PERFECT COMPLEXES AND COMPLETION

PAUL BALMER AND BEREN SANDERS

Abstract. Let $\hat{R}$ be the I-adic completion of a commutative ring R with respect to a finitely generated ideal I. We give a necessary and sufficient criterion for the category of perfect complexes over $\hat{R}$ to be equivalent to the subcategory of dualizable objects in the derived category of I-complete complexes of R-modules. Our criterion is always satisfied when R is noetherian. When specialized to R local and noetherian and to I the maximal ideal, our theorem recovers a recent result of Benson, Iyengar, Krause and Pevtsova.

Contents

1. Introduction

Let R be a commutative ring and let $I \subset R$ be a finitely generated ideal. We write $Y \coloneqq V(I) \subseteq \mathrm{Spec}(R)$ for the corresponding closed subset with quasi-compact complement. It is well-known that the I-adic completion $\hat{R}_I \coloneqq \lim_n R/I^n$ only depends on Y. We sometimes write $\hat{R}_Y$, or just $\hat{R}$, for this ring.

It is legitimate to ask whether the derived category $\mathrm{D}(\hat{R})$ of the completion can be recovered from the derived category of R in purely tensor-triangular terms. The question is tantalizing since the term 'completion' also refers to a standard construction in stable homotopy theory. Let us remind the reader.

Suppose that $\mathcal{T}$ is a 'big' tensor-triangulated category. The exact hypothesis will play an interesting role, so we give some details. We want the category $\mathcal{T}$ to be compactly generated in the sense of Neeman [Nee01] and to admit a good notion of 'small' object, namely, we want the compact objects $\mathcal{T}^c$ and the dualizable objects $\mathcal{T}^d$ to coincide: $\mathcal{T}^c = \mathcal{T}^d$. There is a profusion of such 'big' tt-categories in mathematics, see [BF11] or [HPS97]. The derived category $\mathcal{T} = \mathrm{D}(R)$ is an example; its small objects $\mathcal{T}^c = \mathcal{T}^d = \mathrm{D}^{\mathrm{perf}}(R)$ are precisely the *perfect complexes*, *i.e.*, the bounded complexes of finitely generated projective R-modules.

For any 'big' tt-category $\mathcal{T}$, consider the spectrum $\mathrm{Spc}(\mathcal{T}^c)$ of the small part $\mathcal{T}^c$. Choose a closed subset $Y \subseteq \mathrm{Spc}(\mathcal{T}^c)$ with quasi-compact complement and let

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$\mathcal{T}^c_Y = \{ c \in \mathcal{T}^c \,|\, \mathrm{supp}(c) \subseteq Y \}$ be the tt-ideal of small objects supported on Y. In the literature (see Greenlees [Gre01, § 2]), the double right-orthogonal of $\mathcal{T}^c_Y$ in $\mathcal{T}$

$$\mathcal{T}^\wedge_Y := ((\mathcal{T}^c_Y)^\perp)^\perp \tag{1.1}$$

is called the subcategory of *Y-complete objects* in $\mathcal{T}$. See more in Remark 2.6. The tt-category $\mathcal{T}^\wedge_Y$ comes with a tt-functor called *completion*

$$(-)^\wedge_Y\colon\ \mathcal{T} \to \mathcal{T}^\wedge_Y \tag{1.2}$$

which is simply a localization, left adjoint to the inclusion $\mathcal{T}^\wedge_Y \hookrightarrow \mathcal{T}$.

Unfortunately, if we plug $\mathcal{T} := \mathrm{D}(R)$ into this abstract theory, using the identification $\mathrm{Spc}(\mathcal{T}^c) \cong \mathrm{Spec}(R)$, it essentially never happens that $\mathcal{T}^\wedge_Y$ is the derived category of $\hat{R}_Y$, except in the trivial case where the ring splits $R \simeq R_1 \times R_2$ and $I = R_2$ (so $\hat{R}_I = R_1$). It even fails for the p-adics: $\mathrm{D}(\mathbb{Z})^\wedge_{(p)} \not\simeq \mathrm{D}(\hat{\mathbb{Z}}_p)$. So 'derived category' and 'completion' do not commute. And it gets worse. Except in the split case, $\mathcal{T}^\wedge_Y$ is not the derived category of *any* commutative ring whatsoever.

The reason for this is well-known: The tt-category $\mathcal{T}^\wedge_Y$ is not a legitimate 'big' tt-category. It is generated by compact-dualizable objects but its $\otimes$-unit, although obviously dualizable, is only compact in split cases (Remark 2.12). In other words, for every 'big' tt-category $\mathcal{T}$ and every Y the inclusion

$$(\mathcal{T}^\wedge_Y)^c \hookrightarrow (\mathcal{T}^\wedge_Y)^d \tag{1.3}$$

is typically a proper inclusion. This distinction, between *compact* and *dualizable*, only makes sense in *tensor*-triangular geometry and is difficult to extract from the mere triangulated structure. When the two categories in (1.3) are different, the larger one is better. Indeed, only $(\mathcal{T}^\wedge_Y)^d$ is an actual tt-category and only $(\mathcal{T}^\wedge_Y)^d$ receives the original $\mathcal{T}^c = \mathcal{T}^d$ under the completion tt-functor $(-)^\wedge_Y$ in (1.2).

A significant observation of Benson–Iyengar–Krause–Pevtsova [BIKP23, § 4] is that although we do not recover $\mathrm{D}(\hat{R}_Y)$ as the Y-completion of $\mathcal{T} = \mathrm{D}(R)$, we can still hope to recover its small objects. They prove that when $(R, \mathfrak{m})$ is a local and noetherian ring and $Y = \{\mathfrak{m}\}$, the category $(\mathcal{T}^\wedge_Y)^d$, whose praise we sang above, recovers the derived category of *perfect* complexes over the $\mathfrak{m}$-adic completion $\hat{R}_\mathfrak{m}$. Interestingly, BIKP did not conjecture that their result would hold for any commutative ring R and any finitely generated ideal I. And indeed it fails in that generality, as we shall see in Remark 5.3. On the other hand, we shall prove that the noetherian result is not restricted to local rings but is rather a global fact:

1.4. **Theorem** (Corollary 5.2)**.** *Suppose that R is noetherian. Then there is a canonical equivalence of tt-categories between the derived category $\mathrm{D}^{\mathrm{perf}}(\hat{R}_Y)$ of perfect complexes over the completion and the category $(\mathrm{D}(R)^\wedge_Y)^d$ of dualizable objects in the tt-category of Y-complete complexes of R-modules, which makes the following diagram commute*

$$\begin{array}{ccccc} & \overset{(-)^\wedge_Y}{\swarrow} & \mathrm{D}^{\mathrm{perf}}(R) & \overset{\hat{R}\otimes_R -}{\searrow} & \\ (\mathrm{D}(R)^\wedge_Y)^d & & \xrightarrow{\ \cong\ } & & \mathrm{D}^{\mathrm{perf}}(\hat{R}_Y). \end{array} \tag{1.5}$$

This statement is an easy consequence of our results concerning possibly non-noetherian rings. In that generality, the above formulation fails, as already mentioned, and we need to add a new condition. We want to express that condition in 'concrete' terms and will do so by means of Koszul complexes.

But before wheeling in Koszul complexes, let us point out another problem that we shall face in the non-noetherian setting: The two notions of completion discussed so far can yield different ring objects, that is, the tt-completion $\hat{\mathbb{1}}_Y$ and the classical completion $\hat{R}_Y$ might not be isomorphic in $\mathcal{T} = \mathrm{D}(R)$. We shall see that this problem has the 'same' solution as the first one. This second result can even be formulated at the level of abstract 'big' tt-categories, not just for $\mathrm{D}(R)$; see Theorem 2.16.

So let us now come to our criterion. For a sequence $\underline{s} = (s_1, \ldots, s_r)$ of generators of the ideal I, consider the usual Koszul complex $\mathrm{kos}_R(\underline{s}) = \otimes_{i=1}^r \mathrm{cone}(s_i \colon \mathbb{1} \to \mathbb{1})$. In degree zero, the homology of this complex is simply R/I and therefore does not change if we replace R by $\hat{R}$. We say that the sequence $\underline{s}$ is *Koszul-complete* if this holds not only in degree zero but in all degrees: $\mathrm{H}_\bullet(\mathrm{kos}_R(\underline{s})) \cong \mathrm{H}_\bullet(\mathrm{kos}_{\hat{R}}(\underline{s}))$; see Definition 3.14. This condition always holds when R is noetherian, as we verify in Proposition 3.17. We can now state our main result.

1.6. **Theorem** (Theorem 5.1)**.** *Let R be a commutative ring and let $\underline{s} = (s_1, \ldots, s_r)$ be a sequence of elements with $Y \coloneqq V(s_1, \ldots, s_r)$. Then the following are equivalent:*

(i) *The sequence $\underline{s}$ is Koszul-complete:* $\mathrm{H}_\bullet(\mathrm{kos}_R(\underline{s})) \cong \mathrm{H}_\bullet(\mathrm{kos}_{\hat{R}}(\underline{s}))$.

(ii) *There is an equivalence of tt-categories between* $\mathrm{D}^{\mathrm{perf}}(\hat{R}_Y)$ *and* $(\mathrm{D}(R)^\wedge_Y)^d$ *which makes the diagram in* (1.5) *commute.*

(iii) *There exists an isomorphism of ring objects $\hat{\mathbb{1}}_Y \cong \hat{R}_Y$ in $\mathrm{D}(R)$. In other words, tt-completion of the unit along Y recovers classical ring completion.*

As we shall see, our proof is more involved than the proof given in [BIKP23] in the special case where R is local and noetherian and $Y = \{\mathfrak{m}\}$ is the closed point. As is already apparent from its title, the article [BIKP23] adopts a very local focus; noetherianity of R is used at every step and the residue field $R/\mathfrak{m}$ is used for dimension arguments. None of those tools are available to us. We give a completely independent proof that does not assume their special case. This being said, we owe to [BIKP23] the insight that such statements could even be true.

Let us return to the original question of recovering $\hat{\mathcal{T}} \coloneqq \mathrm{D}(\hat{R})$ from $\mathcal{T} = \mathrm{D}(R)$. In the Koszul-complete case (*e.g.* if R is noetherian), we have recovered the small objects $\hat{\mathcal{T}}^c$ from $\mathcal{T}$. Those readers who want to think of $\mathcal{T}$ and its acolytes $\mathcal{T}^\wedge_Y$ and $(\mathcal{T}^\wedge_Y)^d$ as homotopy categories of some ∞-categories can now construct $\hat{\mathcal{T}}$ as the *Ind-completion* of $\hat{\mathcal{T}}^c$. In the non-Koszul-complete case, *i.e.*, when the two rings $\hat{\mathbb{1}}_Y$ and $\hat{R}_Y$ are different, we need to make a choice. Since I-adic completion has long been a source of headaches outside the noetherian world, now might be the time to let go of $\hat{R}_Y$, to opt for the better-behaved $(\mathcal{T}^\wedge_Y)^d$, and to simply Ind-complete the latter to define $\hat{\mathcal{T}}$ in full generality, as recently proposed in Naumann–Pol–Ramzi [NPR24]. We comment further on this topic in the final Remark 5.4.

The outline of the paper is relatively straightforward. In Section 2, we review the abstract notion of completion in 'big' tt-categories $\mathcal{T}$. In Section 3, we specialize the discussion to $\mathcal{T} = \mathrm{D}(R)$ and we contrast tt-completion with classical I-adic completion. The technical heart of the paper beats in Section 4 where we prove that when $R \cong \hat{R}_Y$ is classically complete, an object of $\mathrm{D}(R)^\wedge_Y$ is dualizable in that category if and only if it is a perfect complex. This is Corollary 4.26, which holds unconditionally and may be of independent interest. In the final Section 5, we prove Theorem 1.6 and its corollaries.

1.7. *Acknowledgements.* The authors would like to thank Greg Stevenson for numerous discussions about completion, over many years.

2. Tensor-triangular completion

In this section, $\mathcal{T}$ stands for a rigidly-compactly generated 'big' tensor-triangulated category, as recalled in the introduction. See details in [BF11] if necessary, or in [HPS97] where such $\mathcal{T}$ are called 'unital algebraic stable homotopy categories'. We denote the internal hom in $\mathcal{T}$ by $[-,-]\colon \mathcal{T}^{\mathrm{op}}\times\mathcal{T}\to\mathcal{T}$.

