ON THE SURJECTIVITY OF THE MAP OF SPECTRA ASSOCIATED TO A TENSOR-TRIANGULATED FUNCTOR

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Abstract. We prove a few results about the map $\operatorname{Spc}(F)$ induced on tensor-triangulated spectra by a tensor-triangulated functor $F$. First, $F$ is conservative if and only if $\operatorname{Spc}(F)$ is surjective on closed points. Second, if $F$ detects tensor-nilpotence of morphisms then $\operatorname{Spc}(F)$ is surjective on the whole spectrum. In fact, surjectivity of $\operatorname{Spc}(F)$ is equivalent to $F$ detecting the nilpotence of some class of morphisms, namely those morphisms which are nilpotent on their cone.

1. Introduction

Hypotheses 1.1. Throughout the paper, $F: \mathcal{K} \to \mathcal{L}$ is a tensor-triangulated functor between essentially small tensor-triangulated categories $\mathcal{K}$ and $\mathcal{L}$. Assume that $\mathcal{K}$ is rigid, i.e. every object has a dual (Remark 2.1).

Consider the induced map on spectra
$$\varphi = \operatorname{Spc}(F): \operatorname{Spc}(\mathcal{L}) \to \operatorname{Spc}(\mathcal{K})$$
in the sense of tensor-triangulated geometry [Bal05, Bal10b, Ste16]. Our first result is a characterization of conservativity of $F$.

Theorem 1.2. Under Hypotheses 1.1, the following properties are equivalent:
(a) The functor $F: \mathcal{K} \to \mathcal{L}$ is conservative, i.e. it detects isomorphisms.
(b) The induced map $\varphi: \operatorname{Spc}(\mathcal{L}) \to \operatorname{Spc}(\mathcal{K})$ is surjective on closed points, i.e. for every closed point $\mathcal{P}$ in $\operatorname{Spc}(\mathcal{K})$, there exists $\mathcal{Q}$ in $\operatorname{Spc}(\mathcal{L})$ such that $\varphi(\mathcal{Q}) = \mathcal{P}$.

We can remove the assumption that $\mathcal{K}$ is rigid, at the cost of replacing (a) by:
(a') $F$ detects $\otimes$-nilpotence of objects, i.e. $F(x) = 0 \Rightarrow x^{\otimes n} = 0$ for some $n \geq 1$.

Our main results are dedicated to surjectivity of $\varphi$ on the whole of $\operatorname{Spc}(\mathcal{K})$.

Theorem 1.3. Under Hypotheses 1.1, suppose that the functor $F: \mathcal{K} \to \mathcal{L}$ detects $\otimes$-nilpotence of morphisms, i.e. every $f: x \to y$ in $\mathcal{K}$ such that $F(f) = 0$ satisfies $f^{\otimes n} = 0$ for some $n \geq 1$. Then the induced map $\varphi: \operatorname{Spc}(\mathcal{L}) \to \operatorname{Spc}(\mathcal{K})$ is surjective.

This result is clearly a corollary of (b)$\Rightarrow$(a) in the following more technical result:

Theorem 1.4. Under Hypotheses 1.1, the following properties are equivalent:
(a) The morphism $\varphi: \operatorname{Spc}(\mathcal{L}) \to \operatorname{Spc}(\mathcal{K})$ is surjective.
(b) The functor $F : \mathcal{K} \to \mathcal{L}$ detects $\otimes$-nilpotence of morphisms which are already $\otimes$-nilpotent on their cone, i.e. every $f : x \to y$ in $\mathcal{K}$ such that $F(f) = 0$ and such that $f^\otimes m \otimes \text{cone}(f) = 0$ for some $m \geq 1$ satisfies $f^\otimes n = 0$ for some $n \geq 1$.

At this point, the Devinatz-Hopkins-Smith [DHS88] Nilpotence Theorem might come to some readers’ mind. This celebrated result asserts that a morphism between finite objects in the topological stable homotopy category $\text{SH}$ must be $\otimes$-nilpotent if it vanishes on complex cobordism. Hopkins and Smith used the Nilpotence Theorem in the subsequent work [HS98] to prove the Chromatic Tower Theorem. A reformulation of the latter, in terms of $\text{Spc}(\text{SH}^c)$, can be found in [Bal10a, §9]. From the Nilpotence Theorem it follows that every prime of $\text{SH}^c$ is the kernel of some Morava $K$-theory. This implication is analogous to the surjectivity of Theorem 1.3 in the special case of $\text{SH}$.

Let us stress however that the scope of Theorems 1.2 and 1.3 is broader than the topological example. In fact, $\text{SH}$ plays among general tensor-triangulated categories the same role that $\mathbb{Z}$ plays among general commutative rings. Commutative algebra is not only the study of $\mathbb{Z}$, and tt-geometry is not only the study of $\text{SH}$. For the reader who never heard of tensor-triangulated categories and yet had the fortitude to read thus far, let us recall that tt-categories also appear in algebraic geometry (e.g. derived categories of schemes), in representation theory (e.g. derived and stable categories of finite groups), in noncommutative topology (e.g. $KK$-categories of $C^*$-algebras), in motivic theory (e.g. stable $\mathbb{A}^1$-homotopy and derived categories of motives), and in equivariant analogues (e.g. equivariant stable homotopy theory). A good introduction can be found in [HPS97, §1.2]. Tensor-triangular geometry is an umbrella theory for all those examples. In particular, computing $\text{Spc}(\mathcal{K})$ is the fundamental problem for every tt-category $\mathcal{K}$ out there; see [Bal05, Thm. 4.10].

