MODEL THEORETIC REFORMULATION OF THE BAUM-CONNES AND FARRELL-JONES CONJECTURES

PAUL BALMER AND MICHEL MATTHEY

ABSTRACT. The Isomorphism Conjectures are translated into the language of homotopical algebra, where they resemble Thomason's descent theorems.

1. INTRODUCTION AND STATEMENT OF THE RESULTS

In [8], Thomason establishes that algebraic K-theory satisfies Zariski and Nisnevich *descent*. This is now considered a profound algebraico-geometric property of K-theory. In [1, 2], we have introduced the sister notion of *codescent*. Here, we prove that each one of the so-called Isomorphism Conjectures (see [3, 5]) among

(1) the Baum-Connes Conjecture,
(2) the real Baum-Connes Conjecture,
(3) the Bost Conjecture,
(4) the Farrell-Jones Conjecture in K-theory,
(5) the Farrell-Jones Conjecture in L-theory,

is equivalent to the codescent property for a suitable K- or L-theory functor.

For a (discrete) group G, these conjectures aim at computing, in geometrical and topological terms, the groups $K_*^{\mathrm{top}}(C_r^* G)$, $KO_*^{\mathrm{top}}({}_{\mathbb{R}}C_r^* G)$, $K_*^{\mathrm{top}}(\ell^1 G)$, $K_*^{\mathrm{alg}}(RG)$ and $L_*^{\langle -\infty \rangle}(\Lambda G)$ respectively, where R and Λ are associative rings with units, and Λ is equipped with an involution. Davis and Lück [4] express these conjectures as follows (the equivalence with the original statements is due to Hambleton-Pedersen [6]). First, fix one of the Conjectures (1)–(5) and denote by $K_*(G)$ the corresponding K- or L-group among the five listed above (for (4) and (5), R and Λ are understood). Denote by $\mathcal{C} := \mathrm{Or}(G)$ the orbit category of G, whose objects are the quotients G/H with H running among the subgroups of G, and the morphisms are the left-G-maps. Let $\mathcal{D} := \mathrm{Or}(G, \mathcal{VC})$ be the full subcategory of $\mathrm{Or}(G)$ on those objects G/H for which H is virtually cyclic. We sometimes write $\mathcal{C}_G$ and $\mathcal{D}_G$ to stress the dependence on the group G. Then, a suitable functor $X_G \colon \mathcal{C} \longrightarrow \mathcal{S}$ is constructed, where $\mathcal{S}$ denotes the usual stable model category of spectra (of compactly generated Hausdorff spaces), for which the weak equivalences are the stable ones. This functor X_G has the property that $\pi_*\big(X_G(G/H)\big)$ is canonically isomorphic to $K_*(H)$ for all $H \leqslant G$. Then, the fixed Isomorphism Conjecture for G amounts to the statement that the following composition, called *assembly map*, is a weak equivalence in $\mathcal{S}$:

$$\mu^G\colon \quad \underset{\mathcal{D}}{\mathrm{hocolim}}\ \mathrm{res}^{\mathcal{C}}_{\mathcal{D}} X_G \longrightarrow \underset{\mathcal{C}}{\mathrm{hocolim}}\ X_G \xrightarrow{\sim} \underset{\mathcal{C}}{\mathrm{colim}}\ X_G \xrightarrow{\cong} X_G(G/G)\,.$$

We turn to homotopical algebra. First, we denote by $\mathcal{U}_{\mathcal{S}}(\mathcal{C}, \mathcal{D})$ the model category $\mathcal{S}^{\mathcal{C}}$ of functors $\mathcal{C} \to \mathcal{S}$, where the weak equivalences and fibrations are the $\mathcal{D}$-*weak equivalences* and $\mathcal{D}$-*fibrations* respectively, *i.e.* they are defined $\mathcal{D}$-objectwise. See

Date: August 6, 2003.
Research supported by Swiss National Science Foundation, grant 620-66065.01.