2.1. *Recollection.* Let $Y\subseteq \mathrm{Spc}(\mathcal{T}^c)$ be a Thomason subset of the tt-spectrum, for instance a closed subset with quasi-compact open complement. The choice of the Thomason subset Y is equivalent to the choice of the thick $\otimes$-ideal of $\mathcal{T}^c$:

$$\mathcal{T}^c_Y=\big\{\,x\in\mathcal{T}^c\,\big|\,\mathrm{supp}(x)\subseteq Y\ \text{in}\ \mathrm{Spc}(\mathcal{T}^c)\,\big\}.$$

A central player in this article is the (smashing) localizing $\otimes$-ideal

$$\mathcal{T}_Y:=\mathrm{Loc}(\mathcal{T}^c_Y)$$

generated by $\mathcal{T}^c_Y$ in $\mathcal{T}$. This tt-category $\mathcal{T}_Y$ of *objects of* $\mathcal{T}$ *supported on* Y is sometimes called the *torsion* part. (See more on its tensor-structure in Remark 2.6.) The compact objects in $\mathcal{T}_Y$ are the above $\mathcal{T}^c_Y$ by [Nee92]; removing any ambiguity, $(\mathcal{T}_Y)^c=(\mathcal{T}^c)_Y=:\mathcal{T}^c_Y$. The category $\mathcal{T}$ admits a recollement with respect to $\mathcal{T}_Y$ and its orthogonal $(\mathcal{T}_Y)^\perp=\big\{\,u\in\mathcal{T}\,\big|\,\mathrm{Hom}_{\mathcal{T}}(t,u)=0,\,\forall\,t\in\mathcal{T}_Y\,\big\}$. As explained in [BF11], the functors that appear in this recollement can all be described in terms of two objects $\mathbb{e}_Y$ and $\mathbb{f}_Y$ that fit in the so-called *idempotent triangle*

$$\mathbb{e}_Y\to\mathbb{1}\to\mathbb{f}_Y\to\Sigma\mathbb{e}_Y\,. \tag{2.2}$$

The latter is uniquely characterized by $\mathbb{e}_Y\in\mathcal{T}_Y$ and $\mathbb{f}_Y\in(\mathcal{T}_Y)^\perp$. We have $\mathbb{e}_Y\otimes\mathbb{f}_Y=0$, forcing $\mathbb{e}_Y^{\otimes 2}\cong\mathbb{e}_Y$ and $\mathbb{f}_Y\cong\mathbb{f}_Y^{\otimes 2}$. The recollement is then given by:

$$\begin{array}{c}\mathcal{T}_Y=\mathbb{e}_Y\otimes\mathcal{T}\\ \mathrm{incl}\downarrow\quad \uparrow{\scriptstyle \mathbb{e}_Y\otimes-}\quad \downarrow{\scriptstyle[\mathbb{e}_Y,-]}\\ \mathcal{T}\\ {\scriptstyle\mathbb{f}_Y\otimes-}\downarrow\quad \uparrow\mathrm{incl}\quad \downarrow{\scriptstyle[\mathbb{f}_Y,-]}\\ (\mathcal{T}_Y)^\perp=\mathbb{f}_Y\otimes\mathcal{T}\end{array} \tag{2.3}$$

2.4. *Recollection.* The Bousfield–Neeman theory of localization of triangulated categories in terms of orthogonal subcategories can be found in [Nee01, Chapter 9]. The three vertical sequences

$$\mathcal{S}\overset{F}{\rightarrowtail}\mathcal{T}\overset{G}{\twoheadrightarrow}\mathcal{U}$$

that appear in (2.3) are 'exact' in the sense that G identifies $\mathcal{U}$ with the Verdier quotient $\mathcal{T}/F(\mathcal{S})$. When F has a right (respectively, left) adjoint then so does G, and the quotient $\mathcal{T}/F(\mathcal{S})$ becomes equivalent to the subcategory $\mathcal{S}^\perp$ (respectively, ${}^\perp\mathcal{S}$) of $\mathcal{T}$ via that adjoint.

In particular, for the middle sequence $(\mathcal{T}_Y)^\perp\rightarrowtail\mathcal{T}\twoheadrightarrow\mathcal{T}_Y$ in (2.3), since the inclusion $(\mathcal{T}_Y)^\perp\rightarrowtail\mathcal{T}$ has adjoints on both sides, the Verdier quotient $\mathcal{T}/\mathcal{T}_Y^\perp$ identifies both with ${}^\perp((\mathcal{T}_Y)^\perp)=\mathcal{T}_Y$ and with $((\mathcal{T}_Y)^\perp)^\perp$. The former are the Y-torsion objects and the latter are the Y-complete ones. This equivalence $\mathcal{T}_Y\cong(\mathcal{T}_Y)^{\perp\perp}$ goes back to Dwyer–Greenlees [DG02] in the case of derived categories.

2.5. *Definition.* Following the literature, we define $\mathcal{T}_Y^\wedge := (\mathcal{T}_Y)^{\perp\perp}$ and call it the subcategory of *Y-complete* objects (or *tt-complete along* Y). It is sometimes denoted $\mathcal{T}_{\mathrm{comp}}$. Since $(\mathcal{T}_Y)^\perp = (\mathcal{T}_Y^c)^\perp$, this definition agrees with (1.1). As explained above, $\mathcal{T}_Y^\wedge$ is equivalent to the Verdier quotient $\mathcal{T}/(\mathcal{T}_Y)^\perp$ of $\mathcal{T}$ by the localizing $\otimes$-ideal $(\mathcal{T}_Y)^\perp$ and in particular $\mathcal{T}_Y^\wedge$ inherits from $\mathcal{T}$ the structure of a tensor-triangulated category making the localization functor $\mathcal{T} \twoheadrightarrow \mathcal{T}_Y^\wedge$ into a tt-functor.

Let us unpack Recollection 2.4 explicitly in our setting.

2.6. *Remark.* The generalized Dwyer–Greenlees equivalence $\mathcal{T}_Y \cong (\mathcal{T}_Y)^{\perp\perp}$ is given here by $[\mathbb{e}_Y, -]\colon \mathcal{T}_Y \xrightarrow{\sim} (\mathcal{T}_Y)^{\perp\perp}$ with inverse given by $\mathbb{e}_Y \otimes -$. In other words, the top part of (2.3) can be written in either of the following equivalent forms:

$$
\begin{array}{ccc}
\mathcal{T}_Y = \mathbb{e}_Y \otimes \mathcal{T} & \underset{\mathbb{e}_Y\otimes -}{\overset{[\mathbb{e}_Y,-]}{\underset{\cong}{\rightleftarrows}}} & (\mathcal{T}_Y)^{\perp\perp} = [\mathbb{e}_Y, \mathcal{T}] \\
\text{incl}\downarrow \ \uparrow\!\!\uparrow \mathbb{e}_Y\otimes - \ \downarrow [\mathbb{e}_Y,-] & & \mathbb{e}_Y\otimes - \downarrow \ \uparrow\!\!\uparrow [\mathbb{e}_Y,-] \ \downarrow \text{incl} \\
\mathcal{T} & = & \mathcal{T} \\
\uparrow \text{incl} & & \uparrow \text{incl} \\
(\mathcal{T}_Y)^\perp & = & (\mathcal{T}_Y)^\perp
\end{array}
\tag{2.7}
$$

and the localization $\mathcal{T} \twoheadrightarrow \mathcal{T}/(\mathcal{T}_Y)^\perp$ mentioned in Definition 2.5 is simply the tt-functor $[\mathbb{e}_Y, -]\colon \mathcal{T} \twoheadrightarrow (\mathcal{T}_Y)^{\perp\perp}$ on the right-hand side of (2.7). It can equivalently be realized via the functor $\mathbb{e}_Y \otimes -\colon \mathcal{T} \twoheadrightarrow \mathcal{T}_Y$ on the left-hand side of (2.7). Note that the composite $\mathcal{T} \twoheadrightarrow \mathcal{T}/(\mathcal{T}_Y)^\perp \rightarrowtail \mathcal{T}$ of the localization followed by its *right* adjoint will always be $[\mathbb{e}_Y, -]\colon \mathcal{T} \to \mathcal{T}$ in both formulations! This is one reason why people prefer the latter as the concrete realization of the Y-completion of Definition 2.5:

$$
\begin{array}{rcccl}
(-)_Y^\wedge := [\mathbb{e}_Y, -] & : & \mathcal{T} & \twoheadrightarrow & \mathcal{T}_Y^\wedge := (\mathcal{T}_Y)^{\perp\perp} \\
 & & t & \longmapsto & \hat{t}_Y := [\mathbb{e}_Y, t].
\end{array}
\tag{2.8}
$$

Another reason to use $[\mathbb{e}_Y, -]$ is that if we write the object $\mathbb{e}_Y$ as a sequential homotopy colimit of objects k_n in $\mathcal{T}_Y^c$ then $[\mathbb{e}_Y, t]$ is a homotopy *limit* of the objects $[k_n, t] \cong k_n^\vee \otimes t$ in $\mathcal{T}_Y$. (See for instance Corollary 4.10.) This gives a good 'completion vibe' to $(-)_Y^\wedge$: The fact that $[\mathbb{e}_Y, t]$ is a limit of the Y-torsion objects $k_n^\vee \otimes t$ is reminiscent of the way I-adic completion $\hat{M}$ is the limit of the I-primary torsion objects M/I^n.

For the torsion part, the localization functor $\mathcal{T} \twoheadrightarrow \mathcal{T}_Y$ is simply $\mathbb{e}_Y \otimes -$ and the tensor in $\mathcal{T}_Y$ is simply the ambient one in $\mathcal{T}$ (making sure to use the correct unit: $\mathbb{e}_Y$). On the other hand, the tensor in $(\mathcal{T}_Y)^{\perp\perp}$ is a *completed tensor*, meaning that one needs to apply $[\mathbb{e}_Y, -]$ after tensoring Y-complete objects in $\mathcal{T}$. This completed tensor is nothing remarkable though. The equivalence $[\mathbb{e}_Y, -]\colon \mathcal{T}_Y \xrightarrow{\sim} (\mathcal{T}_Y)^{\perp\perp}$ preserves the tensor. After all, we are just discussing the quotient of $\mathcal{T}$ by a $\otimes$-ideal: $\mathcal{T}_Y^\wedge \cong \mathcal{T}/\mathcal{T}_Y^\perp$.

2.9. *Remark.* By Definition 2.5, an object $t \in \mathcal{T}$ is tt-complete (along Y) when it belongs to $(\mathcal{T}_Y)^{\perp\perp}$, that is, when the unit map $t \to \hat{t}_Y = [\mathbb{e}_Y, t]$ is an isomorphism in $\mathcal{T}$. This is equivalent to the vanishing of $[\mathbb{f}_Y, t]$ in $\mathcal{T}$, by applying $[-, t]$ to (2.2).

2.10. *Remark.* In view of Remark 2.6, everything 'tt' that we discuss in this article, like compact or dualizable objects, can be done interchangeably with the tt-category $(\mathcal{T}_Y)^{\perp\perp}$ or the tt-equivalent $\mathcal{T}_Y$. Following tradition, we tend to *state* our results about completion in terms of $\mathcal{T}_Y^{\wedge} = (\mathcal{T}_Y)^{\perp\perp}$. However we often *prove* them in the more explicit $\mathcal{T}_Y$.

2.11. *Remark.* The analogy between $\operatorname{holim}_n k_n^\vee \otimes t$ and $\lim_n M/I^n M$ in Remark 2.6 may have caused more cognitive harm than good over the years, in particular in creating the illusion that $\mathcal{T}_Y^{\wedge}$ recovers the usual I-adic completion. Let us clear this up.

2.12. *Remark.* By construction, $\mathcal{T}_Y$ is compactly generated by $\mathcal{T}_Y^c$, which consists of dualizable objects as these are the images under the tt-functor $e_Y \otimes -\colon \mathcal{T} \to \mathcal{T}_Y$ of $\mathcal{T}_Y^c \subseteq \mathcal{T}^d$. But its $\otimes$-unit $\mathbb{1}_{\mathcal{T}_Y} = e_Y$ might not be compact.