After this motivational digression, let us return to the development of our results. It is interesting to know whether the converse of Theorem 1.3 holds true in glorious generality: Does surjectivity of $\text{Spc}(F)$ alone guarantee that $F$ detects $\otimes$-nilpotence of morphisms? By Theorem 1.4, this problem can be reduced as follows.

**Question 1.5.** Under Hypotheses 1.1, if $\varphi : \text{Spc}(\mathcal{L}) \to \text{Spc}(\mathcal{K})$ is surjective and if $f : x \to y$ satisfies $F(f) = 0$, is $f$ necessarily $\otimes$-nilpotent on its cone?

We do not know any counter-example. In fact, we can give a positive answer under the assumption that $F : \mathcal{K} \to \mathcal{L}$ admits a right adjoint. Since $\mathcal{K}$ and $\mathcal{L}$ are essentially small (typically the ‘compact’ objects of some big ambient category), existence of such a right adjoint is rather restrictive. In the context of [BDS16], it would be equivalent to having ‘Grothendieck-Neeman’ duality. To give an example, this right adjoint exists in the case of a finite separable extension, see [Bal16b]. The following are generalizations of some of the results in [Bal16a].

**Theorem 1.6.** Under Hypotheses 1.1, suppose that $F : \mathcal{K} \to \mathcal{L}$ admits a right adjoint $U : \mathcal{L} \to \mathcal{K}$. Then the map $\varphi : \text{Spc}(\mathcal{L}) \to \text{Spc}(\mathcal{K})$ is surjective if and only if the functor $F : \mathcal{K} \to \mathcal{L}$ detects $\otimes$-nilpotence of morphisms.

Again, this is a special case of a sharper, slightly more technical result.

**Theorem 1.7.** Under Hypotheses 1.1, suppose that $F : \mathcal{K} \to \mathcal{L}$ admits a right adjoint $U : \mathcal{L} \to \mathcal{K}$ and consider the image $U(1) \in \mathcal{K}$ of the $\otimes$-unit. Then the image of the map $\varphi : \text{Spc}(\mathcal{L}) \to \text{Spc}(\mathcal{K})$ is exactly the support of the object $U(1)$:

$$\text{im}(\text{Spc}(F)) = \text{supp}(U(1)).$$
An example of the latter, not covered by the separable extensions of [Bal16a], can be obtained by ‘modding out’ coefficients in motivic categories, see [VSF00, Chap. 5]. For instance, if \( \mathcal{K} = \text{DM}_{\text{gm}}(X; Z) \xrightarrow{F} \text{DM}_{\text{gm}}(X; Z/p) = \mathcal{L} \) then we have \( \text{im}(\text{Sp}(F)) = \text{supp}(\mathcal{Z}/p) \). From these techniques, one can easily reduce the computation of the (yet unknown) spectrum of the integral derived category of geometric motives \( \text{DM}_{\text{gm}}(X, Z) \) to the case of field coefficients:

\[
\text{Spc}(\text{DM}_{\text{gm}}(X; Z)) = \text{im}(\text{Sp}(\text{DM}_{\text{gm}}(X; \mathbb{Q}))) \sqcup \bigcup_p \text{im}(\text{Sp}(\text{DM}_{\text{gm}}(X; \mathcal{Z}/p))).
\]

These considerations will be pursued elsewhere.

In the presence of a ‘big’ ambient category, our condition of detecting \( \otimes \)-nilpotence could also be related to conservativity, as discussed in [MNN17, Thm. 4.19].

Let us now state a direct consequence of Theorem 1.3, that was apparently never noticed despite its importance and simplicity. It is the case where \( F \) is faithful.

**Corollary 1.8.** Suppose that \( \mathcal{K} \subset \mathcal{L} \) is a rigid tensor-triangulated subcategory. Then every prime \( \mathcal{P} \in \text{Spc}(\mathcal{K}) \) is the intersection of a prime \( \mathcal{Q} \in \text{Spc}(\mathcal{L}) \) with \( \mathcal{K} \).

A special sub-case of interest is that of ‘cellular’ subcategories, i.e. those \( \mathcal{K} \subseteq \mathcal{L} \) generated by a collection of ‘nice’ objects of \( \mathcal{L} \), typically \( \otimes \)-invertible ones (spheres). Such cellular subcategories \( \mathcal{K} \) are commonly studied when the ambient \( \mathcal{L} \) appears out-of-reach of known methods. For instance, Dell’Ambrogio [Del10] used this approach for equivariant \( KK \)-theory, and later with Tabuada [DT12] for non-commutative motives. Peter [Pet13] discusses the case of mixed Tate motives. Similarly, Heller-Ormsby [HO16] consider cellular subcategories in their recent study of tt-geometry in stable motivic homotopy theory. In all cases, Corollary 1.8 says that whatever can be detected via these cellular subcategories \( \mathcal{K} \) is actually relevant information about the bigger and more mysterious ambient category \( \mathcal{L} \). In particular, surjectivity of the comparison homomorphisms introduced in [Bal10a] can be tested on the cellular subcategory:

**Corollary 1.9.** Let \( u \in \mathcal{L} \) be a \( \otimes \)-invertible object and \( \mathcal{K} \) the full thick triangulated subcategory of \( \mathcal{L} \) generated by \( \{ u^{\otimes n} \mid n \in \mathbb{Z} \} \), which is supposed rigid(1). Note that the graded rings \( R_{\mathcal{K}, u}^* \) and \( R_{\mathcal{L}, u}^* \) associated to \( u \) are the same in \( \mathcal{K} \) and in \( \mathcal{L} \):

\[
R_{\mathcal{K}, u}^* \overset{\text{def}}{=} \text{Hom}_{\mathcal{K}}(1, u^{\otimes \bullet}) = \text{Hom}_{\mathcal{L}}(1, u^{\otimes \bullet}) \overset{\text{def}}{=} R_{\mathcal{L}, u}^*.
\]

If the comparison map \( \rho_{\mathcal{K}, u}^* \) for \( \mathcal{K} \) (recalled below) is surjective for the ‘cellular’ subcategory \( \mathcal{K} \) then the comparison map \( \rho_{\mathcal{L}, u}^* \) for the ambient \( \mathcal{L} \) is also surjective:

\[
\begin{array}{cccc}
\text{Spc}(\mathcal{L}) & \xrightarrow{\text{Cor. 1.8}} & \text{Spc}(\mathcal{K}) & \ni \\
\rho_{\mathcal{L}, u}^* & \circ \quad & \rho_{\mathcal{K}, u}^* & \ni \\
\text{Spec}^*(R_{\mathcal{L}, u}^*) & \xrightarrow{\rho_{\mathcal{L}, u}^*} & \text{Spec}^*(R_{\mathcal{K}, u}^*) & \ni \\
\rho_{\mathcal{K}, u}^*(\mathcal{P}) & \overset{\text{def}}{=} & \{ f \in R_{\mathcal{K}, u}^* \mid \text{cone}(f) \notin \mathcal{P} \}.
\end{array}
\]

For an introduction to these comparison maps and their importance, the reader is invited to consult the above references [Bal10a, Del10, DT12, HO16] or [San13].

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1 This is automatic if \( \mathcal{L} \) lives in a ‘big’ ambient category with internal hom, where rigid objects are closed under triangles. See [HPS97, Thm. A.2.5 (a)].
Acknowledgments: I am thankful to Beren Sanders for observing in a previous version of this article that my proof of surjectivity of \( \varphi \) reduced to \( F \) detecting nilpotence of morphisms of the form \( \eta_x \otimes y : y \to x^\vee \otimes x \otimes y \). Beren’s idea led me to the ‘morphisms which are nilpotent on their cone’ and to Theorem 1.4. I also thank Ivo Dell’Ambrogio, Martin Gallauer, Jeremiah Heller and Kyle Ormsby for their comments.

2. The proofs

The tensor \( \otimes : \mathcal{K} \times \mathcal{K} \longrightarrow \mathcal{K} \) is exact in each variable and \( 1 \) stands for the \( \otimes \)-unit in \( \mathcal{K} \). Recall that a tt-ideal \( \mathcal{I} \subseteq \mathcal{K} \) is a triangulated, thick, \( \otimes \)-ideal subcategory, i.e., it is non-empty, is closed under taking cones, direct summands and under tensoring by any object of \( \mathcal{K} \). For \( \mathcal{E} \subseteq \mathcal{K} \), we denote by \( \langle \mathcal{E} \rangle \subseteq \mathcal{K} \) the tt-ideal it generates.

A proper tt-ideal \( \mathcal{P} \subsetneq \mathcal{K} \) is prime if \( x \otimes y \in \mathcal{P} \) implies \( x \in \mathcal{P} \) or \( y \in \mathcal{P} \). The spectrum \( \text{Spc}(\mathcal{K}) = \{ \mathcal{P} \subseteq \mathcal{K} | \mathcal{P} \text{ is prime} \} \) has a topology whose basis of open is given by the subsets \( U(x) = \{ \mathcal{P} \in \text{Spc}(\mathcal{K}) | x \in \mathcal{P} \} \), for every \( x \in \mathcal{K} \). The closed complement \( \text{supp}(x) = \{ \mathcal{P} \in \text{Spc}(\mathcal{K}) | x \notin \mathcal{P} \} \) is called the support of the object \( x \). A tensor-triangulated functor \( F : \mathcal{K} \longrightarrow \mathcal{L} \) induces a continuous map \( \varphi = \text{Spc}(F) : \text{Spc}(\mathcal{L}) \longrightarrow \text{Spc}(\mathcal{K}) \) given explicitly by \( \varphi(\mathcal{Q}) = F^{-1}(\mathcal{Q}) \), for every prime \( \mathcal{Q} \subset \mathcal{L} \).

Remark 2.1. Our assumption that the tensor category \( \mathcal{K} \) is rigid, means that there exists an exact functor called the dual \( (-)^\vee : \mathcal{K}^{\text{op}} \longrightarrow \mathcal{K} \) that provides an adjoint to tensoring with any object \( x \in \mathcal{K} \) as follows:

\[
\begin{array}{ccc}
\mathcal{K} & \xrightarrow{x \otimes -} & x^\vee \otimes - \\
\text{K} & \downarrow & \downarrow \\
\text{K} & \end{array}
\]

Some authors call such objects \( x \) strongly dualizable, e.g., [HPS97]. The adjunction (2.2) comes with units (coevaluation) and counits (evaluation)

\[
\eta_x : 1 \to x^\vee \otimes x \quad \text{and} \quad \epsilon_x : x \otimes x^\vee \to 1
\]

which satisfy the relation

\[
(\epsilon_x \otimes x) \circ (x \otimes \eta_x) = 1_x.
\]

It follows from (2.4) that \( x \) is a direct summand of \( x \otimes x^\vee \otimes x \cong x \otimes x \cong x^\vee \).

