo } 4. \end{cases}$$

See [**13**, Thm 5.1] for a more general statement involving coherent Witt groups.

8.14. More on transfer. In Theorem 8.7 we saw the existence of a homomorphism $\mathrm{W}^i(R/J) \overset{\sim}{\longrightarrow} \mathrm{W}_J^{i+r}(R)$. This was generalized in [**14**] to morphisms $R \longrightarrow S$ such that S is finitely generated as an R-module (instead of $R \longrightarrow R/J$).

8.15. The spectral sequence. Consider a regular scheme X of finite Krull dimension d. There is a filtration of the derived category $\mathrm{D^b}(\mathrm{VB}_X)$ by the codimension of the support of the homology:

$$0 = \mathcal{J}_{d+1} \subset \mathcal{J}_d \subset \cdots \subset \mathcal{J}_1 \subset \mathcal{J}_0 = \mathrm{D^b}(\mathrm{VB}_X).$$

Explicitly, for any $p \geq 0$, let us denote by $X^{(p)}$ the points of X of codimension p; in the affine case, these are the prime ideals of height p. We define $\mathcal{J}_p$ to be the union of the subcategories $\mathrm{D}^{\mathrm{b}}_Z(\mathrm{VB}_X)$ where the points of Z belong to $X^{(q)}$ for $q \geq p$. In words, $\mathcal{J}_p$ is the subcategory of those complexes whose total homology has support of codimension greater or equal to p. As mentioned in 6.7, such a filtration produces a spectral sequence. This is developed in [**8**] and was generalized to coherent Witt groups in [**12**, § 3], where a version "with support" is also given. I will not repeat this here since it has been repeated several times under several forms in [**5**], [**6**] and [**9**]. Let us simply say the following. There is a local-to-global spectral sequence converging to the derived Witt groups of X and whose first page is very special. The q-th line of the $E_1^{p,q}$-page of this spectral sequence is zero for all $q \in \mathbb{Z}$ except for $q \equiv 0$ modulo 4. For such a $q \equiv 0$, the q-th line is isomorphic to a complex of the following form:

$$\bigoplus_{x \in X^{(0)}} \mathrm{W}(\kappa(x)) \to \bigoplus_{x \in X^{(1)}} \mathrm{W}(\kappa(x)) \to \cdots \to \bigoplus_{x \in X^{(d-1)}} \mathrm{W}(\kappa(x)) \to \bigoplus_{x \in X^{(d)}} \mathrm{W}(\kappa(x))$$

in the range $0 \leq p \leq d$ and is zero elsewhere. Above, $\kappa(x)$ is the residue field at $x \in X$. Such a complex is called a *Gersten-Witt complex* for the scheme X. The *augmented Gersten-Witt complex* is the above complex augmented at the beginning by the natural localization homomorphism

$$\mathrm{W}(X) \longrightarrow \bigoplus_{x \in X^{(0)}} \mathrm{W}(\kappa(x)) .$$

In [**9**, Def. 3.1] for instance, we recall an explicit "triangular" definition of this complex without local choices. The above expression in terms of residue fields is conceptually clearer, since Witt groups of fields do really classify quadratic forms. Nevertheless, the above presentation of the Gersten-Witt complex with residue fields is cumbersome when the differentials must be made explicit or when something needs to be proven about this complex (like it being exact). In such cases, the underlying "triangular" complex described in the above quoted references is easier to handle. An immediate application of this spectral sequence is the following result.

8.16. Theorem. *Let X be regular of Krull dimension at most 4. Then the natural homomorphism $\mathrm{W}(X) \to \mathrm{W}_{\mathrm{nr}}(X)$ is an epimorphism from the Witt group of X onto the unramified Witt group of X, which is defined to be the kernel of the first differential in the Gersten-Witt complex. If X is of dimension at most 3, it is even an isomorphism. In particular, if X is connected of dimension at most 3 with function field Q, we have a natural injection $\mathrm{W}(X) \hookrightarrow \mathrm{W}(Q)$.*
This is [**8**, Cor. 10.2 and 10.3].

9. The Gersten Conjecture

9.1. Statement. The *Gersten Conjecture for Witt groups* is due to William Pardon [**21**]. It claims the existence and more importantly the *exactness* of the Gersten-Witt complex (see 8.15) for $X = \mathrm{Spec}(R)$ where R is *regular* and *local*. The result is also conjectured to be true for R semi-local regular.

9.2. The low dimensional case. Directly from the spectral sequence and from the vanishing of $\mathrm{W}^i(R)$ for $i \not\equiv 0$ modulo 4, we obtained the Conjecture for local regular rings of dimension at most 4 in [**8**, Thm 10.4]. This is true in the semi-local case as well, because the same vanishing holds, see [**7**].

9.3. The essentially smooth case. The above Conjecture is analogous to the Gersten Conjecture in K-theory for instance. There are general strategies to attack such Gersten-type conjectures in the geometric case, that is, when R is essentially smooth, *i.e.* is a local ring of a smooth scheme over some ground field k. This was formalized in [**10**] into a proof which basically works for any cohomology theory which is homotopy invariant and excisive. This led to the first proof of the Conjecture in [**5**, Thm 4.3] for R semi-local essentially smooth over an infinite ground field (of characteristic different from 2, of course).

9.4. Panin's trick. With the above result, the Gersten-Witt Conjecture essentially reached the level the K-theoretic Gersten Conjecture had reached with Quillen long ago (plus the low-dimensional cases 9.2, which seem special to Witt groups).

Ivan Panin discovered more recently a proof of the K-theoretic Gersten Conjecture for R local merely containing a field. This is a non-trivial consequence of a result of Popescu, which asserts that any regular k-algebra over some field k is a limit of smooth k-algebras.