details in [1, § 3], for instance. For a diagram $X \in \mathcal{S}^{\mathcal{C}}$, we let $\xi_X : QX \longrightarrow X$ be the cofibrant replacement of X in $\mathcal{U}_{\mathcal{S}}(\mathcal{C},\mathcal{D})$. As in [1, § 4], we say that X *satisfies $\mathcal{D}$-codescent* if the map $\xi_X(c)$ is a weak equivalence in $\mathcal{S}$ for every $c \in \mathcal{C}$; if this is only fulfilled at some $c_0 \in \mathcal{C}$, we say that X *satisfies $\mathcal{D}$-codescent at c_0*. For a conceptual approach to codescent and a parallel with descent, see [1, §§ 1 and 5]. Let $\mathcal{U}_{\mathcal{S}}(\mathcal{C})$ be the model structure on $\mathcal{S}^{\mathcal{C}}$ with the $\mathcal{C}$-weak equivalences and $\mathcal{C}$-fibrations; we define $\mathcal{U}_{\mathcal{S}}(\mathcal{D})$ on $\mathcal{S}^{\mathcal{D}}$ similarly. We denote by $\mathrm{Ho}_{\mathcal{S}}(\mathcal{C})$ and $\mathrm{Ho}_{\mathcal{S}}(\mathcal{D})$ the homotopy category of $\mathcal{U}_{\mathcal{S}}(\mathcal{C})$ and $\mathcal{U}_{\mathcal{S}}(\mathcal{D})$ respectively. As in [1, Prop. 13.2], we have the derived adjunction of the Quillen adjunction $\mathrm{ind}_{\mathcal{D}}^{\mathcal{C}} \colon \mathcal{U}(\mathcal{D}) \rightleftarrows \mathcal{U}(\mathcal{C}) : \mathrm{res}_{\mathcal{D}}^{\mathcal{C}}$, namely

$$L\mathrm{ind}_{\mathcal{D}}^{\mathcal{C}} \colon \mathrm{Ho}_{\mathcal{S}}(\mathcal{D}) \rightleftarrows \mathrm{Ho}_{\mathcal{S}}(\mathcal{C}) : \mathrm{Res}_{\mathcal{D}}^{\mathcal{C}} .$$

For the sequel, fix a group G and one of the Isomorphism Conjectures (1)–(5); let $X_G \in \mathcal{S}^{\mathcal{C}}$ be the corresponding functor. Keep the other notations as above.

Theorem 1.1. *The following statements are equivalent:*

(i) *G satisfies the considered Isomorphism Conjecture;*
(ii) *the corresponding functor $X_G \in \mathcal{U}_{\mathcal{S}}(\mathcal{C},\mathcal{D})$ satisfies $\mathcal{D}$-codescent at $G/G \in \mathcal{C}$.*

Theorem 1.2. *For subgroups $L \leqslant H \leqslant G$, the following statements are equivalent:*

(i) *$X_H \in \mathcal{U}_{\mathcal{S}}(\mathcal{C}_H,\mathcal{D}_H)$ satisfies $\mathcal{D}_H$-codescent at $H/L \in \mathcal{C}_H$;*
(ii) *$X_G \in \mathcal{U}_{\mathcal{S}}(\mathcal{C}_G,\mathcal{D}_G)$ satisfies $\mathcal{D}_G$-codescent at $G/L \in \mathcal{C}_G$.*

In fact, by general results of [1] (without invoking 1.1 above), if X_G satisfies $\mathcal{D}_G$-codescent, then X_H satisfies $\mathcal{D}_H$-codescent for every subgroup $H \leqslant G$.

Main Theorem. *The following statements are equivalent:*

(i) *every subgroup H of G satisfies the considered Isomorphism Conjecture;*
(ii) *the corresponding functor $X_G \in \mathcal{U}_{\mathcal{S}}(\mathcal{C},\mathcal{D})$ satisfies $\mathcal{D}$-codescent;*
(iii) *up to isomorphism, the image of X_G in $\mathrm{Ho}_{\mathcal{S}}(\mathcal{C})$ belongs to $L\mathrm{ind}_{\mathcal{D}}^{\mathcal{C}}\big(\mathrm{Ho}_{\mathcal{S}}(\mathcal{D})\big)$.*

Note that the usual Baum-Connes and Bost Conjectures are stated with *finite* subgroups instead of virtually cyclic ones, but this is known to be equivalent. So, in these cases, we could as well set $\mathcal{D}_G := \mathrm{Or}(G, \mathcal{F}in)$ instead of $\mathrm{Or}(G, \mathcal{VC})$.

Remark 1.3. Let $X \in \mathcal{S}^{\mathcal{C}}$ be a diagram and let $\zeta_X \colon \mathcal{Q}X \longrightarrow X$ be an arbitrary *cofibrant approximation* of X in $\mathcal{U}_{\mathcal{S}}(\mathcal{C},\mathcal{D})$, namely, ζ_X is merely a $\mathcal{D}$-weak equivalence and $\mathcal{Q}X$ is cofibrant in $\mathcal{U}_{\mathcal{S}}(\mathcal{C},\mathcal{D})$. Then, X satisfies $\mathcal{D}$-codescent at some object $c \in \mathcal{C}$ if and only if $\zeta_X(c)$ is a weak equivalence in $\mathcal{S}$, see [1, Prop. 6.5]. This illustrates the flexibility of the codescent-type reformulation of the Isomorphism Conjectures, namely, every such cofibrant approximation of X_G yields a possibly very different assembly map that can be used to test the considered conjecture.