It is a general fact that any tensor functor \( F : \mathcal{K} \to \mathcal{L} \) preserves rigidity, since we can use \( F(x^\vee) \) as \( F(x) \) with \( F(\eta_x) \) and \( F(\epsilon_x) \) as units and counits. See for instance [FHM03, Prop.3.1]. In particular, although we do not assume \( \mathcal{L} \) rigid, every object we use below will be rigid as long as it comes from \( \mathcal{K} \).

Remark 2.5. In a not-necessarily rigid tt-category, an object \( x \) with empty support, \( \text{supp}(x) = \emptyset \), is \( \otimes \)-nilpotent, i.e., \( x \otimes n = 0 \) for some \( n \geq 1 \). See [Bal05, Cor.2.4]. When \( x \) is rigid, \( x \otimes n = 0 \) forces \( x = 0 \) since \( x \) is a summand of \( x \otimes (x^\vee)^{\otimes (n-1)} \).

We begin with Theorem 1.2, which is relatively straightforward. We only need a few standard facts from basic tt-geometry, which do not use rigidity, namely:
(A) Given a $\otimes$-multiplicative class $S$ of objects in $\mathcal{K}$ (i.e. $1 \in S$ and $x, y \in S \Rightarrow x \otimes y \in S$) and a tt-ideal $\mathfrak{I} \subset \mathcal{K}$ such that $\mathfrak{I} \cap S = \emptyset$, then there exists a prime $\mathcal{P} \in \text{Spc}(\mathcal{K})$ such that $\mathfrak{I} \subseteq \mathcal{P}$ and $\mathcal{P} \cap S = \emptyset$. This fact uses that $\mathcal{K}$ is essentially small and is proven in [Bal05, Lemma 2.2].

(B) A point $\mathcal{P} \in \text{Spc}(\mathcal{K})$ is closed if and only if $\mathcal{P}$ is a minimal prime for inclusion in $\mathcal{K}$ (i.e. $\mathcal{P}' \subseteq \mathcal{P} \Rightarrow \mathcal{P}' = \mathcal{P}$). See [Bal05, Prop. 2.9].

(C) Any non-empty closed subset, for instance $\{\mathcal{P}\}$ for a point $\mathcal{P}$, or $\text{supp}(x)$ for a non-trivial object $x$, contains a closed point. See [Bal05, Cor. 2.12].

(D) For $F: \mathcal{K} \to \mathcal{L}$ and $\varphi = \text{Spc}(F): \text{Spc}(\mathcal{L}) \to \text{Spc}(\mathcal{K})$, and every object $x \in \mathcal{K}$, we have $\text{supp}(F(x)) = \varphi^{-1}(\text{supp}(x))$ in $\text{Spc}(\mathcal{L})$. See [Bal05, Prop. 3.6].

**Proof of Theorem 1.2.** Suppose that $F: \mathcal{K} \to \mathcal{L}$ is conservative and let $\mathcal{P} \in \text{Spc}(\mathcal{K})$ be a closed point, i.e. a minimal prime. Consider its complement $S = \mathcal{K} \setminus \mathcal{P}$. Since $\mathcal{P}$ is prime, $S$ is $\otimes$-multiplicative in $\mathcal{K}$ and does not contain zero. Since $F$ is a conservative tensor functor, the same holds for the class $F(S)$ in $\mathcal{L}$. (Recall that for a triangulated functor $F$, conservativity is equivalent to $F(x) = 0 \Rightarrow x = 0$, since a morphism is an isomorphism if and only if its cone is zero.) By the general fact (A) recalled above, for the $\otimes$-multiplicative class $F(S)$ and for the tt-ideal $\mathfrak{I} = 0$ in $\mathcal{L}$, there exists a prime $\mathcal{Q} \in \text{Spc}(\mathcal{L})$ such that $\mathcal{Q} \cap F(S) = \emptyset$. This relation implies that $F^{-1}(\mathcal{Q}) \subseteq \mathcal{P}$. By minimality of the closed point $\mathcal{P}$, see (B), this inclusion $F^{-1}(\mathcal{Q}) \subseteq \mathcal{P}$ forces $\mathcal{P} = F^{-1}(\mathcal{Q}) = \varphi(\mathcal{Q})$.

Conversely, suppose that $\varphi: \text{Spc}(\mathcal{L}) \to \text{Spc}(\mathcal{K})$ is surjective on closed points and let $x \in \mathcal{K}$ be such that $F(x) = 0$. We want to show that $x = 0$. Suppose ab absurdo that $x \neq 0$. Then we have $\text{supp}(x) \neq \emptyset$. By (C), we know that there exists a closed point $\mathcal{P} \in \text{supp}(x)$, which by assumption belongs to the image of $\varphi$, say $\mathcal{P} = \varphi(\mathcal{Q})$. But then $\mathcal{Q} \in \varphi^{-1}(\text{supp}(x)) = \text{supp}(F(x))$ by (D). This last statement contradicts $\text{supp}(F(x)) = \text{supp}(0) = \emptyset$. So $x = 0$ as claimed. \hfill \Box

**Remark 2.6.** The proof also gives a statement for $\mathcal{K}$ not rigid. In that case, the property $\text{supp}(x) = \emptyset$ does not necessarily imply that $x = 0$ but that $x$ is $\otimes$-nilpotent, as an object. See Remark 2.5. Surjectivity of $\varphi$ onto closed points is therefore equivalent to $F$ detecting $\otimes$-nilpotence of objects. See Theorem 1.2 (a').