The only option for e_Y to be compact is for Y to be open. Indeed, if e_Y is compact in $\mathcal{T}_Y$ then $e_Y \in (\mathcal{T}_Y)^c = (\mathcal{T}^c)_Y \subseteq \mathcal{T}^c$ and therefore the triangle (2.2) belongs to $\mathcal{T}^c$ and the right-idempotent f_Y is compact as well; the relation $e_Y \otimes f_Y = 0$ gives $\operatorname{Spc}(\mathcal{T}^c) = \operatorname{supp}(e_Y) \sqcup \operatorname{supp}(f_Y) = Y \sqcup Y'$, where $Y' = \operatorname{supp}(f_Y)$ is closed; hence Y is open. Actually, more is true in that case. We must have $\mathbb{1} \cong e_Y \oplus f_Y$ and the whole category $\mathcal{T} = \mathcal{T}_Y \times \mathcal{T}_{Y'} = \mathcal{T}_Y^{\wedge} \times \mathcal{T}_{Y'}$ decomposes, making the entire story rather trivial. Compare with Stevenson [Ste13].

Of course, the same is true for the tt-equivalent category $\mathcal{T}_Y^{\wedge}$ of Y-complete objects. See Remark 2.10. The tt-category $\mathcal{T}_Y^{\wedge}$ is compactly generated and its compact objects are dualizable but its $\otimes$-unit is compact only if $\mathcal{T}$ splits as above.

2.13. *Example.* For instance, if $\mathcal{T} = \mathrm{D}(R)$ is the derived category of a commutative ring R, the units of $\mathcal{T}_Y$ and of $\mathcal{T}_Y^{\wedge}$ are compact (inside those categories) if and only if $R \simeq R_1 \times R_2$ is the product of two rings, etc. In particular, the derived category $\mathrm{D}(\hat{\mathbb{Z}}_p)$ of the p-adics cannot be obtained as $\mathrm{D}(\mathbb{Z})^{\wedge}_{(p)}$, nor as $\mathrm{D}(\mathbb{Z}_{(p)})^{\wedge}_{(p)}$, and not even as the p-complete objects $\mathrm{D}(\hat{\mathbb{Z}}_p)^{\wedge}_{(p)}$ over $\hat{\mathbb{Z}}_p$ itself! It really never works!

2.14. *Remark.* In conclusion, calling $\mathcal{T}_Y^{\wedge}$ the Y-completion of $\mathcal{T}$ is a little bit of a misnomer. We revisit this topic in Remark 5.4.

We now want to assume that $\mathcal{T}$ admits a tt-functor to another tt-category $\mathcal{S}$ that plays the role of $\mathcal{S} = \mathrm{D}(\hat{R})$ when $\mathcal{T} = \mathrm{D}(R)$. Let us recall the basics.

2.15. *Recollection.* Consider a tensor-exact functor $f^*\colon \mathcal{T} \to \mathcal{S}$ between two 'big' tt-categories. Following [HPS97], we say that f^* is a *geometric* functor if it preserves arbitrary coproducts. Since f^* is a tensor functor, it preserves dualizable objects, hence compact objects. By Neeman [Nee96], f^* admits a right adjoint $f_*\colon \mathcal{S} \to \mathcal{T}$ which preserves coproducts and the adjunction $f^* \dashv f_*$ satisfies a projection formula. See details in [BDS16].

Denote by $f := \operatorname{Spc}(f^*)\colon \operatorname{Spc}(\mathcal{S}^c) \to \operatorname{Spc}(\mathcal{T}^c)$ the map induced by the tt-functor $f^*\colon \mathcal{T}^c \to \mathcal{S}^c$ on small objects. Let $Y' = f^{-1}(Y)$ be the preimage of our Thomason subset Y; this Y' is also a Thomason subset and $f^*\colon \mathcal{T} \to \mathcal{S}$ maps $\mathcal{T}_Y$ into $\mathcal{S}_{Y'}$. Moreover $f^*(e_Y) \cong e_{Y'}$ by [BF11, Theorem 6.3].

2.16. **Theorem.** *Let $f^*\colon \mathcal{T} \to \mathcal{S}$ be a geometric tt-functor and let $Y' = f^{-1}(Y)$ as in Recollection 2.15. Suppose that $\mathcal{S}$ satisfies the following two hypotheses:*

($\mathcal{S}$1) *The unit of $\mathcal{S}$ is tt-complete along Y', that is, $\mathbb{1} \cong \hat{\mathbb{1}}_{Y'}$ in $\mathcal{S}$.*

($\mathcal{S}$2) *The right adjoint $f_*\colon \mathcal{S} \to \mathcal{T}$ is conservative, or equivalently, $\mathcal{S}^c$ is generated, as a thick triangulated subcategory, by the image $f^*(\mathcal{T}^c)$.*

Then the following three properties are equivalent:

(i) *There exists an isomorphism* $\hat{\mathbb{1}}_Y \simeq f_*(\mathbb{1}_{\mathcal{S}})$ *of ring objects in* $\mathcal{T}$.

(ii) *The functor* $f^*\colon \mathcal{T} \to \mathcal{S}$ *is fully faithful on* $\mathcal{T}_Y$.

(iii) *The functor* $f^*\colon \mathcal{T} \to \mathcal{S}$ *restricts to an equivalence* $\mathcal{T}_Y \xrightarrow{\sim} \mathcal{S}_{Y'}$.

Suppose furthermore that Y *is closed and that* $k \in \mathcal{T}^c$ *is a compact object such that* $\mathrm{supp}(k) = Y$. *Then the above three properties are equivalent to:*

(iv) *The unit* $k \to f_*(f^*(k))$ *is an isomorphism.*

2.17. *Remark.* The proof will show that (i)⇒(ii) and (iii)⇒(ii) hold without Hypotheses ($\mathcal{S}1$) and ($\mathcal{S}2$), and so does (ii)⇔(iv) when Y is closed. Hypothesis ($\mathcal{S}1$) is used to prove (iii)⇒(i) and Hypothesis ($\mathcal{S}2$) is used to prove (ii)⇒(iii).

Proof of Theorem 2.16. Suppose (i): We have a commutative triangle in $\mathcal{T}$

$$\begin{array}{ccc} & \mathbb{1} & \\ {}^{\eta^{\otimes}}\swarrow & & \searrow^{\eta_{\mathbb{1}}} \\ \hat{\mathbb{1}}_Y & \xrightarrow{\ \simeq\ } & f_*(\mathbb{1}_{\mathcal{S}}) \end{array}$$

where $\eta^{\otimes}\colon \mathbb{1} \to [e_Y, e_Y]$ is the unit of the tensor-hom adjunction and $\eta_{\mathbb{1}}$ is the unit of the $f^* \dashv f_*$ adjunction at $\mathbb{1}$. Since the former becomes an isomorphism under $e_Y \otimes -$ so does the latter: $e_Y \otimes \eta_{\mathbb{1}}$ is an isomorphism. By the identification of the projection formula, the unit $\eta_t\colon t \to f_* f^*(t) \cong t \otimes f_*(\mathbb{1}_{\mathcal{S}})$ is nothing but $t \otimes \eta_{\mathbb{1}}$. We have shown that η_t is an isomorphism for $t = e_Y$ and therefore for all $t \in e_Y \otimes \mathcal{T} = \mathcal{T}_Y$. This proves that $(f^*)_{|\mathcal{T}_Y}\colon \mathcal{T}_Y \to \mathcal{S}$ is fully faithful, as claimed in (ii).

Suppose that f^* is fully faithful on $\mathcal{T}_Y = \mathrm{Loc}(\mathcal{T}^c_Y)$ as in (ii). The essential image of $(f^*)_{|\mathcal{T}_Y}\colon \mathcal{T}_Y \to \mathcal{S}$ is then generated as a localizing subcategory by $f^*(\mathcal{T}^c_Y)$. Thanks to Hypothesis ($\mathcal{S}2$) the latter is a tensor-ideal in $\mathcal{S}^c$ and it coincides with $\mathcal{S}^c_{Y'}$ by [BF11, Theorem 6.3] again. In short, $f^*(\mathcal{T}_Y) = \mathrm{Loc}(\mathcal{S}^c_{Y'}) = \mathcal{S}_{Y'}$ as claimed in (iii).

Now suppose (iii). We have a diagram

$$\begin{array}{ccc} \mathcal{T}_Y & \xrightarrow[\cong]{G := (f^*)_{|\mathcal{T}_Y}} & \mathcal{S}_{Y'} \\ {\scriptstyle \mathrm{incl}}\downarrow \ \uparrow{\scriptstyle e_Y \otimes -} & & {\scriptstyle \mathrm{incl}}\downarrow \ \uparrow{\scriptstyle e_{Y'} \otimes -} \\ \mathcal{T} & \xrightarrow{\ f^*\ } & \mathcal{S} \end{array}$$

where the square with inclusions commutes by definition of G. The other square also commutes, *i.e.*, we have an isomorphism of tt-functors from $\mathcal{T}$ to $\mathcal{S}_{Y'}$

$$G \circ (e_Y \otimes -) \cong (e_{Y'} \otimes -) \circ f^*$$

since f^* is a tensor functor and $e_{Y'} \cong f^*(e_Y)$. Taking right adjoints in the latter isomorphism of tensor functors and using that G^{-1} is the right adjoint of G, we obtain an isomorphism of lax-monoidal functors $\mathcal{S}_{Y'} \to \mathcal{T}$:

$$[e_Y, -] \circ G^{-1} \cong f_* \circ [e_{Y'}, -].$$

Evaluating at the unit $e_{Y'} = G(e_Y)$ of $\mathcal{S}_{Y'}$ we obtain an isomorphism of ring objects

$$[e_Y, e_Y] \cong f_*([e_{Y'}, e_{Y'}])$$

in $\mathcal{T}$. By Hypothesis ($\mathcal{S}1$), the object $[e_{Y'}, e_{Y'}] = \hat{\mathbb{1}}_{Y'}$ on the right-hand side agrees with $\mathbb{1}_{\mathcal{S}}$ and this gives us (i).

Finally, Condition (iv) is equivalent to the unit $\eta_t\colon t \to f_*(f^*(t))$ being an isomorphism for all t in the localizing $\otimes$-ideal of $\mathcal{T}$ generated by k, which is precisely $\mathcal{T}_Y$. Thus (iv) is equivalent to $f^*\colon \mathcal{T}_Y \to \mathcal{S}$ being fully faithful as in (ii). $\square$

2.18. *Remark.* In Theorem 2.16, we cannot drop Hypothesis ($\mathcal{S}1$) for a trivial reason: Take $f^* = \mathrm{Id}_{\mathcal{T}}$ in a tt-category $\mathcal{T}$ whose unit is not tt-complete along Y; in that case (i) fails but the other properties are trivially true.

We cannot drop Hypothesis ($\mathcal{S}2$) either. To see this, consider the easy example of $Y = \mathrm{Spc}(\mathcal{T}^c)$ itself, in which case $\mathbb{e}_Y = \mathbb{1}$ and tt-completeness is a void condition. In that situation, a mere isomorphism $f_*(\mathbb{1}_{\mathcal{S}}) \simeq \mathbb{1}_{\mathcal{T}}$ does not force f^* to be an equivalence, as for instance with $f^*\colon \mathrm{D}(R) \to \mathrm{D}(\mathbb{P}^1_R)$. In that example, (i), (ii) and (iv) hold true but (iii) fails.

2.19. *Remark.* Suppose that Y is closed in Theorem 2.16. Since (i)–(iii) are independent of k we see that if Property (iv) holds for one choice of k in $\mathcal{T}^c_Y$ then it holds for all other choices. In other words, (iv) only depends on Y.