2. The proofs

Let $\mathcal{G}\mathrm{pds}^{\mathrm{f}}$ be the category of groupoids with *faithful* functors. For the considered conjecture, by [4, 7], there exists a *homotopy functor* $\mathcal{X} \colon \mathcal{G}\mathrm{pds}^{\mathrm{f}} \longrightarrow \mathcal{S}$, *i.e.* $\mathcal{X}$ takes equivalences of groupoids to weak equivalences, such that X_G is the composite

$$X_G \colon \quad \mathcal{C} = \mathrm{Or}(G) \xrightarrow{\iota} \mathcal{G}\mathrm{pds}^{\mathrm{f}} \xrightarrow{\mathcal{X}} \mathcal{S}.$$

The functor ι takes G/H to its *G-transport groupoid* $\overline{G/H}^{G}$ with the set G/H as objects and with $\{g \in G \mid g g_1 H = g_2 H\}$ as morphisms from $g_1 H$ to $g_2 H$. Moreover, the functor $\mathcal{X}$ takes values in cofibrant spectra, so that X_G is $\mathcal{C}$-objectwise cofibrant.

Let $\mathcal{C}\mathrm{at}$ be the category of small categories and $\mathrm{s}\mathcal{S}\mathrm{ets}$ that of simplicial sets. Denote by $\otimes_{\mathcal{D}}\colon \mathrm{s}\mathcal{S}\mathrm{ets}^{\mathcal{D}^{\mathrm{op}}} \times \mathcal{S}^{\mathcal{D}} \longrightarrow \mathcal{S}$ the tensor product over $\mathcal{D}$ induced by the simplicial model structure on $\mathcal{S}$, where $K \in \mathrm{s}\mathcal{S}\mathrm{ets}$ "acts" on $E \in \mathcal{S}$ by $|K|_{+} \wedge E$.

Proof of Theorem 1.1. *A priori*, to test whether X_G satisfies $\mathcal{D}$-codescent at some $c \in \mathcal{C}$ requires a thorough understanding of the usually mysterious cofibrant *replacement* of X_G. A key point here is the freedom to use *any* cofibrant *approximation* instead, see Remark 1.3. We provide in [2, § 6] a general construction of cofibrant approximations in $\mathcal{U}_{\mathcal{S}}(\mathcal{C}, \mathcal{D})$, one of which is exactly suited for our present purposes [2, Cor. 6.9]. Evaluated at the terminal object $G/G \in \mathcal{C}$, this cofibrant approximation $\zeta_{X_G} : \mathcal{Q}X_G \longrightarrow X_G$ is a certain map (described at the end of the proof)

$$\zeta_{X_G}(G/G)\colon \quad \mathcal{Q}X_G(G/G) = B(?\searrow\mathcal{D})^{\mathrm{op}} \underset{?\in\mathcal{D}}{\otimes} \mathrm{res}^{\mathcal{C}}_{\mathcal{D}} X_G(?) \longrightarrow X_G(G/G)\,.$$

Indeed, using the notations of [2, Not. 6.1], this follows from the canonical identification $(?\searrow\mathcal{D}\searrow^{\mathcal{C}} G/G)^{\mathrm{op}} = (?\searrow\mathcal{D})^{\mathrm{op}}$ of diagrams in $\mathcal{C}\mathrm{at}^{\mathcal{D}^{\mathrm{op}}}$ and from the fact that X_G is $\mathcal{C}$-objectwise cofibrant. By definition of the homotopy colimit, we have

$$B(?\searrow\mathcal{D})^{\mathrm{op}} \underset{?\in\mathcal{D}}{\otimes} \mathrm{res}^{\mathcal{C}}_{\mathcal{D}} X_G(?) = \underset{\mathcal{D}}{\mathrm{hocolim}}\; \mathrm{res}^{\mathcal{C}}_{\mathcal{D}} X_G\,.$$