**Remark 2.7.** In complete generality, if a closed point $\mathcal{P} \in \text{Spc}(\mathcal{K})$ belongs to the image of $\varphi: \text{Spc}(\mathcal{L}) \to \text{Spc}(\mathcal{K})$, say $\mathcal{P} = \varphi(\mathcal{Q})$, then $\mathcal{P}$ is also the image of a closed point $\mathcal{Q}'$, which can be chosen in the closure of $\mathcal{Q}$. Indeed, there exists a closed point $\mathcal{Q}' \in \{\mathcal{Q}\}$ by (C) and continuity of $\varphi$ implies $\varphi(\mathcal{Q}') \in \{\mathcal{P}\} = \{\mathcal{P}\}$.

\hfill * * *

We now turn to the slightly more tricky Theorem 1.4. Let us clarify the following:

**Definition 2.8.** A morphism $f: x \to y$ is called $\otimes$-nilpotent if $f^{\otimes n}: x^{\otimes n} \to y^{\otimes n}$ is zero for some $n \geq 1$. We say that $f: x \to y$ is $\otimes$-nilpotent on an object $z$ in $\mathcal{K}$ if there exists $n \geq 1$ such that $f^{\otimes n} \otimes z$ is the zero morphism $x^{\otimes n} \otimes z \to y^{\otimes n} \otimes z$. In particular, $f$ is $\otimes$-nilpotent on its cone if there exists $n \geq 1$ such that $f^{\otimes n} \otimes \text{cone}(f) = 0$.

The following useful fact was already observed in [Bal10a, Prop. 2.12]:

**Proposition 2.9.** Let $f: x \to y$ be a morphism in $\mathcal{K}$. Then

$$\{ z \in \mathcal{K} \mid f \text{ is } \otimes\text{-nilpotent on } z \}$$

forms a tt-ideal, even if $\mathcal{K}$ is not rigid.
Closure under direct summands and $\otimes$ is clear from the definition. The trick for closure under cones, is that if $f^{\otimes n} \otimes z_2 = 0$ for $i = 1, 2$ and if $z_1 \to z_2 \to z_3 \to \Sigma z_1$ is an exact triangle, then $f^{\otimes (n_1 + n_2)} \otimes z_3$ will vanish. This is the place where the same statement would fail with ‘$f$ vanishes on $z$’ (instead of ‘$f$ $\otimes$-nilpotent on $z$’).

**Proposition 2.10.** Let $\xi : w \to 1$ be a morphism in $\mathcal{K}$ (not necessarily rigid) such that $\xi \otimes \text{cone}(\xi) = 0$. Then the cone of $\xi^{\otimes n}$ generates the same tt-ideal, for all $n$:

$$\langle \text{cone}(\xi) \rangle = \{ z \in \mathcal{K} \mid \xi \otimes \text{nilpotent on } z \} = \langle \text{cone}(\xi^{\otimes n}) \rangle$$

**Proof.** The assumption $\xi \otimes \text{cone}(\xi) = 0$ implies that the object $\text{cone}(\xi)$ belongs to $\{ z \in \mathcal{K} \mid \xi \otimes \text{-nilpotent on } z \}$, which is a tt-ideal by Proposition 2.9. On the other hand, if the morphism $\xi^{\otimes n} \otimes z$ is zero then the exact triangle

$$w^{\otimes n} \otimes z \xrightarrow{\xi^{\otimes n} \otimes z = 0} z \to \text{cone}(\xi^{\otimes n}) \otimes z \to \Sigma w^{\otimes n} \otimes z$$

implies that $z$ is a summand of $\text{cone}(\xi^{\otimes n}) \otimes z$. Hence $z$ belongs to $\langle \text{cone}(\xi^{\otimes n}) \rangle$. Finally, in the Verdier quotient $\mathcal{K}/\langle \text{cone}(\xi) \rangle$, the morphism $\xi$ is an isomorphism, hence so is $\xi^{\otimes n}$. Therefore $\text{cone}(\xi^{\otimes n}) \in \langle \text{cone}(\xi) \rangle$. In short, we have obtained

$$\langle \text{cone}(\xi) \rangle \subseteq \{ z \in \mathcal{K} \mid \xi^{\otimes n} \otimes z = 0 \text{ for some } n \geq 1 \} \subseteq \bigcup_{n \geq 1} \langle \text{cone}(\xi^{\otimes n}) \rangle \subseteq \langle \text{cone}(\xi) \rangle$$

This proves the claim. Compare [Bal10, §2]. \qed

We can now establish the key observation of the paper:

**Corollary 2.11.** Let $x \in \mathcal{K}$ be a rigid object in a (not necessarily rigid) tt-category $\mathcal{K}$. Choose $\xi_x$ a ‘homotopy fiber’ of the coevaluation morphism $\eta_x$ of (2.3), i.e. choose an exact triangle in $\mathcal{K}$

(2.12)\[ w_x \xrightarrow{\xi_x} \Sigma \xrightarrow{\eta_x} x^{\vee} \otimes x \to \Sigma w_x \]

for a morphism $\xi_x$. Then the tt-ideal $\langle x \rangle$ generated by our object is exactly the subcategory on which $\xi_x$ is $\otimes$-nilpotent:

(2.13)\[ \langle x \rangle = \{ z \in \mathcal{K} \mid \xi_x^{\otimes n} \otimes z = 0 \text{ for some } n \geq 1 \} \]

Moreover, for every $n \geq 1$ the morphism $\xi_x^{\otimes n}$ is $\otimes$-nilpotent on its cone.