2.20. **Corollary.** *With notation as in Theorem 2.16, let $\hat{f}^*\colon \mathcal{T}^\wedge_Y \to \mathcal{S}^\wedge_{Y'}$ be the tt-functor $t \mapsto (f^*(t))^\wedge_{Y'} = [\mathbb{e}_{Y'}, f^*(t)]$ obtained by Y'-completion of f^*. Then the equivalent conditions of Theorem 2.16 are furthermore equivalent to*

(ii') *The functor $\hat{f}^*\colon \mathcal{T}^\wedge_Y \to \mathcal{S}^\wedge_{Y'}$ is fully faithful.*

(iii') *The functor $\hat{f}^*\colon \mathcal{T}^\wedge_Y \to \mathcal{S}^\wedge_{Y'}$ is an equivalence.*

Proof. In view of Remark 2.6, we can transport results from $\mathcal{T}_Y$ to $\mathcal{T}^\wedge_Y$. We claim that the following diagram of tt-functors commutes

$$\begin{array}{ccc} \mathcal{T}_Y = \mathbb{e}_Y \otimes \mathcal{T} & \xleftarrow[\cong]{\mathbb{e}_Y \otimes -} & (\mathcal{T}_Y)^{\perp\perp} = \hat{\mathcal{T}}_Y \\ {\scriptstyle f^*}\downarrow & & \downarrow{\scriptstyle \hat{f}^*} \\ \mathcal{S}_{Y'} = \mathbb{e}_{Y'} \otimes \mathcal{S} & \xrightarrow[\cong]{[\mathbb{e}_{Y'}, -]} & (\mathcal{S}_{Y'})^{\perp\perp} = \hat{\mathcal{S}}_{Y'} \end{array}$$

where the horizontal tt-equivalences are the ones of (2.7). Commutativity uses $\mathbb{e}_{Y'} \cong f^*(\mathbb{e}_Y)$ and direct computation $[\mathbb{e}_{Y'}, f^*(\mathbb{e}_Y \otimes t)] \cong [\mathbb{e}_{Y'}, f^*(\mathbb{e}_Y) \otimes f^*(t)] \cong [\mathbb{e}_{Y'}, \mathbb{e}_{Y'} \otimes f^*(t)] \cong [\mathbb{e}_{Y'}, f^*(t)]$ which is the formula for $\hat{f}^* = (-)^\wedge_{Y'} \circ f^*$. So we can rephrase Properties (ii) and (iii) about $f^*\colon \mathcal{T}_Y \to \mathcal{S}_{Y'}$ in terms of $\hat{f}^*$. $\square$

At this stage, the reader probably wants some explicit example. Here it comes.

3. Ring completion and Koszul complexes

We now turn our attention to commutative algebra. Let R be a commutative ring, which need not be noetherian. We consider the 'big' tt-category $\mathcal{T} = \mathrm{D}(R)$. We write $I \subset R$ for a finitely generated ideal and $Y = V(I)$ for the associated Thomason closed subset of $\mathrm{Spec}(R) \cong \mathrm{Spc}(\mathcal{T}^c)$. We recall the idempotent triangle (2.2) $\mathbb{e}_Y \to \mathbb{1} \to \mathbb{f}_Y \to \Sigma\mathbb{e}_Y$ of Recollection 2.1. (See more about this in Remark 3.9.)

3.1. *Remark.* A tt-completion of $\mathcal{T}$ in the sense of Section 2 depends on the choice of a tt-ideal of compact objects, which is necessarily of the form $\mathcal{T}^c_Y$ for a unique *Thomason* subset Y of $\mathrm{Spc}(\mathcal{T}^c) \cong \mathrm{Spec}(R)$; see [Tho97]. If, following tradition, one also wants Y to be *closed* then Y will be given by $Y = V(I)$ for some *finitely generated* ideal I as above. In particular, non-finitely generated ideals I are never part of the tt-completion picture, even for non-noetherian rings R.

3.2. *Recollection.* It is well-known that $\mathcal{T}_Y = \mathrm{Loc}(\mathrm{D}^{\mathrm{perf}}(R)_Y)$ coincides with the subcategory $\mathrm{D}_Y(R)$ of complexes of R-modules supported on Y, that is, whose restriction to the open complement $U := \mathrm{Spec}(R) \smallsetminus Y$ is acyclic. See [Nee92]. The tt-complete complexes of Definition 2.5, *i.e.*, those complexes belonging to

$$\mathcal{T}_Y^{\wedge} = (\mathcal{T}_Y)^{\perp\perp} = \left\{ t \in \mathrm{D}(R) \,\middle|\, t \cong [e_Y, t] \right\}$$

are often called *derived I-complete* or *derived Y-complete*. The category $\mathcal{T}_Y^{\wedge} = \mathrm{D}(R)_Y^{\wedge}$ is often denoted $\mathrm{D}(R)_{\mathrm{comp}}$ or $\mathrm{D}(R;I)_{\mathrm{comp}}$ in the literature. Finally, recall the tt-equivalence $\mathrm{D}_Y(R) \cong \mathrm{D}(R)_Y^{\wedge}$ of Remark 2.6.

3.3. *Remark.* There is a vast literature comparing classical I-adic completion $\hat{M}_I = \lim_n M/I^n M$, its derived functors, and tt-completion $[e_Y, -]$, for R-modules and for complexes. It is rich and technical and we cannot do it justice here. The interested reader will find a more detailed account in the Stack Project [Sta20, Section 15.91], referring to earlier work such as [GM92, DG02, PSY14], to which one could add [Sch03, Pos23] for the derived functors of completion. We recall some facts.

3.4. *Terminology.* An R-module M is called *classically I-complete* if $M \cong \hat{M}_I$. It is called *I-separated* if $\cap_n I^n M = 0$.

3.5. **Proposition** ([Sta20, Proposition 15.91.5]). *An R-module M is classically I-complete if and only if it is I-separated and derived I-complete (Recollection 3.2). In particular, if $R \cong \lim_n R/I^n$ is classically I-complete then $\mathbb{1} \cong [e_Y, \mathbb{1}] = \hat{\mathbb{1}}_Y$, that is, the unit of $\mathrm{D}(R)$ is tt-complete along Y.* □

Let us make the principal case explicit for the benefit of the reader.

3.6. *Example.* Let $s \in R$ and $Y = V(s)$. In that case, localization away from $V(s)$ amounts to passing from R to $R[1/s]$ and the right idempotent $f_s = f_Y$ is simply $\mathbb{1}[1/s] = \mathrm{hocolim}(\mathbb{1} \xrightarrow{s} \mathbb{1} \xrightarrow{s} \cdots)$, that is, the complex

$$f_s = \Big(\cdots \to 0 \to \coprod_{n\in\mathbb{N}} R \xrightarrow{\mathrm{id}-\tau} \coprod_{n\in\mathbb{N}} R \to 0 \to \cdots \Big) \tag{3.7}$$

concentrated in homological degrees one and zero, where τ maps the n-th copy of R to the $(n+1)$-st copy of R, via multiplication by s. (To be clear $\mathbb{N} = \{0, 1, 2, \ldots\}$.) In other words, this complex is a projective resolution of $R[1/s]$. The canonical map $\mathbb{1} \to f_s$ is the inclusion of R in the 0-th copy of R in degree zero. Consequently $e_{V(s)}$ can also be made explicit: It is the homotopy fiber of this map $\mathbb{1} \to f_s$.

For any module M, for instance $M = R$, we know that M is derived s-complete if and only if $[f_s, M] = 0$. Since we described f_s as a bounded complex of projectives in (3.7), we can compute $[f_s, M]$ by applying $\mathrm{Hom}_R(-, M)$ degreewise. We get

$$[f_s, M] = \Big(\cdots \to 0 \to \prod_{n\in\mathbb{N}} M \xrightarrow{(\mathrm{id}-\tau)^*} \prod_{n\in\mathbb{N}} M \to 0 \to \cdots \Big) \tag{3.8}$$

concentrated in homological degrees zero and -1; the map $(\mathrm{id} - \tau)^* \colon M^{\mathbb{N}} \to M^{\mathbb{N}}$ is easily checked to send $(x_n)_{n\in\mathbb{N}}$ to $(x_n - s x_{n+1})_{n\in\mathbb{N}}$.

In conclusion, M is derived s-complete if and only if the map $(\mathrm{id} - \tau)^*$ in (3.8) is an isomorphism. It is easy to verify that this holds if M is classically I-complete. In fact, this holds if M is classically I-complete for any ideal I that contains s. (For surjectivity of $(1 - \tau)^*$, show that every $(y_n)_n$ is the image of $(x_n)_n$ defined by $x_n = \sum_{\ell=0}^{\infty} s^{\ell} y_{\ell+n}$. For injectivity, use $\cap_n s^n M \subseteq \cap_n I^n M = 0$.)

3.9. *Remark.* We gave explicit formulas for the idempotents $\mathbb{e}_{V(s)}$ and $\mathbb{f}_{V(s)}$ in the monogenic case above. For a general ideal $I = \langle s_1, \ldots, s_r \rangle$ we have $\mathbb{e}_{V(I)} = \mathbb{e}_{V(s_1)} \otimes \cdots \otimes \mathbb{e}_{V(s_r)}$ and one can recover $\mathbb{f}_{V(I)}$ as the mapping cone of $\mathbb{e}_{V(I)} \to \mathbb{1}$. In summary, the idempotents in the triangle (2.2) can be made very concrete in $\mathcal{T} = \mathrm{D}(R)$. See also Proposition 4.9 for another description.

3.10. *Remark.* We have seen in Proposition 3.5 that classically I-complete implies derived I-complete. (We also proved it in the monogenic case at the end of Example 3.6 and one can jazz that up into a complete proof using an induction argument on the number of generators, thanks to the Mayer–Vietoris properties of idempotents [BF11, Theorem 5.18]. We leave this to the interested reader.) As can be expected, the converse is wrong: There are modules M that are derived I-complete but not classically I-complete. See Yekutieli [Yek11, Example 3.20].

3.11. *Example.* Kedlaya [Ked24, Example 6.1.4] gives a simple example over the ring $\mathcal{O} = \hat{\mathbb{Z}}_p$ of p-adics. Take $L = \{ (x_i)_i \in \mathcal{O}^{\mathbb{N}} \mid \lim_i x_i = 0 \}$ to be the $\mathcal{O}$-module of zero-converging sequences and let $M = \mathrm{Coker}(\ell\colon L \to L)$ be the cokernel of the monomorphism $\ell((x_i)_i) \coloneqq (p^i x_i)_i$. Then M is derived p-complete, as L is (*e. g.* by the criterion of Example 3.6). However, M is not p-separated: For instance, the sequence $(p^i)_i$ is a non-zero element of $\cap_n p^n M$.

There are also rings R that are derived complete but not classically complete.

3.12. *Example.* Let $\mathcal{O}$ be an s-complete ring for some $s \in \mathcal{O}$ and let M be a derived s-complete module that is not s-separated (for instance $s = p$ in $\mathcal{O} = \hat{\mathbb{Z}}_p$ and M as in Example 3.11). Define the $\mathcal{O}$-algebra $R = \mathcal{O} \oplus M$ in which M squares to zero. Then R is derived s-complete but not classically s-complete. The latter is easy, since $\cap_n s^n \cdot R$ contains $\cap_n s^n M \neq 0$ in the second factor. So R is not s-separated. To see that the derived completeness of $\mathcal{O}$ and of M over $\mathcal{O}$ implies that the square-zero extension R is derived s-complete as an R-module, we can use the discussion of Example 3.6. It suffices to show that the map $(\mathrm{id} - \tau)^*$ of (3.8) is an isomorphism $R^{\mathbb{N}} \to R^{\mathbb{N}}$. This map separates into two factors, one involving $\mathcal{O}^{\mathbb{N}}$ and one involving $M^{\mathbb{N}}$, which are both isomorphisms by the derived s-completeness of $\mathcal{O}$ and of M over $\mathcal{O}$.

3.13. *Remark.* In summary, we have seen that if the ring homomorphism $R \to \hat{R}$ is an isomorphism then so is the map $\mathbb{1} \to \hat{\mathbb{1}}_Y$ in $\mathrm{D}(R)$. We have seen that the converse fails (Example 3.12). Consequently, there is no ring isomorphism $\hat{\mathbb{1}}_Y \simeq \hat{R}$ in general. Hammering the point: For the ring of Example 3.12, the derived completion $\hat{\mathbb{1}}_Y$ is not recovering the classical I-adic completion $\hat{R}$. We want a simple formulation for when these two notions of completion agree.