So, it suffices to show that $\zeta_{X_G}(G/G)$ coincides with the assembly map μ^G. In the notations of [2, Not. 5.1], we have $\underline{\mathrm{mor}_{\mathcal{D},\mathcal{C}}(?, G/G)} = *$ in $\mathrm{s}\mathcal{S}\mathrm{ets}^{\mathcal{D}^{\mathrm{op}}}$ (the constant diagram with value the point). By [2, Lem. 5.3], the spectrum $* \otimes_{\mathcal{D}} \mathrm{res}^{\mathcal{C}}_{\mathcal{D}} X_G$ identifies with $\mathrm{ind}^{\mathcal{C}}_{\mathcal{D}} \mathrm{res}^{\mathcal{C}}_{\mathcal{D}} X_G(G/G)$. Letting ϵ denote the counit of the adjunction $(\mathrm{ind}^{\mathcal{C}}_{\mathcal{D}}, \mathrm{res}^{\mathcal{C}}_{\mathcal{D}})$, it is routine to verify that there is a canonical commutative diagram

$$\begin{tikzcd}
\mathrm{hocolim}_{\mathcal{D}} \mathrm{res}^{\mathcal{C}}_{\mathcal{D}} X_G \arrow[rr, equal] \arrow[d, equal] & & B(?\searrow\mathcal{D})^{\mathrm{op}} \otimes_{\mathcal{D}} \mathrm{res}^{\mathcal{C}}_{\mathcal{D}} X_G(?) \arrow[d] \\
\mathrm{hocolim}_{\mathcal{D}} \mathrm{res}^{\mathcal{C}}_{\mathcal{D}} X_G \arrow[r] \arrow[d] & \mathrm{colim}_{\mathcal{D}} \mathrm{res}^{\mathcal{C}}_{\mathcal{D}} X_G \arrow[r, "\cong"] \arrow[d] & * \otimes_{\mathcal{D}} \mathrm{res}^{\mathcal{C}}_{\mathcal{D}} X_G \arrow[d, "\epsilon_{X_G}(G/G)"] \\
\mathrm{hocolim}_{\mathcal{C}} X_G \arrow[r, "\sim"] & \mathrm{colim}_{\mathcal{C}} X_G \arrow[r, "\cong"] & X_G(G/G)
\end{tikzcd}$$

The composition of the first column followed by the last row is the assembly map μ^G. The composition in the last column is $\zeta_{X_G}(G/G)$, see [2, Cor. 6.9]. $\square$

More generally, one can prove that the "$(X, \mathcal{F}, G)$-Isomorphism Conjecture" of [4, Def. 5.1] is equivalent to X satisfying $\mathrm{Or}(G, \mathcal{F})$-codescent at G/G, for any objectwise cofibrant diagram $X \in \mathcal{S}^{\mathrm{Or}(G)}$ and any family $\mathcal{F}$ of subgroups of G.

For $g \in G$ and $H \leqslant G$, we write ${}^{g}H := gHg^{-1}$. In the orbit category $\mathrm{Or}(G) = \mathcal{C}_G$, for an element $g \in G$ such that ${}^{g}H \leqslant K$ for some subgroups H and K of G, we designate by the right coset Kg the morphism $G/H \longrightarrow G/K$ taking $\tilde{g}H$ to $\tilde{g}g^{-1}K$.

Proof of Theorem 1.2. Consider the functor $\Phi\colon \mathcal{C}_H \longrightarrow \mathcal{C}_G$ taking a coset $H/L \in \mathcal{C}_H$ to G/L. For any $L \leqslant H$, we have canonical equivalences of groupoids in $\mathcal{G}\mathrm{pds}^{\mathrm{f}}$

$$\overline{H/L}^{H} \overset{\sim}{\longleftarrow} \overline{L} \overset{\sim}{\longrightarrow} \overline{G/L}^{G}\,,$$

where $\overline{L}$ is L viewed as a one-object groupoid. Since $\mathcal{X}$ is a homotopy functor, one checks that there is a canonical zig-zag of two $\mathcal{C}_H$-weak equivalences between X_H and $\Phi^* X_G = X_G \circ \Phi$ in $\mathcal{U}_{\mathcal{S}}(\mathcal{C}_H)$. By weak invariance of codescent [1, Prop. 6.10], X_H and $\Phi^* X_G$ satisfy $\mathcal{D}_H$-codescent at exactly the same objects H/L of $\mathcal{C}_H$.