**Proof.** Consider the exact triangle obtained by tensoring (2.12) with $x$:

$$x \otimes w_x \xrightarrow{x \otimes \xi_x} x \xrightarrow{x \otimes \eta_x} x^{\vee} \otimes x \xrightarrow{x \otimes \xi_x} \Sigma x \otimes w_x$$

By the unit-counit relation (2.4), the morphism $x \otimes \eta_x$ is a monomorphism. This forces $x \otimes \xi_x = 0$. Hence $\xi_x \otimes \text{cone}(\xi_x) \cong \xi_x \otimes x^{\vee} \otimes x = 0$ and we can apply Proposition 2.10 to $\xi = \xi_x$. It gives us (2.13) since $\langle \text{cone}(\xi_x) \rangle = \langle x^{\vee} \otimes x \rangle = \langle x \rangle$ by rigidity of $x$. The ‘moreover part’ also follows from Proposition 2.10 where we proved that $\xi$ is $\otimes$-nilpotent on cone($\xi^{\otimes n}$). \qed

The above result allows us to translate questions about tt-ideals into a $\otimes$-nilpotence problem. We isolate a surjectivity argument that we shall use twice.

**Lemma 2.14.** Under Hypotheses 1.1, choose for every $x \in \mathcal{K}$ an exact triangle as in (2.12). Let $\mathcal{P} \in \text{Spc}(\mathcal{K})$ be a prime. Suppose that $\mathcal{P}$ satisfies the following technical condition:

(2.15)\[ \text{For all } x \in \mathcal{P}, \text{ all } s \in \mathcal{K} \setminus \mathcal{P} \text{ and all } n \geq 1, \text{ we have } F(\xi_x^{\otimes n} \otimes s) \neq 0. \]

Then $\mathcal{P}$ belongs to the image of $\varphi : \text{Spc}(\mathcal{L}) \to \text{Spc}(\mathcal{K})$. 


Proof. Consider the complement $S = \mathcal{K} \setminus \mathcal{P}$. Let $J \subseteq \mathcal{L}$ be the tt-ideal generated by $F(\mathcal{P})$, just viewed as a class of objects in $\mathcal{L}$. We claim that $J = \langle F(\mathcal{P}) \rangle$ equals $J' := \{ y \in \mathcal{L} \mid \text{there exists } x \in \mathcal{P} \text{ such that } y = \langle F(x) \rangle \}.$

Indeed, since we have $F(\mathcal{P}) \subseteq J' \subseteq J$ directly from the definitions, it suffices to show that $J'$ is a tt-ideal. It is clearly thick and a $\otimes$-ideal. For closure under cones, if $y_1 \rightarrow y_2 \rightarrow y_3 \rightarrow \Sigma y_1$ is exact in $\mathcal{L}$ and $y_i \in \langle F(x_i) \rangle$ for $x_i \in \mathcal{P}$ and $i = 1, 2$, then $y_3 \in \langle y_1, y_2 \rangle \subseteq \langle F(x_1), F(x_2) \rangle = \langle F(x_1 \oplus x_2) \rangle$ and $x_1 \oplus x_2$ still belongs to $\mathcal{P}$.

Now, for every object $x \in \mathcal{K}$, the tt-functor $F : \mathcal{K} \rightarrow \mathcal{L}$ sends an exact triangle over the unit $\eta_x$ as in (2.12) to an exact triangle in $\mathcal{L}$:

$$F(w_x) \xrightarrow{F(\xi_x)} \mathbb{1} \xrightarrow{\eta_{F(x)}} F(x) \xrightarrow{F(\zeta_x)} \Sigma F(w_x).$$

Here we use that $F(\eta_x) = \eta_{F(x)}$ which is another way of saying that $F$ preserves duals. See Remark 2.1. Using this last exact triangle in Corollary 2.11 for the rigid object $F(x)$ in the tt-category $\mathcal{L}$, we see that $\langle F(x) \rangle = \{ y \in \mathcal{L} \mid F(\xi_x)^{\otimes n} \otimes y = 0 \text{ for some } n \geq 1 \}$.

Combining this with the description of $J = \langle F(\mathcal{P}) \rangle$ as $J'$ above, we obtain $\langle F(\mathcal{P}) \rangle = \{ y \in \mathcal{L} \mid F(\xi_x)^{\otimes n} \otimes y = 0 \text{ for some } n \geq 1 \text{ and some } x \in \mathcal{P} \}.$

It follows that if $s \in S = \mathcal{K} \setminus \mathcal{P}$ then $F(s)$ cannot belong to $J = \langle F(\mathcal{P}) \rangle$. Indeed, if $F(s) \in \langle F(\mathcal{P}) \rangle$ then by the above there exists $x \in \mathcal{P}$ and $n \geq 1$ such that $0 = F(\xi_x)^{\otimes n} \otimes F(s) \cong F(\xi_x)^{\otimes n} \otimes s$ since $F$ is a $\otimes$-functor. This contradicts (2.15).