3.14. *Definition.* We say that a sequence $\underline{s} = (s_1, \ldots, s_r)$ in R is *Koszul-complete* if the canonical map $R \to \hat{R} = \hat{R}_I$ for $I = \langle s_1, \ldots, s_r \rangle$ induces a quasi-isomorphism

$$\mathrm{kos}(\underline{s}) \to \mathrm{kos}(\underline{s}) \otimes_R \hat{R}$$

where $\mathrm{kos}(\underline{s}) = \otimes_{i=1}^r \mathrm{cone}(s_i\colon \mathbb{1} \to \mathbb{1})$ is the usual Koszul complex. Note that the above tensor could equivalently be a derived tensor since the Koszul complex is perfect. Once we prove in Remark 3.21 that this property only depends on $Y = V(I)$, not on the choice of $\underline{s}$, we could also simply say that Y is *Koszul-complete*.

3.15. *Remark.* The induced map in homology $\mathrm{H}_i(\mathrm{kos}_R(\underline{s})) \to \mathrm{H}_i(\mathrm{kos}_R(\underline{s}) \otimes_R \hat{R}) \cong \mathrm{H}_i(\mathrm{kos}_{\hat{R}}(\underline{s}))$ is an isomorphism for $i = 0$, since both sides are R/I in that case. Thus $\underline{s}$ being Koszul-complete means that these maps are isomorphisms for $i = 1, \ldots, r$.

3.16. *Example.* This property does not hold in general, already for $r = 1$. For instance, it would fail for the ring of Example 3.12. One can give a simpler example in the same vein. Let p be a prime and consider the $\mathbb{Z}_{(p)}$-algebra $R = \mathbb{Z}_{(p)} \oplus (\mathbb{Q}/\mathbb{Z}_{(p)})$ with the second term squaring to zero. We use $s_1 = p$. Observe that $\mathrm{H}_1(\mathrm{kos}(p)) = \mathrm{Ker}(R \xrightarrow{p} R)$ is non-zero; it is $\mathbb{Z}/p$ embedded in $\mathbb{Q}/\mathbb{Z}_{(p)}$ via $1/p$. Since $R/p^n \cong \mathbb{Z}/p^n$ for all $n \geq 1$, the completion $\hat{R}_{(p)} \cong \hat{\mathbb{Z}}_{(p)}$ is a domain. So $\mathrm{kos}(p) \otimes_R \hat{R}_{(p)}$ has trivial H_1 and the map $\mathrm{kos}(s) \to \mathrm{kos}(s) \otimes_R \hat{R}_{(p)}$ is not injective on H_1.

This notion of Koszul-completeness is only relevant in the non-noetherian world:

3.17. **Proposition.** *Suppose that the ring R is noetherian. Then every sequence $\underline{s} = (s_1, \ldots, s_r)$ is Koszul-complete in the sense of Definition 3.14.*

Proof. Although probably well-known, we give a proof for completeness. Write $I = \langle s_1, \ldots, s_r \rangle$ and recall that because R is noetherian, we have $M \otimes_R \hat{R} \cong \hat{M}_I$ for every finitely generated R-module M. We can apply this to one of our homology groups $M = \mathrm{H}_i(\mathrm{kos}(\underline{s}))$ for $1 \leq i \leq r$, which is finitely generated because R is noetherian. On the other hand, the homology of the Koszul complex is killed by $s_1, \ldots, s_r$, since the maps $s_i \cdot -$ are homotopic to zero on $\mathrm{cone}(s_i)$ hence on the Koszul complex itself. So our R-module $M = \mathrm{H}_i(\mathrm{kos}(\underline{s}))$ satisfies $IM = 0$ and thus $\hat{M}_I \cong M$. Therefore $\mathrm{H}_i(\mathrm{kos}(\underline{s})) = M \cong \hat{M}_I \cong M \otimes_R \hat{R} = \mathrm{H}_i(\mathrm{kos}(\underline{s})) \otimes_R \hat{R} \cong \mathrm{H}_i(\mathrm{kos}(\underline{s}) \otimes_R \hat{R})$ where the last isomorphism holds because $\hat{R}$ is R-flat in the noetherian case. □

3.18. *Example.* If $\underline{s}$ is regular and $\hat{R}$ is flat over R then $\mathrm{kos}(s) \to R/I$ is a quasi-isomorphism, hence $\mathrm{kos}(s) \otimes \hat{R} \to R/I \otimes \hat{R}$ is an isomorphism by flatness. But $R/I \otimes_R \hat{R} \cong \hat{R}/I \cong R/I$ so s is Koszul-complete, even without R being noetherian.

3.19. *Remark.* A general fact about classical completion with respect to a finitely generated ideal I is that the completion is itself always complete; that is, the completion $\hat{R}$ is always classically I'-complete for the extended (finitely generated) ideal $I' = I\hat{R}$; see [Sta20, Lemma 05GG]. This does not require R to be noetherian.

3.20. **Theorem.** *Let R be a commutative ring, $I \subseteq R$ a finitely generated ideal and $Y = V(I)$. Let $f\colon \mathrm{Spec}(\hat{R}) \to \mathrm{Spec}(R)$ and let $Y' = f^{-1}(Y) = V(I')$ be the preimage of Y. (See Remark 3.19.) Then the following conditions are equivalent:*

(i) *There exists an isomorphism of ring objects $[\mathbb{e}_Y, \mathbb{1}] \simeq \hat{R}$ in $\mathrm{D}(R)$.*

(ii) *The functor $f^*\colon \mathrm{D}(R) \to \mathrm{D}(\hat{R})$ restricts to an equivalence $\mathrm{D}_Y(R) \xrightarrow{\sim} \mathrm{D}_{Y'}(\hat{R})$.*

(iii) *The completed extension-of-scalars $\hat{f}^*\colon \mathrm{D}(R)^\wedge_Y \to \mathrm{D}(\hat{R})^\wedge_{Y'}$ is an equivalence.*

(iv) *Every sequence $\underline{s}$ that generates I is Koszul-complete (Definition 3.14).*

(v) *There exists a Koszul-complete sequence $\underline{s}$ that generates I.*

Proof. It suffices to apply Theorem 2.16 and Corollary 2.20 to $\mathcal{T} = \mathrm{D}(R)$ and $\mathcal{S} = \mathrm{D}(\hat{R})$ and $f^*\colon \mathcal{T} \to \mathcal{S}$ the extension-of-scalars and to the object $k = \mathrm{kos}(\underline{s})$, which generates $\mathcal{T}^c_Y$ by [Tho97]. The two hypotheses ($\mathcal{S}1$) and ($\mathcal{S}2$) hold. The latter is obvious since $\hat{R} = f^*(R)$ generates $\mathcal{S}^c = \mathrm{D}^{\mathrm{perf}}(\hat{R})$. The former holds by Proposition 3.5 since $\hat{R}$ is I'-complete, where $I' = I\hat{R}$ is finitely generated (Remark 3.19). □

3.21. *Remark.* Following up on Remark 2.19, we see from Theorem 3.20 that once we know that an ideal I is generated by a Koszul-complete sequence then every other sequence of generators is automatically Koszul-complete. In fact, the same holds for any sequence defining the same closed subset Y.

3.22. **Corollary.** *If R is noetherian then we have tt-equivalences on Y-torsion $\mathrm{D}_Y(R) \xrightarrow{\sim} \mathrm{D}_{Y'}(\hat{R})$ and on Y-complete $\mathrm{D}(R)^{\wedge}_{Y} \simeq \mathrm{D}(\hat{R})^{\wedge}_{Y'}$ subcategories.*

Proof. This is an immediate consequence of Proposition 3.17 and Theorem 3.20. □

4. Dualizable objects in the complete case

We keep our setup: R is a commutative ring and $Y \coloneqq V(I) \subset \mathrm{Spec}(R)$ for a finitely generated ideal I. The goal of this technical section is to characterize the dualizable objects in the derived category $\mathrm{D}_Y(R)$ of complexes supported on Y, when $R \cong \hat{R}$ is classically I-adically complete. It will be achieved in Theorem 4.17. Most of the discussion holds for any commutative ring R.

4.1. *Remark.* We use homological indexing for complexes and view R-modules as complexes concentrated in degree zero. We denote the (left-derived) tensor product in $\mathcal{T} = \mathrm{D}(R)$ by $\otimes$ and write $\otimes_R$ for the ordinary tensor product of R-modules. Recall that if t and u are right-bounded complexes, say, t belongs to $\mathrm{D}_{\geq a}(R) := \big\{\, t \in \mathrm{D}(R) \,\big|\, \mathrm{H}_i(t) = 0 \text{ for all } i < a \,\big\}$ and $u \in \mathrm{D}_{\geq b}(R)$, then $t \otimes u \in \mathrm{D}_{\geq a+b}(R)$ is also right-bounded and the *rightmost* homology modules satisfy

$$\mathrm{H}_{a+b}(t \otimes u) \cong \mathrm{H}_a(t) \otimes_R \mathrm{H}_b(u)$$

by the usual canonical truncations and right-exactness of $\otimes_R$.

4.2. *Recollection.* Let $\cdots \to t_{n+1} \xrightarrow{f_{n+1}} t_n \xrightarrow{f_n} t_{n-1} \to \cdots$ be a sequence of morphisms in a triangulated category $\mathcal{T}$, indexed by $n \in \mathbb{N}$. If $u \in \mathcal{T}$ is an object such that every map $\mathrm{Hom}_{\mathcal{T}}(\Sigma u, f_n)\colon \mathrm{Hom}_{\mathcal{T}}(\Sigma u, t_n) \to \mathrm{Hom}_{\mathcal{T}}(\Sigma u, t_{n-1})$ is surjective, then there is a canonical isomorphism

$$\mathrm{Hom}_{\mathcal{T}}(u, \mathrm{holim}_n\, t_n) \cong \lim_n \mathrm{Hom}_{\mathcal{T}}(u, t_n).$$

This a standard Mittag-Leffler argument, combined with applying the homological functor $\mathrm{Hom}_{\mathcal{T}}(u, -)$ to the exact triangle

$$\mathrm{holim}_n\, t_n \to \prod_n t_n \xrightarrow{1-\tau} \prod_n t_n \to \Sigma\, \mathrm{holim}_n\, t_n \tag{4.3}$$

that defines the homotopy limit. (As usual, τ is the unique map to the product such that $\mathrm{proj}_n \circ \tau = f_{n+1} \circ \mathrm{proj}_{n+1}$ for all $n \in \mathbb{N}$.)

We also remind the reader of the standard trick of lifting idempotents:

4.4. **Proposition.** *We can lift finitely generated projective modules, up to isomorphism, along the ring quotient $\hat{R} \twoheadrightarrow R/I$.*

Proof. It suffices to show that every idempotent matrix in $\mathrm{M}_\ell(R/I)$ lifts to an idempotent matrix in $\mathrm{M}_\ell(\hat{R})$, for any $\ell \geq 1$. For each $n \geq 1$ the ring homomorphism

$$\mathrm{M}_\ell(R/I^{n+1}) \twoheadrightarrow \mathrm{M}_\ell(R/I^n)$$

has nilpotent kernel, thus idempotents lift. Hence we can construct the desired idempotent in $\lim_n \mathrm{M}_\ell(R/I^n) \cong \mathrm{M}_\ell(\lim_n R/I^n) = \mathrm{M}_\ell(\hat{R})$. □

After these recollections, we gather in one place all the notation we shall use.

4.5. *Notation.* Take $s_1, \dots, s_r \in R$ that generate I. Let $n \geq 1$.

(1) Write $I^{(n)}$ for $\langle s_1^n, \dots, s_r^n \rangle$. Of course $I^{(1)} = I$ and $I^{(n)} \subseteq I^{(n-1)}$.