Fix an object $H/K \in \mathcal{D}_H$. Let $E_{H/K} \subset G$ be a set of representatives for the quotient $H\backslash\{g \in G \,|\, {}^gK \leqslant H\}$. Let $M\gamma\colon \Phi(H/K) = G/K \longrightarrow G/M = \Phi(H/M)$ be a morphism in $\mathcal{C}_G$ with $M \leqslant H$ (and $\gamma \in G$). It is straightforward that there is a unique pair (g, Mh) with $g \in E_{H/K}$ and $Mh \in \mathrm{mor}_{\mathcal{C}_H}(H/{}^gK, H/M)$ (namely characterized by $Hg = H\gamma$ and $Mh = M\gamma g^{-1}$) such that $M\gamma$ decomposes in $\mathcal{C}_G$ as

$$\begin{tikzcd} G/K \arrow[r, "{}^gKg"] \arrow[rr, bend left, "M\gamma"] & G/{}^gK \arrow[r, "Mh"] & G/M\,. \end{tikzcd}$$

Since $\Phi(\mathcal{D}_H) \subset \mathcal{D}_G$, this precisely says that Φ is a *left glossy morphism of pairs of small categories* in the sense of [1, Defs. 7.3 and 8.1]. By left glossy invariance of codescent [1, Thm. 9.14], Φ^*X_G satisfies $\mathcal{D}_H$-codescent at some $H/L \in \mathcal{C}_H$ if and only if X_G satisfies $\mathcal{D}_1$-codescent at $G/L \in \mathcal{C}_G$, where $\mathcal{D}_1 := \Phi(\mathcal{D}_H)$. Set $\mathcal{D}_2 := \mathcal{D}_G$ and fix $H/L \in \mathcal{C}_H$. For $i = 1, 2$, consider the full subcategory $\mathcal{E}_i$ of $\mathcal{D}_i$ given by

$$\mathcal{E}_i := \big\{ G/K \in \mathcal{D}_i \,\big|\, \mathrm{mor}_{\mathcal{C}_G}(G/K, G/L) \neq \varnothing \big\}\,.$$

By the Pruning Lemma [1, Thm. 11.5], X_G satisfies $\mathcal{D}_1$-codescent at G/L if and only if it satisfies $\mathcal{E}_1$-codescent at G/L. Since $L \leqslant H$, every object of $\mathcal{E}_1$ is isomorphic, inside $\mathcal{C}_G$, to some object of $\mathcal{E}_2$ and conversely; in other words, $\mathcal{E}_1$ and $\mathcal{E}_2$ are *essentially equivalent* in $\mathcal{C}_G$, in the sense of [1, Def. 3.12]. So, by [1, Prop. 10.1], X_G satisfies $\mathcal{E}_1$-codescent at G/L if and only if it satisfies $\mathcal{E}_2$-codescent at G/L. By the Pruning Lemma again, X_G satisfies $\mathcal{E}_2$-codescent at G/L if and only if it satisfies $\mathcal{D}_2$-codescent at G/L, *i.e.* $\mathcal{D}_G$-codescent at G/L.

In total, we have proven that X_H satisfies $\mathcal{D}_H$-codescent at an object $H/L \in \mathcal{C}_H$ if and only if X_G satisfies $\mathcal{D}_G$-codescent at G/L, as was to be shown. □

Proof of the Main Theorem. The equivalence between (i) and (ii) follows from Theorems 1.1 and 1.2; (ii) and (iii) are equivalent by [1, Thm. 13.5]. □

References

[1] P. Balmer and M. Matthey. Codescent theory I : Foundations, *Preprint*, 2003.

[2] P. Balmer and M. Matthey, Codescent theory II : Cofibrant approximations, *Preprint*, 2003.

[3] P. Baum, A. Connes, and N. Higson. Classifying spaces for proper actions and K-theory of group C^*-algebras, In *C^*-algebras : 1943-1993*, Contemp. Math. **167**, 241–291, 1994.

[4] J.F. Davis and W. Lück. Spaces over a category and assembly maps in isomorphism conjectures in K- and L-theory, *K-theory* **15** (1998), 201–252.

[5] F.T. Farrell and L.E. Jones. Isomorphism conjectures in algebraic K-theory, *J. Amer. Math. Soc.* **6** (1993), 249–297.

[6] I. Hambleton and E.K. Pedersen. Identifying assembly maps in K- and L-theory, *Preprint* (2002), to appear in *Math. Ann.*

[7] M. Joachim. K-homology of C^*-categories and symmetric spectra representing K-homology, *Preprint* (2002), to appear in *Math. Ann.*

[8] R.W. Thomason and T. Trobaugh. Higher algebraic K-theory of schemes and of derived categories. In *The Grothendieck Festschrift III*, 247–435, Progr. Math. 88, Birkhäuser, 1990.

Department of Mathematics, ETH Zentrum, CH-8092 Zürich, Switzerland

E-mail address: `paul.balmer@math.ethz.ch` *and* `michel.matthey@math.ethz.ch`

URL: `http://www.math.ethz.ch/~balmer` *and* `http://www.math.ethz.ch/~matthey`