In [**9**], we show how to transpose this to the Gersten-Witt case. So, the status of the conjecture is now essentially the following:

9.5. Theorem. *Let R be a (semi-) local regular ring containing a field of characteristic different from* 2. *Then the Gersten-Witt Conjecture holds for R.*

Proof. To be honest, in [**9**], this result is only proven for R local containing a field. The essentially smooth case [**5**] is for R semi-local. It remains to check that the proof of [**9**] goes through for semi-local rings as well. A useful fact was not clear at the time of [**9**] and prevented us from writing the result in the semi-local case: we did not know whether the shifted groups $\mathrm{W}^i(R)$ vanish for R semi-local when $i \not\equiv 0 \bmod 4$. This is now settled and will appear in [**7**]. (Checking that the rest of the proof of [**9**] goes through is an interesting diploma thesis subject in 2003.) □

10. Miscellaneous

10.1. Two multiplicative structures on triangular Witt groups. It seems believable that as soon as the triangulated categories with duality under consideration also carry a reasonable tensor product, then we obtain a graded theory:

$$\mathrm{W}^i \times \mathrm{W}^j \longrightarrow \mathrm{W}^{i+j}.$$

This is indeed the case but is not as simple at it might seem at first sight. The reader will find a detailed exposition of this in [**16**]. This is used to prove the following result and this multiplicative structure is likely to be extremely useful in the future. Note that this multiplicative structure extends the usual one on $\mathrm{W}^0(X) = \mathrm{W}(X)$, obtained by tensor product of vector bundles.

10.2. Theorem. *Let X be a regular scheme (see 7.1) of finite Krull dimension d. Assume that X is connected and that Q is its function field. Then the kernel*

$$J := \mathrm{Ker}\big(\mathrm{W}(X) \longrightarrow \mathrm{W}(Q)\big)$$

is a nilpotent ideal in the ring $\mathrm{W}(X)$. *One can even choose the exponent N such that $J^N = 0$ to be $N = [\frac{d}{4}] + 1$ (at least when X is defined over a field).*
See [**6**] for a precise statement. It is also proven there that the bound $[\frac{d}{4}] + 1$ is the best possible one by computing the following example.

10.3. Example. Let $d \geq 1$. Consider the following scheme over $\mathbb{R}$:

$$X := \underbrace{\mathbb{P}^1_{\mathbb{R}} \times_{\mathbb{R}} \cdots \times_{\mathbb{R}} \mathbb{P}^1_{\mathbb{R}}}_{d \text{ times}}.$$

Then the total $\mathbb{Z}/4$-graded Witt group $\mathrm{W}(X) \oplus \mathrm{W}^1(X) \oplus \mathrm{W}^2(X) \oplus \mathrm{W}^3(X)$ is isomorphic to $\mathbb{Z}[\varepsilon_1, \ldots, \varepsilon_d]/\varepsilon_i^2$ with ε_i in degree $1 \in \mathbb{Z}/4$ for all $i = 1, \ldots, d$. In particular, for any $k \le [\frac{d}{4}]$, the element $\varepsilon_1 \cdot \varepsilon_2 \ldots \cdot \varepsilon_{4k}$ is not zero and belongs to $\mathrm{W}^{4k}(X) = \mathrm{W}^0(X) = \mathrm{W}(X)$, and even more it belongs to the kernel of $\mathrm{W}(X) \to \mathrm{W}(Q)$. See [**6**, § 4].

10.4. A projective bundle theorem. Charles Walter obtained a very general projective bundle Theorem describing the Witt groups of $\mathbb{P}(E)$ in terms of the Witt groups of X, where E is a vector bundle over X. This is done for all shifted Witt groups and all possible dualities on $\mathbb{P}(E)$. Unfortunately, the result is not always a closed formula and involves a long exact sequence, non-split in general and originating in the (triangular) localization long exact sequence of 6.6 above. We give here a "split case" for simplicity.

10.5. Theorem. (C. Walter) *Let X be a scheme and consider $\mathbb{P}^r_X$ the r-th projective space over X. Let m be an integer (whose parity is our only concern). Consider $\mathcal{O}(m)$ the m-th multiple of the tautological line bundle over $\mathbb{P}^r$. We can consider the corresponding Witt groups with twisted dualities with respect to this line bundle, see 8.8. We have for any $i \in \mathbb{Z}$:*

$$\mathrm{W}^i(\mathbb{P}^r_X, \mathcal{O}(m)) = \left\{ \begin{array}{cc} \mathrm{W}^i(X) & \textit{for } m \textit{ even} \\ 0 & \textit{for } m \textit{ odd} \end{array} \right\} \bigoplus \left\{ \begin{array}{cc} 0 & \textit{for } m+r \textit{ even} \\ \mathrm{W}^{i-r}(X) & \textit{for } m+r \textit{ odd.} \end{array} \right\}$$

The above formula summarizes 4 possibilities according to the 4 "parities" of (m, r) in $\mathbb{Z}/2 \times \mathbb{Z}/2$. For instance, for m even and r odd, the formula reads $\mathrm{W}^i(\mathbb{P}^r_X) = \mathrm{W}^i(X) \oplus \mathrm{W}^{i-r}(X)$. This result of [**28**] generalizes the pioneer result of Arason [**1**] in the field case, as well as intermediate results of Gille [**11**] and [**12**].

10.6. Representability. Once we have a cohomology theory which satisfies homotopy invariance and Nisnevich excision, everyone is very tempted to believe that this theory is representable in the $\mathbb{A}^1$-homotopy theory of Morel and Voevodsky. This is indeed the case for Witt groups and will appear in Hornbostel [**17**], where the details are to be found.

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