(2) We denote the Koszul complex for $(s_1^n, \dots, s_r^n)$ by

$$k^{(n)} := \otimes_{i=1}^r \operatorname{cone}(s_i^n) = \otimes_{i=1}^r (\cdots \to 0 \to R \xrightarrow{s_i^n} R \to 0 \cdots)$$

with non-zero entries in nonnegative homological degrees. (Each factor lives in degrees 1 and 0 and the product ranges between degree r and 0.) We have maps $q_n \colon k^{(n)} \to k^{(n-1)}$ given on each tensor factor $\operatorname{cone}(s_i^n) \to \operatorname{cone}(s_i^{n-1})$ by

$$\begin{array}{ccccccc} \cdots 0 & \longrightarrow & R & \xrightarrow{s_i^n} & R & \longrightarrow & 0 \cdots \\ & & {\scriptstyle s_i}\downarrow & & \| & & \\ \cdots 0 & \longrightarrow & R & \xrightarrow{s_i^{n-1}} & R & \longrightarrow & 0 \cdots \end{array}$$

(3) There is a canonical map $p_n \colon k^{(n)} \to R/I^{(n)}$ given by the projection $R \twoheadrightarrow R/I^{(n)}$ in degree zero and the obvious square with the above q_n commutes:

$$\begin{array}{ccc} k^{(n)} & \xrightarrow{p_n} & R/I^{(n)} \\ {\scriptstyle q_n}\downarrow & & \downarrow{\scriptstyle \mathrm{can}} \\ k^{(n-1)} & \xrightarrow{p_{n-1}} & R/I^{(n-1)}. \end{array} \tag{4.6}$$

We define the complex $\ell^{(n)} \in \mathrm{D}(R)$ as the homotopy fiber of p_n, *i.e.*, by the exact triangle in $\mathrm{D}(R)$:

$$\ell^{(n)} \to k^{(n)} \xrightarrow{p_n} R/I^{(n)} \to \Sigma \ell^{(n)}. \tag{4.7}$$

Since $\mathrm{H}_0(k^{(n)}) = R/I^{(n)}$, the long exact sequence in homology tells us that

$$\ell^{(n)} \in \mathrm{D}_{\geq 1}(R).$$

(4) We let $\iota_n \colon \operatorname{Spec}(R/I^{(n)}) \hookrightarrow \operatorname{Spec}(R)$ denote the closed immersion associated to the ring surjection $R \twoheadrightarrow R/I^{(n)}$ and let $\iota_n^* \colon \mathrm{D}(R) \to \mathrm{D}(R/I^{(n)})$ denote the corresponding extension-of-scalars. Since its right adjoint, restriction-of-scalars $(\iota_n)_*$, preserves homology, we get via the projection formula

$$\mathrm{H}_i(\iota_n^*(t)) \cong \mathrm{H}_i((\iota_n)_* \iota_n^*(t)) \cong \mathrm{H}_i((R/I^{(n)}) \otimes t)$$

for every $i \in \mathbb{Z}$ and $t \in \mathrm{D}(R)$. By Remark 4.1, the tt-functor ι_n^* preserves right-boundedness: $\iota_n^*(\mathrm{D}_{\geq 0}(R)) \subseteq \mathrm{D}_{\geq 0}(R/I^{(n)})$. Moreover, when $t \in \mathrm{D}_{\geq 0}(R)$ is concentrated in non-negative degrees we obtain

$$\mathrm{H}_0(\iota_n^*(t)) \cong \mathrm{H}_0((R/I^{(n)}) \otimes t) \cong (R/I^{(n)}) \otimes_R \mathrm{H}_0(t) \cong \mathrm{H}_0(t)/I^{(n)} \cdot \mathrm{H}_0(t). \tag{4.8}$$

We now give a series of preparatory lemmas. Note that the Koszul complexes $k^{(n)}$ of Notation 4.5 (2) are dualizable and the maps $q_n \colon k^{(n)} \to k^{(n-1)}$ described in (2) correspond to maps $q_n^\vee \colon (k^{(n-1)})^\vee \to (k^{(n)})^\vee$.

4.9. **Proposition.** *The homotopy colimit of the sequence*

$$(k^{(1)})^\vee \xrightarrow{q_2^\vee} \cdots \xrightarrow{q_n^\vee} (k^{(n)})^\vee \xrightarrow{q_{n+1}^\vee} (k^{(n+1)})^\vee \to \cdots$$

in $\mathrm{D}(R)$ *is isomorphic to the left idempotent* e_Y.

Proof. This is well-known. Since $\mathbb{e}_Y = \mathbb{e}_{V(s_1)} \otimes \cdots \otimes \mathbb{e}_{V(s_r)}$, it is possible to reduce to the case $r = 1$. For $s \in R$, the right-idempotent $\mathbb{f}_{V(s)}$ is simply $R[1/s]$ in degree zero, hence $\mathbb{e}_{V(s)}$ is $(0 \to R \to R[1/s] \to 0)$ in homological degrees 0 and -1, which is easily seen to be the homotopy colimit of

$$\begin{array}{ccccccccc}
\mathrm{cone}(s)^\vee = & & \cdots 0 & \longrightarrow & R & \xrightarrow{s} & R & \longrightarrow & 0\cdots \\
\downarrow & & & & \| & & \downarrow{\scriptstyle s} & & \\
\mathrm{cone}(s^2)^\vee = & & \cdots 0 & \longrightarrow & R & \xrightarrow{s^2} & R & \longrightarrow & 0\cdots \\
\downarrow & & & & \| & & \downarrow{\scriptstyle s} & & \\
\mathrm{cone}(s^3)^\vee = & & \cdots 0 & \longrightarrow & R & \xrightarrow{s^3} & R & \longrightarrow & 0\cdots \\
\downarrow & & & & \| & & \downarrow{\scriptstyle s} & & \\
\vdots & & & & \vdots & & \vdots & &
\end{array}$$

□

4.10. **Corollary.** *Let $t \in \mathrm{D}(R)$ be an object. Then the homotopy limit of the sequence*

$$\cdots \to k^{(n+1)} \otimes t \xrightarrow{q_{n+1} \otimes 1} k^{(n)} \otimes t \xrightarrow{q_n \otimes 1} \cdots \xrightarrow{q_2 \otimes 1} k^{(1)} \otimes t$$

is isomorphic to $[\mathbb{e}_Y, t]$.

Proof. Apply $[-, t]$ to Proposition 4.9 and use $[(k^{(n)})^\vee, t] \cong k^{(n)} \otimes t$. □

We now turn to the objects $\ell^{(n)}$ defined by the exact triangles (4.7).

4.11. **Lemma.** *For every $n \geq 1$ we have $I^{(n)} \cdot \mathrm{H}_i(\ell^{(n)}) = 0$ for all $i \geq 1$.*

Proof. As the Koszul complex $k^{(n)}$ and the module $R/I^{(n)}$ have the same homology in degree 0 and as R/I is concentrated in degree 0, we have $\mathrm{H}_i(\ell^{(n)}) = \mathrm{H}_i(k^{(n)})$ for all $i \geq 1$. But the map s_i^n is zero on $\mathrm{cone}(s_i^n)$ and *a fortiori* on $\otimes_i \mathrm{cone}(s_i^n) = k^{(n)}$. The induced map in homology is then zero as well. □

4.12. **Lemma.** *If $t \in \mathrm{D}_{\geq 0}(R)$ satisfies $\mathrm{H}_0((R/I^{(n)}) \otimes t) = 0$ then $\mathrm{H}_0(k^{(n)} \otimes t) = 0$ as well and $\mathrm{H}_1(p_n \otimes 1)\colon \mathrm{H}_1(k^{(n)} \otimes t) \to \mathrm{H}_1((R/I^{(n)}) \otimes t)$ is an isomorphism.*

Proof. Since $t \in \mathrm{D}_{\geq 0}(R)$, we can use (4.8) and the assumption $\mathrm{H}_0((R/I^{(n)}) \otimes t) = 0$ to deduce that $\mathrm{H}_0(t) = I^{(n)} \cdot \mathrm{H}_0(t)$. It follows that $\mathrm{H}_1(\ell^{(n)}) \otimes_R \mathrm{H}_0(t) = 0$ since $I^{(n)} \cdot \mathrm{H}_1(\ell^{(n)}) = 0$ by Lemma 4.11. Since $\ell^{(n)} \in \mathrm{D}_{\geq 1}(R)$ and $t \in \mathrm{D}_{\geq 0}(R)$, we know that $\ell^{(n)} \otimes t \in \mathrm{D}_{\geq 1}(R)$ and, by Remark 4.1 again, the rightmost homology is $\mathrm{H}_1(\ell^{(n)} \otimes t) \cong \mathrm{H}_1(\ell^{(n)}) \otimes_R \mathrm{H}_0(t)$. As the latter vanishes, we have in fact

$$\ell^{(n)} \otimes t \in \mathrm{D}_{\geq 2}(R).$$

Plugging this into the homology long exact sequence associated to the exact triangle (4.7) $\otimes t$, we get isomorphisms $\mathrm{H}_i(k^{(n)} \otimes t) \xrightarrow{\sim} \mathrm{H}_i((R/I^{(n)}) \otimes t)$ for $i = 0, 1$. □

4.13. **Lemma.** *Let $N \subset R$ be a nilideal and $a \in \mathbb{Z}$. Let $c \in \mathrm{D}^{\mathrm{perf}}(R)$ be a perfect complex and $d \in \mathrm{D}^{\mathrm{perf}}(R/N)$ its image modulo N. Suppose that $d \in \mathrm{D}_{\geq a}(R/N)$ has homology only in degree above a. Then $c \in \mathrm{D}_{\geq a}(R)$ has the same right-boundedness.*

Proof. We can assume $c \neq 0$. Let $\mathrm{H}_i(c)$ be the rightmost non-zero homology group of c. We need to show that $i \geq a$. Since c is perfect, its rightmost homology group $\mathrm{H}_i(c)$ is a finitely generated R-module and, since N is a nilideal, Nakayama's Lemma guarantees that $(R/N) \otimes_R \mathrm{H}_i(c)$ remains non-zero. By Remark 4.1, this non-zero module is $\mathrm{H}_i((R/N) \otimes c) \cong \mathrm{H}_i(d)$. Hence $i \geq a$. □

We now turn to a key lemma, where derived completion appears.

4.14. **Lemma.** *Let $t \in \mathrm{D}_{\geq 0}(R)$ be such that $\iota_n^*(t) \in \mathrm{D}(R/I^{(n)})$ is perfect for every $n \geq 1$. If $\mathrm{H}_0((R/I) \otimes t) = 0$ then $\mathrm{H}_0([\mathbb{e}_Y, t]) = 0$.*

Proof. We first claim that $\mathrm{H}_0(\iota_n^*(t)) = 0$ for all $n \geq 1$. The $n = 1$ case is the hypothesis $\mathrm{H}_0(\iota_1^*(t)) \cong \mathrm{H}_0((R/I) \otimes t) = 0$. Let $n \geq 2$ be such that $\mathrm{H}_0(\iota_{n-1}^*(t)) = 0$. Write $\bar{R} = R/I^{(n)}$ and $N = I^{(n-1)}/I^{(n)} \subseteq \bar{R}$. The homomorphism $\bar{R} \twoheadrightarrow \bar{R}/N = R/I^{(n-1)}$ has nilpotent kernel N and sends $\iota_n^*(t)$ to $\iota_{n-1}^*(t)$:

$$\begin{array}{ccc} \iota_n^*(t) & \in & \mathrm{D}^{\mathrm{perf}}(R/I^{(n)}) = \mathrm{D}^{\mathrm{perf}}(\bar{R}) \\ \downarrow & & \downarrow \\ \iota_{n-1}^*(t) & \in & \mathrm{D}^{\mathrm{perf}}(R/I^{(n-1)}) = \mathrm{D}^{\mathrm{perf}}(\bar{R}/N). \end{array}$$

Both of these objects are perfect complexes by assumption. We can therefore use Lemma 4.13 to conclude that $\mathrm{H}_0(\iota_n^*(t)) = 0$, which proves the claim by induction.