In short, we have shown that the $\otimes$-multiplicative class $F(S) = F(\mathcal{K} \setminus \mathcal{P})$ does not meet the tt-ideal $J = \langle F(\mathcal{P}) \rangle$, in the tt-category $\mathcal{L}$. By the existence trick (A) again, there exists a prime $\mathcal{Q}$ satisfying the following two relations: $J \subseteq \mathcal{Q}$ and $F(S) \cap \mathcal{Q} = \emptyset$. Unpacking the definition of $S = \mathcal{K} \setminus \mathcal{P}$ and $J = \langle F(\mathcal{P}) \rangle$, these two relations mean respectively $\mathcal{P} \subseteq F^{-1}(\mathcal{Q})$ and $F^{-1}(\mathcal{Q}) \subseteq \mathcal{P}$. Hence $\mathcal{P} = F^{-1}(\mathcal{Q}) = \varphi(\mathcal{Q})$ as wanted.

We are now ready to prove our main result.

Proof of Theorem 1.4.

(a)$\Rightarrow$(b): Suppose that $\varphi : \text{Spc}(\mathcal{L}) \rightarrow \text{Spc}(\mathcal{K})$ is surjective and let $f : x \rightarrow y$ be a morphism such that $F(f) = 0$ and which is $\otimes$-nilpotent on its cone, say $f^{\otimes n} \otimes \text{cone}(f) = 0$. It follows from the exact triangle $x \xrightarrow{f} y \rightarrow \text{cone}(f) \rightarrow \Sigma x$ in $\mathcal{K}$ and from $F(f) = 0$ that $F(\text{cone}(f)) \cong F(y) \oplus \Sigma F(x)$ in $\mathcal{L}$. Taking supports, we have $\text{supp}(F(\text{cone}(f))) = \text{supp}(F(x)) \cup \text{supp}(F(y))$. By (D), this translates into

$$\varphi^{-1}(\text{supp}(\text{cone}(f))) = \varphi^{-1}(\text{supp}(x)) \cup \varphi^{-1}(\text{supp}(y)) = \varphi^{-1}(\text{supp}(x)) \cup \text{supp}(y)).$$

Since $\varphi$ is surjective, this implies $\text{supp}(\text{cone}(f)) = \text{supp}(x) \cup \text{supp}(y)$. Therefore $x, y \in \langle \text{cone}(f) \rangle$. But we assumed that $f$ is $\otimes$-nilpotent on $\text{cone}(f)$ and it follows from Proposition 2.9 that $f$ is also $\otimes$-nilpotent on $x$ and on $y$. This means that there exists $n \geq 1$ such that $f^{\otimes n} \otimes x = 0 : x^{\otimes (n+1)} \rightarrow y^{\otimes n} \otimes x$. But then $f^{\otimes (n+1)}$ decomposes as $x^{\otimes n+1} \xrightarrow{f^{\otimes n+1}} y^{\otimes n} \otimes x \xrightarrow{y^{\otimes n} \otimes f} y^{\otimes (n+1)}$

and is therefore also zero, that is, $f^{\otimes (n+1)} = 0$ as wanted.
(b) ⇒ (a): Suppose that \( F : \mathcal{K} \to \mathcal{L} \) detects \( \otimes \)-nilpotence of those morphisms which are already zero on their cone. Let \( \mathcal{P} \in \text{Spc}(\mathcal{K}) \) be a prime and let us show that property (2.15) in Lemma 2.14 is satisfied. Let \( g = \xi_x^{\otimes n} \otimes s \) be the morphism in (2.15) for some objects \( x \in \mathcal{P} \) and \( s \in \mathcal{K} \setminus \mathcal{P} \) and for \( n \geq 1 \). Suppose \( ab \) absurdo that \( F(g) = 0 \). The cone of \( g = \xi_x^{\otimes n} \otimes s \) is simply \( \text{cone}(\xi_x^{\otimes n}) \otimes s \). By Corollary 2.11, \( \xi_x^{\otimes n} \) is \( \otimes \)-nilpotent on its cone. Hence \( g \) is \( \otimes \)-nilpotent on its cone as well. We can therefore apply our assumption (b) to \( g \) and deduce from the (absurd) assumption \( F(g) = 0 \) that \( g = \xi_x^{\otimes n} \otimes s \) is \( \otimes \)-nilpotent. In other words, \( \xi_x \) is \( \otimes \)-nilpotent on \( \xi_x^{\otimes m} \) for some \( m \geq 1 \). By Corollary 2.11 again, this implies that \( s^{\otimes m} \) belongs to \( \langle x \rangle \subseteq \mathcal{P} \), and therefore \( s \in \mathcal{P} \) since \( \mathcal{P} \) is prime, a contradiction with the choice of \( s \) in \( S = \mathcal{K} \setminus \mathcal{P} \). In short, we have verified property (2.15) of Lemma 2.14 for the prime \( \mathcal{P} \), which tells us that \( \mathcal{P} \) belongs to the image of \( \varphi \) as claimed. \( \square \)

---

Let us now prove Theorems 1.6 and 1.7. We therefore assume the existence of an adjoint \( U : \mathcal{L} \to \mathcal{K} \) to our tensor-triangulated functor \( F \):

\[
\begin{array}{ccc}
\mathcal{K} & \xrightarrow{F} & \mathcal{L} \\
\downarrow{\mathcal{U}} & & \downarrow{\mathcal{U}} \\
\mathcal{L} & \underset{\text{L}}{\xrightarrow{\text{U}}} & \mathcal{K}
\end{array}
\]

By general theory, \( U \) must satisfy a projection formula

\[
U(F(x) \otimes z) \cong x \otimes U(z)
\]

for all \( x \in \mathcal{K} \) and \( z \in \mathcal{L} \). The latter is an easy consequence of rigidity of \( x \) and the adjunctions (2.2) and (2.16). See for instance [FHM03, Prop. 3.2].