Our second claim is that the map $\mathrm{H}_1((R/I^{(n)}) \otimes t) \to \mathrm{H}_1((R/I^{(n-1)}) \otimes t)$ is surjective for every $n \geq 2$. Keeping the notation $\bar{R} = R/I^{(n)}$ and $N = I^{(n-1)}/I^{(n)}$ above, the perfect complex $\iota_n^*(t)$ belongs to $\mathrm{D}_{\geq 1}(R/I^{(n)})$ by the already proved first claim, *i.e.*, the perfect complex $\iota_n^*(t)$ has rightmost non-zero homology in degree one (or higher). Thus we can invoke Remark 4.1 again, for the ring $\bar{R}$, to get that

$$\mathrm{H}_1(\iota_{n-1}^*(t)) \cong (\bar{R}/N) \otimes_{\bar{R}} \mathrm{H}_1(\iota_n^*(t)).$$

Hence the map $\mathrm{H}_1((R/I^{(n)}) \otimes t) \cong \mathrm{H}_1(\iota_n^*(t)) \twoheadrightarrow \mathrm{H}_1(\iota_{n-1}^*(t)) \cong \mathrm{H}_1((R/I^{(n-1)}) \otimes t)$ is indeed surjective.

Combining with (4.6) and Lemma 4.12 our two claims in degree zero and one also hold if we replace $(R/I^{(n)}) \otimes t$ by $k^{(n)} \otimes t$, namely we have

$$\mathrm{H}_0(k^{(n)} \otimes t) \cong 0$$

for all $n \geq 1$ and the maps induced by $q_n \colon k^{(n)} \to k^{(n-1)}$

$$\mathrm{H}_1(k^{(n)} \otimes t) \to \mathrm{H}_1(k^{(n-1)} \otimes t)$$

are surjective for all $n \geq 2$. The latter implies that $\mathrm{H}_0(\mathrm{holim}_n(k^{(n)} \otimes t)) \cong \lim_n \mathrm{H}_0(k^{(n)} \otimes t) = 0$; see Recollection 4.2 for $u = R$ in $\mathcal{T} = \mathrm{D}(R)$. But Corollary 4.10 tells us that this homotopy limit, $\mathrm{holim}_n(k^{(n)} \otimes t)$, is precisely the object $[\mathbb{e}_Y, t]$. □

4.15. **Lemma.** *Let $b \geq a$ be integers. Let $d \in \mathrm{D}(R)$ be such that $\mathrm{kos}(s_1, \ldots, s_r) \otimes d$ has homology concentrated in degrees between b and a. Then:*

(a) *For every $n \geq 1$, the complex $k^{(n)} \otimes d$ has homology concentrated in degrees between b and a.*

(b) *The complex $[\mathbb{e}_Y, d]$ has homology concentrated in degrees between b and $a - 1$. In particular, $[\mathbb{e}_Y, d] \in \mathrm{D}_{\geq a-1}(R)$ is right-bounded.*

Proof. Part (a) follows from the more general fact that $\mathrm{kos}(s_1^{\ell_1}, \ldots, s_r^{\ell_r}) \otimes d$ has homology concentrated between degrees b and a, for every $\ell_1, \ldots, \ell_r \geq 1$. The latter is an immediate induction on the ℓ_i once we observe that $\mathrm{cone}(s^{\ell+1})$ is an extension of $\mathrm{cone}(s^\ell)$ and $\mathrm{cone}(s)$ and since the case $\ell_1 = \ldots = \ell_r = 1$ is the hypothesis. Part (b) then follows from Corollary 4.10 and the homology long exact sequence for the exact triangle defining the homotopy limit; see (4.3). □

4.16. **Lemma.** *Let $n \geq 1$ and $t \in \mathrm{D}(R)$. The tt-functor $\iota_n^* \colon \mathrm{D}(R) \to \mathrm{D}(R/I^{(n)})$ sends the canonical maps $\mathbb{e}_Y \otimes t \to t$ and $t \to [\mathbb{e}_Y, t]$ to isomorphisms. In particular, $\iota_n^*(\mathbb{e}_Y) \cong \mathbb{1}$.*

Proof. The cones of these maps are $\mathbb{f}_Y \otimes t$ and $[\mathbb{f}_Y, t]$ and these objects are killed by the Koszul objects $k^{(n)}$ since $\mathrm{supp}(k^{(n)}) = Y$ and since $k^{(n)}$ is dualizable in $\mathcal{T}$. So it suffices to observe that $\iota_n^*(k^{(n)}) \cong \oplus_{i=0}^r \mathbb{1}^{\binom{n}{i}}$, so $\mathbb{1}$ kills $\iota_n^*(\mathbb{f}_Y \otimes t)$ and $\iota_n^*([\mathbb{f}_Y, t])$. □

Recall from Recollection 3.2 the tt-category $\mathrm{D}_Y(R) \subseteq \mathrm{D}(R)$ of complexes supported on Y and the tt-equivalent category $\mathrm{D}(R)^\wedge_Y = (\mathrm{D}_Y(R))^{\perp\perp} \subseteq \mathrm{D}(R)$ of derived complete complexes along Y.

4.17. **Theorem.** *Let R be a commutative ring and let $I \subset R$ be a finitely generated ideal such that $R \cong \hat{R}$ is I-adically complete and let $Y = V(I)$.*

(a) *A complex $d \in \mathrm{D}_Y(R)$ supported on Y is dualizable in the category $\mathrm{D}_Y(R)$ if and only if d belongs to the thick subcategory generated by the unit $\mathbb{e}_Y$.*

(b) *A derived complete complex $d \in \mathrm{D}(R)^\wedge_Y$ is dualizable in the category $\mathrm{D}(R)^\wedge_Y$ if and only if d is a perfect complex.*

Proof. We only prove (a), as (b) will follow by the tt-equivalence $\mathcal{T}_Y \cong \mathcal{T}^\wedge_Y$ of Remark 2.6 and the fact that the unit $\mathbb{1}_{\mathcal{T}^\wedge_Y}$ is $\hat{\mathbb{1}}_Y \cong R$ by Proposition 3.5 under the inclusion $\mathcal{T}^\wedge_Y = \mathcal{T}_Y^{\perp\perp} \rightarrowtail \mathcal{T} = \mathrm{D}(R)$. Since dualizable objects always form a thick subcategory containing the unit, we only need to prove that if d is a dualizable object in $\mathcal{T}_Y$ then $d \in \mathrm{thick}\langle \mathbb{e}_Y \rangle$ in $\mathcal{T}_Y$. (Since $\mathcal{T}_Y \subset \mathcal{T}$ is a thick subcategory, this is the same as $\mathrm{thick}\langle \mathbb{e}_Y \rangle$ in $\mathcal{T}$.) If $d = 0$ there is nothing to prove, so we could assume that $d \neq 0$. We are going to proceed by induction on the 'homological amplitude' of the image $\iota_1^*(d)$ of d in $\mathrm{D}(R/I)$ and we will construct an exact triangle in $\mathcal{T}_Y$

$$d' \to \mathbb{e}_Y \otimes P \xrightarrow{g} d \to \Sigma d' \tag{4.18}$$

where P is a finitely-generated projective R-module and where either $d' = 0$ or the 'homological amplitude' of $\iota_1^*(d')$ is less than that of d. For this 'homological amplitude' to make sense, we need to know that $\iota_1^*(d)$ is perfect. In fact, we have more generally for the ideals $I^{(n)}$ of Notation 4.5:

(4.19) *for every $n \geq 1$ the complex $\iota_n^*(d) \in \mathrm{D}(R/I^{(n)})$ is perfect.*

To see this, note that the functor $(\iota_n^*)_{|\mathcal{T}_Y} : \mathcal{T}_Y \hookrightarrow \mathcal{T} = \mathrm{D}(R) \to \mathrm{D}(R/I^{(n)})$ sends the tensor product to the tensor product, since both the inclusion $\mathcal{T}_Y \hookrightarrow \mathcal{T}$ and extension-of-scalars ι_n^* do. However, the inclusion $\mathcal{T}_Y \hookrightarrow \mathcal{T}$ does not preserve the unit ($\mathbb{e}_Y \not\mapsto \mathbb{1}$). Luckily, the second functor ι_n^* corrects this issue, by Lemma 4.16. This proves (4.19) since a tensor functor preserves dualizable objects and the dualizable objects in the derived category of a ring are just the perfect complexes. The 'homological amplitude' of $\iota_1^*(d)$ is simply the length of the interval where its homology is non-zero, *i.e.*, the smallest number $b - a$ where $\mathrm{H}_i(\iota_1^*(d)) = 0$ for all $i > b$ and all $i < a$. By convention, let us say that this amplitude is -1 if all those groups are zero, that is, when $\iota_1^*(d) = 0$. We shall see that this only happens for $d = 0$.

For our dualizable object $d \in \mathcal{T}_Y$, define the following object of $\mathcal{T} = \mathrm{D}(R)$:

$$t := [\mathbb{e}_Y, d].$$

Note that $[\mathbb{e}_Y, t] = t$ and $\mathbb{e}_Y \otimes t \cong d$ and in particular $t \neq 0$ when $d \neq 0$. By Lemma 4.16 again, this object t has the same image as d in $\mathrm{D}(R/I^{(n)})$ under ι_n^*:

$$\iota_n^*(t) \cong \iota_n^*(d) \tag{4.20}$$

for every $n \geq 1$. In particular, by (4.19) the object $\iota_n^*(t)$ is compact in $\mathrm{D}(R/I^{(n)})$:

$$\iota_n^*(t) \in \mathrm{D}^{\mathrm{perf}}(R/I^{(n)}) \quad \text{for all } n \geq 1. \tag{4.21}$$

On the other hand, it is a general fact in a tensor-triangulated category $\mathcal{S}$ with coproducts and cocontinuous tensor, like $\mathcal{S} = \mathcal{T}_Y$, that if k is compact and d is dualizable then $k \otimes d$ is compact. This is immediate from $\mathrm{Hom}_{\mathcal{S}}(k \otimes d, -) \cong \mathrm{Hom}_{\mathcal{S}}(k, d^\vee \otimes -)$. In particular, in our case, we see that $\mathrm{kos}(s_1, \ldots, s_r) \otimes d$ is compact in $\mathcal{T}_Y$, hence in $\mathcal{T}$ since $(\mathcal{T}_Y)^c = (\mathcal{T}^c)_Y$ by construction of $\mathcal{T}_Y$. We can therefore invoke Lemma 4.15 for our d to conclude that $t = [\mathbb{e}_Y, d]$ is right-bounded. Up to replacing d by a (de)suspension, we can assume that

$$t = [\mathbb{e}_Y, d] \text{ belongs to } \mathrm{D}_{\geq 0}(R), \text{ and } \mathrm{H}_0(t) \neq 0 \text{ when } d \neq 0. \tag{4.22}$$

By (4.21) and (4.22), the object t satisfies the hypotheses of Lemma 4.14, hence

$$\mathrm{H}_0(\iota_1^* t) = 0 \quad \text{if and only if} \quad d = 0. \tag{4.23}$$

Indeed, the vanishing of $\mathrm{H}_0(\iota_1^* t) \cong \mathrm{H}_0((R/I) \otimes t)$ implies $\mathrm{H}_0([\mathbb{e}_Y, t]) = 0$ by Lemma 4.14. But $[\mathbb{e}_Y, t] = t$ and $\mathrm{H}_0([\mathbb{e}_Y, t]) = \mathrm{H}_0(t) = 0$ only happens for $d = 0$ by (4.22). The converse in (4.23) is clear since t was defined to be $[\mathbb{e}_Y, d]$.

So let us proceed with constructing the exact triangle (4.18), when $d \neq 0$, that is, when $\mathrm{H}_0(t) \neq 0$. The object $t \in \mathrm{D}_{\geq 0}(R)$ is represented by a complex of projective R-modules (not necessarily finitely generated) as in the top row below

$$\begin{array}{llcccccccccc} & t = & \cdots & \longrightarrow & t_2 & \longrightarrow & t_1 & \longrightarrow & t_0 & \longrightarrow & 0 & \longrightarrow 0 \cdots \\ (4.24) & \bar{t} = & \cdots & \longrightarrow & \bar{t}_2 & \longrightarrow & \bar{t}_1 & \longrightarrow & \bar{t}_0 & \longrightarrow & 0 & \longrightarrow 0 \cdots \\ & & & & & & & & \uparrow{\scriptstyle \bar{f}} & & & \\ & & & & & & & & \bar{P} & & & \end{array}$$

and its image under $\iota_1^* = (R/I) \otimes_R -$, displayed in the second row with the shortcut $\bar{t}_i := (R/I) \otimes_R t_i$, is a perfect complex by (4.21). By (4.23) we know that $\bar{t}_1 \to \bar{t}_0$ cannot be surjective, *i.e.*, the 'homological amplitude' of $\iota_1^*(d) \cong \iota_1^*(t) = \bar{t}$ is the biggest integer $b \geq 0$ such that $\mathrm{H}_b(\bar{t}) \neq 0$. Since $\bar{t}$ is quasi-isomorphic to a bounded complex of finitely generated projective R/I-modules, that we can assume to live in non-negative degrees since $\bar{t} \in \mathrm{D}_{\geq 0}(R/I)$, there exists a finitely generated projective R/I-module $\bar{P}$ as in the third row of (4.24) (simply the last entry of that perfect complex) with the property that the homotopy fiber of $\bar{P} \to \bar{t}$

$$\bar{t}' \to \bar{P} \xrightarrow{\bar{f}} \bar{t} \to \Sigma \bar{t}' \tag{4.25}$$

is a perfect complex $\bar{t}'$ with amplitude $b-1$. (Recall that amplitude -1 means $\bar{t}' = 0$.)

By Proposition 4.4, since R is classically complete, we know that $\bar{P} \cong (R/I) \otimes_R P$ for some finitely generated projective R-module P and we can lift the map $\bar{f}\colon \bar{P} \to \bar{t}_0$ to a map $f\colon P \to t_0$ since P is projective and $t_0 \twoheadrightarrow \bar{t}_0$ is a surjection of R-modules. Since t is concentrated in nonnegative degrees, f defines an actual morphism of complexes $f\colon P \to t$ (in degree zero). Define $g := \mathbb{e}_Y \otimes f\colon \mathbb{e}_Y \otimes P \to \mathbb{e}_Y \otimes t \cong d$ and complete it into an exact triangle as announced in (4.18). Note that d' is again dualizable in $\mathcal{T}_Y$. So it only remains to show that $\iota_1^*(d')$ has homological amplitude $b-1$. By Lemma 4.16 we know that the image of the triangle (4.18) under the tt-functor ι_1^* is isomorphic to the exact triangle (4.25). In other words, $\iota_1^*(d') \simeq \bar{t}'$ which has amplitude one less than that of d. This finishes the proof. $\square$

4.26. **Corollary.** *Suppose that $R \cong \hat{R}_Y$ is classically Y-complete. Then the subcategory of dualizable objects inside the category $\mathrm{D}(R)^{\wedge}_{Y}$ of Y-complete complexes coincides with that of perfect complexes $\mathrm{D}^{\mathrm{perf}}(R) = \mathrm{D}(R)^c$ and is tt-equivalent to the subcategory of dualizable objects in $\mathrm{D}_Y(R)$ via the following tt-equivalences:*

$$\begin{array}{ccc} & \mathrm{D}^{\mathrm{perf}}(R) & \\ {}^{e_Y\otimes -}\swarrow\cong & & \Vert\,! \\ (\mathrm{D}_Y(R))^d & \xrightarrow[\cong]{[e_Y,-]} & (\mathrm{D}(R)^{\wedge}_{Y})^d. \end{array}$$

Proof. The right-hand equality is Theorem 4.17 (b). The horizontal equivalence is simply the dualizable part of the tt-equivalence $\mathcal{T}_Y \cong (\mathcal{T}_Y)^{\perp\perp} = \mathcal{T}^{\wedge}_{Y}$ of Remark 2.6. The left-hand tensor functor $e_Y \otimes - : \mathcal{T} \to \mathcal{T}_Y$ preserves dualizable objects hence restricts to a functor $\mathcal{T}^c = \mathcal{T}^d \to (\mathcal{T}_Y)^d$. The composite maps $c \in \mathcal{T}^c$ to $[e_Y, e_Y \otimes c] \cong [e_Y, e_Y] \otimes c = \hat{\mathbb{1}}_Y \otimes c$ which is simply c in our case since $\hat{\mathbb{1}}_Y \cong \mathbb{1}$ by Proposition 3.5. □

5. Main results

We keep our general notation: R is a commutative ring and $Y \subseteq \mathrm{Spec}(R)$ is a closed subset with quasi-compact complement. We write $\mathcal{T} = \mathrm{D}(R)$ for the derived category, $\mathcal{T}_Y = \mathrm{D}_Y(R)$ for the category of complexes supported on Y and $\mathcal{T}^{\wedge}_{Y} = \mathrm{D}(R)^{\wedge}_{Y}$ for the category of derived complete complexes along Y (Recollection 3.2).

We are ready to prove our main result.

5.1. **Theorem.** *Let $\underline{s} = (s_1, \ldots, s_r)$ be a sequence of elements of R such that $Y = V(s_1, \ldots, s_n)$. Then the following are equivalent:*

- (i) *The sequence $\underline{s}$ is Koszul-complete (Definition 3.14).*
- (ii) *There is a canonical tt-equivalence $(\mathrm{D}_Y(R))^d \cong \mathrm{D}^{\mathrm{perf}}(\hat{R}_Y)$ making the following diagram commute:*

$$\begin{array}{ccc} & \mathrm{D}^{\mathrm{perf}}(R) & \\ {}^{e_Y\otimes -}\swarrow & & \searrow{}^{\hat{R}_Y\otimes_R -} \\ (\mathrm{D}_Y(R))^d & \xrightarrow{\cong} & \mathrm{D}^{\mathrm{perf}}(\hat{R}_Y). \end{array}$$

- (iii) *There is a canonical tt-equivalence $(\mathrm{D}(R)^{\wedge}_{Y})^d \cong \mathrm{D}^{\mathrm{perf}}(\hat{R}_Y)$ making the following diagram commute:*

$$\begin{array}{ccc} & \mathrm{D}^{\mathrm{perf}}(R) & \\ {}^{(-)^{\wedge}_{Y}}\swarrow & & \searrow{}^{\hat{R}_Y\otimes_R -} \\ (\mathrm{D}(R)^{\wedge}_{Y})^d & \xrightarrow{\cong} & \mathrm{D}^{\mathrm{perf}}(\hat{R}_Y). \end{array}$$

- (iv) *There exists an isomorphism of ring objects $\hat{\mathbb{1}}_Y \cong \hat{R}_Y$ in $\mathrm{D}(R)$.*

Proof. Let $f : \mathrm{Spec}(\hat{R}) \to \mathrm{Spec}(R)$ be the ring-theoretic completion and let $f^* : \mathcal{T} := \mathrm{D}(R) \to \mathrm{D}(\hat{R}) =: \mathcal{S}$ be extension-of-scalars. Write $Y' = f^{-1}(Y)$ as usual.

(i)⇒(ii): We have a commutative diagram of tt-functors

$$\begin{array}{ccc} \mathcal{T} & \xrightarrow{f^*} & \mathcal{S} \\ {\scriptstyle e_Y\otimes -}\downarrow & & \downarrow{\scriptstyle e_{Y'}\otimes -} \\ \mathcal{T}_Y & \xrightarrow{f^*_{|\mathcal{T}_Y}} & \mathcal{S}_{Y'}. \end{array}$$

The bottom functor is an equivalence by Theorem 3.20. On the other hand, since $\hat{R}$ is classically Y'-complete (Remark 3.19), Corollary 4.26 implies that the right vertical functor is an equivalence when restricted to dualizable objects. We get

$$\begin{array}{ccc} \mathcal{T}^c & \xrightarrow{f^*} & \mathcal{S}^c \\ {\scriptstyle e_Y \otimes -}\downarrow & & \downarrow{\scriptstyle \cong} \\ (\mathcal{T}_Y)^d & \xrightarrow{\cong} & (\mathcal{S}_{f^{-1}(Y)})^d. \end{array}$$

(ii)⇒(i): The hypothesis is that we have a tt-equivalence making the right-hand triangle below commute.

$$\begin{array}{ccccc} & & \mathcal{T}^c & & \\ & \nearrow^{\text{incl}} & {\scriptstyle e_Y \otimes -}\downarrow & \searrow^{f^*|_{\mathcal{T}^c}} & \\ \mathcal{T}^c_Y & \xrightarrow{\text{incl}} & (\mathcal{T}_Y)^d & \xrightarrow{\cong} & \mathcal{S}^c \end{array}$$

The left-hand triangle commutes since $e_Y \otimes -$ is right adjoint to the inclusion $\mathcal{T}_Y \rightarrowtail \mathcal{T}$ and in particular retracts it. The bottom composite is fully faithful, hence for any $a, b \in \mathcal{T}^c_Y$, we have that $\mathcal{T}(a, b) \to \mathcal{S}(f^*(a), f^*(b)) \cong \mathcal{T}(a, f_*(\mathbb{1}) \otimes b)$ is a bijection. Since this is true for all $a \in \mathcal{T}^c_Y$ it is also true for all $a \in \operatorname{Loc}(\mathcal{T}^c_Y) = \mathcal{T}_Y$ and the latter contains b and $f_*(\mathbb{1}) \otimes b$. It follows by Yoneda that $\eta_b \colon b \to f_*(\mathbb{1}) \otimes b$ is an isomorphism for every $b \in \mathcal{T}^c_Y$. Plugging in $b = \operatorname{kos}(\underline{s})$ we get (i).

The equivalence (i)⇔(iv) was already established in Theorem 3.20. The equivalence (ii)⇔(iii) follows again from the tt-equivalence $\mathcal{T}_Y \cong \mathcal{T}^\wedge_Y$ of Remark 2.6. □

5.2. **Corollary.** *Let R be a noetherian commutative ring and let $Y \subseteq \operatorname{Spec}(R)$ be a closed subset. There are canonical equivalences of tt-categories between $\mathrm{D}^{\mathrm{perf}}(\hat{R})$ and the subcategories $(\mathrm{D}_Y(R))^d$ and $(\mathrm{D}(R)^\wedge_Y)^d$ making the following diagram commute*

$$\begin{array}{ccccc} & & \mathrm{D}^{\mathrm{perf}}(R) & & \\ & \swarrow^{e_Y \otimes -} & \downarrow{\scriptstyle (-)^\wedge_Y} & \searrow^{\hat{R} \otimes_R -} & \\ (\mathrm{D}_Y(R))^d \cong & & (\mathrm{D}(R)^\wedge_Y)^d & \xrightarrow{\cong} & \mathrm{D}^{\mathrm{perf}}(\hat{R}). \end{array}$$

Proof. Immediate from Theorem 5.1 and Proposition 3.17 (and Remark 2.6). □

5.3. *Remark.* In view of Theorem 5.1 and Example 3.16 the equivalence of Corollary 5.2 does not hold for the non-noetherian ring $R = \mathbb{Z}_{(p)} \oplus (\mathbb{Q}/\mathbb{Z}_{(p)})$ and $Y = V(p)$.

5.4. *Remark.* We learned in Example 2.13 that $\mathcal{T}^\wedge_Y$ should not be thought of as the tt-analogue of ring-completion. Theorem 2.16 and Theorem 3.20 suggest that $\mathcal{T}^\wedge_Y$ only recovers the part of Y-completion *supported on* Y. Perhaps one should write $\mathcal{T}^\wedge_{Y\,Y}$ to make this point: One decoration $^\wedge_Y$ for Y-completion and another Y for support.

We could also say that the only thing we need to build the full completion $\hat{\mathcal{T}}$ is a rigid tt-category $\hat{\mathcal{T}}^d$ that will serve as the dualizable objects in $\hat{\mathcal{T}}$ and that we can Ind-complete into $\hat{\mathcal{T}}$. Theorem 5.1 suggests that the dualizable objects in $\mathcal{T}^\wedge_Y$ is a possible choice for this $\hat{\mathcal{T}}^c$. This is the definition chosen in [NPR24]. Another, possibly smaller, choice would be to take the smallest tt-subcategory of $(\mathcal{T}^\wedge_Y)^d$ that contains the image of $\mathcal{T}^c$. It is not clear if there is a difference in general between these two choices and Theorem 4.17 tells us that there is none in the case of $\mathrm{D}(R)$. Future investigation of this topic seems worth pursuing in general tt-geometry.

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UCLA Mathematics Department, Los Angeles, CA 90095-1555, USA
Email address: `balmer@math.ucla.edu`

Mathematics Department, UC Santa Cruz, 95064 CA, USA
Email address: `beren@ucsc.edu`
URL: `http://people.ucsc.edu/∼beren/`