**Proof of Theorem 1.7.** Let \( \mathcal{P} \in \text{Spc}(\mathcal{K}) \). We need to show that \( \mathcal{P} \in \text{im}(\varphi) \) if and only if \( \mathcal{P} \in \text{supp}(U(\mathbb{1})) \). The latter means \( U(\mathbb{1}) \notin \mathcal{P} \).

Suppose first that \( \mathcal{P} = \varphi(\mathcal{Q}) \) for some \( \mathcal{Q} \in \text{Spc}(\mathcal{L}) \). Then \( \mathcal{P} = F^{-1}(\mathcal{Q}) \). To show \( U(\mathbb{1}) \notin \mathcal{P} \) it therefore suffices to show that \( FU(\mathbb{1}) \notin \mathcal{Q} \). This is easy since, by the unit-counit relation for (2.16), the object \( FU(\mathbb{1}_\mathcal{L}) \cong FUF(1_\mathcal{K}) \) admits \( F(1_\mathcal{K}) \cong 1_\mathcal{L} \) as a direct summand and \( 1 \) cannot belong to any prime.

The reverse inclusion is the interesting one. So, let \( \mathcal{P} \in \text{supp}(U(\mathbb{1})) \), meaning \( U(\mathbb{1}) \notin \mathcal{P} \). Let us show that \( \mathcal{P} \) satisfies condition (2.15) of Lemma 2.14. Take objects \( x \in \mathcal{P} \) and \( s \in \mathcal{K} \setminus \mathcal{P} \), and suppose \( ab \) absurdo that \( F(g) = 0 \) where \( g = \xi_x^{\otimes n} \otimes s \) for some \( n \geq 1 \) as before. By the projection formula (2.17) for \( z = \mathbb{1} \), the property \( UF(g) = U(0) = 0 \) implies \( g \otimes U(1) = 0 \). Consequently we have an exact triangle

\[
\begin{array}{ccc}
w_x^{\otimes n} \otimes s \otimes U(\mathbb{1}) \xrightarrow{g \otimes U(1) = 0} s \otimes U(\mathbb{1}) & \xrightarrow{\text{cone}(g) \otimes U(1)} & \Sigma w_x^{\otimes n} \otimes s \otimes U(1)
\end{array}
\]

in \( \mathcal{K} \). This proves that \( s \otimes U(\mathbb{1}) \) is a direct summand of \( \text{cone}(g) \otimes U(1) \in \langle \text{cone}(\xi_x^{\otimes n}) \rangle \subseteq \langle \text{cone}(\xi_x^{\otimes n}) \rangle \). By Proposition 2.10, the latter is contained in \( \langle x \rangle \subseteq \mathcal{P} \). In short, we have \( s \otimes U(\mathbb{1}) \in \mathcal{P} \). Since \( \mathcal{P} \) is prime this forces \( s \in \mathcal{P} \) or \( U(\mathbb{1}) \in \mathcal{P} \), which are both absurd. So we have proven (2.15) for \( \mathcal{P} \) and we conclude by Lemma 2.14 again. \( \square \)

**Proof of Theorem 1.6.** In view of Theorem 1.7 it suffices to prove that \( F : \mathcal{K} \to \mathcal{L} \) detects \( \otimes \)-nilpotence if and only if \( \text{supp}(U(\mathbb{1})) = \text{Spc}(\mathcal{K}) \), which means \( (U(\mathbb{1})) = \mathcal{K} \). This is a standard argument, as in [Bal16a, Prop. 3.15] for instance. Let us outline it for completeness. The point is that \( A := U(\mathbb{1}) \) is a ring-object (for \( U \) is lax-monoidal). Let \( J \xrightarrow{\xi} \mathbb{1} \xrightarrow{u} A \to \Sigma J \) be an exact triangle over the unit \( u : \mathbb{1} \to A \).
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We have $A \otimes \xi = 0$ (since $A \otimes u$ is a split monomorphism, retracted by multiplication $A \otimes A \to A$). A morphism $f : x \to y$ satisfies $F(f) = 0$ if and only if the composite $x \xrightarrow{f} y \xrightarrow{u} A \otimes y$ is zero (by adjunction and the projection formula: $A \otimes - \simeq UF(-)$); this is in turn equivalent to the morphism $f : x \to y$ factoring via $\xi \otimes y : J \otimes y \to A \otimes y \to \Sigma J \otimes y$. So we are down to proving that $\xi : J \to 1$ is $\otimes$-nilpotent if and only if $\langle \xi \rangle = K$. This is now immediate from Proposition 2.10, which says that $\langle A \rangle = \{ z \in K \mid \xi \text{ is } \otimes \text{-nilpotent on } z \}$. Indeed, $1 \in \langle A \rangle$ if and only if $\xi$ is $\otimes$-nilpotent on $1$. □

References


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