FUNDAMENTAL ISOMORPHISM CONJECTURE VIA NON-COMMUTATIVE MOTIVES

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Abstract. Given a group, we construct a "fundamental localizing invariant" on its orbit category. We prove that any isomorphism conjecture valid for this fundamental invariant implies the same isomorphism conjecture for all localizing invariants, like non-connective K-theory, Hochschild homology, cyclic homology, and so on. Then, we discuss how to reduce such a fundamental isomorphism conjecture to essentially K-theoretic ones. Finally, we develop the analogue additive results.

Introduction

We warn the allergic reader that this article contains many conjectures and even introduces a new one. However, it is shown that this single new conjecture implies several old ones. The interest of this work does not come from technical difficulties, although it involves rather sophisticated objects. Instead, we believe that the organization and conceptual clarification that we propose can be valuable to some readers. In particular, we hope to invite some mathematicians active in the field of Isomorphism Conjectures to learn about non-commutative motives.

Our first goal is to replace a large collection of so-called isomorphism conjectures by a single, deeper conjecture, that we call the Fundamental Isomorphism Conjecture (or Mamma Conjecture). We explain how one arrives to this conjecture in a natural way. A posteriori, it is easy to show that this single conjecture implies all other isomorphism conjectures under consideration. The non-trivial point is to *discover* such a fundamental conjecture.

Our second goal is to prove that the Mamma Conjecture can actually be translated into more standard ones, involving only K-theory, cum grano salis. Combining the two results, we see that if a group G satisfies the latter class of K-theoretic Isomorphism Conjectures then it satisfies all other isomorphisms conjectures.

We fix a "base" commutative ring R, which can be $\mathbb{Z}$, $\mathbb{Q}$ or $\mathbb{C}$ for instance.

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1. The Mamma Conjecture

1.1. **Assembly properties.** Given a group G, the Farrell-Jones *Isomorphism Conjectures* predict the value of algebraic K- and L-theory of the group ring RG in terms of the values on the virtually cyclic subgroups of G. These conjectures imply well-known conjectures due to Bass, Borel, Kaplansky, Novikov; see a survey in Lück [12].

In [5], Davis and Lück proposed the following unified setting for stating such isomorphism conjectures. Let $\mathcal{F}$ be a family of subgroups of G and $\mathbf{E} : \mathrm{Or}(G) \to \mathsf{Spt}$ a functor from the orbit category of G (whose objects are indexed by subgroups $H < G$ and whose morphisms $H \to H'$ are maps of G-sets $G/H \to G/H'$) to the category of spectra. The $(\mathbf{E}, \mathcal{F}, G)$-*assembly map* is the induced map of spectra

$$\operatorname*{hocolim}_{\mathrm{Or}(G,\mathcal{F})} \mathbf{E} \longrightarrow \operatorname*{hocolim}_{\mathrm{Or}(G)} \mathbf{E} = \mathbf{E}(G)\,, \tag{1.1.1}$$

where $\mathrm{Or}(G,\mathcal{F}) \subset \mathrm{Or}(G)$ is the orbit category restricted on $\mathcal{F}$, that is, the full subcategory on those $H \in \mathcal{F}$. We say that *the functor* $\mathbf{E}$ *has the* $\mathcal{F}$*-assembly property for* G when the map (1.1.1) is a weak equivalence of spectra, *i.e.* when it induces an isomorphism on stable homotopy groups. There is an obvious gain in flexibility in this approach but it is equally obvious that considering arbitrary functors $\mathbf{E} : \mathrm{Or}(G) \to \mathsf{Spt}$ is way too general and does not isolate any particular property which might justify the assembly property to hold for K-theory, for instance. Remedying this drawback is also a motivation for the present work.

When we speak of *the* $(\mathbf{E}, \mathcal{F}, G)$*-isomorphism conjecture*, we refer to the expressed hope that the assembly property holds for a particular choice of $\mathbf{E}$, $\mathcal{F}$ and G. Davis and Lück proved (see also [6] for details on the proof) that the Farrell-Jones Conjecture in K-theory for G is equivalent to the $(\mathbb{K}, \mathcal{VC}, G)$-isomorphism conjecture, where $\mathbb{K}$ is non-connective K-theory and $\mathcal{VC}$ is the family of virtually cyclic subgroups of G; and similarly for L-theory. The first step in their approach is the construction of a functor to R-linear categories, via the category Grp of groupoids:

$$\mathrm{Or}(G) \xrightarrow{\overline{?}} \mathsf{Grp} \xrightarrow{R[-]} R\text{-}\mathsf{cat}\,. \tag{1.1.2}$$

The first functor $\overline{?}$ associates to $H < G$ the transport groupoid $\overline{G/H}$ of the G-set G/H and the second functor $R[-]$ is R-linearization; see details in [5].

1.1.3. *Remark.* From now on, we are going to drop the L-theoretic variant of the game, to avoid dragging along *dualities* on the R-linear categories $R[\overline{G/H}]$. Actually, in full generality, one should not only consider categories with duality but probably more general enrichments, for instance to include topological K-theory of the reduced C^*-algebra (aiming at the Baum-Connes Conjecture). Such variations might well exist but are beyond the scope of the present short article.

However, besides the original Farrell-Jones K-theoretic Isomorphism Conjecture, the literature contains many variations on the theme, replacing the K-theory functor by other functors $\mathbf{E}$, and the category of spectra by other model categories $\mathcal{M}$. It is actually immediate from Davis and Lück's formalism that infinitely many such conjectures can be made. Consider for instance the isomorphism conjecture for homotopy K-theory (KH) [2, § 7], for Hochschild homology (HH) and cyclic homology (HC) [13, § 1], or for topological Hochschild homology (THH) [11, § 6].

This simple idea of letting the functor $\mathbf{E} : \mathrm{Or}(G) \to \mathcal{M}$ float freely generates a profusion of potential isomorphism conjectures:

$$
\begin{tikzcd}
 & \mathsf{Spt} \\
 & \mathsf{Spt} \\
\mathrm{Or}(G) \arrow[ur, "\mathbb{K}"] \arrow[uur, phantom] \arrow[r, "HH"] \arrow[dr, "HC"] \arrow[ddr, "THH"] \arrow[dddr, "\mathbf{E}"] \arrow[uur, "KH"] & \mathrm{Ch}(R) \\
 & \mathrm{Ch}(R) \\
\vdots & \mathsf{Spt} \\
 & \mathcal{M}
\end{tikzcd}
\tag{1.1.4}
$$

We just need $\mathcal{M}$ to be a (reasonable) Quillen model category [15], so that we can speak of homotopy colimits. For instance, $\mathrm{Ch}(R)$ stands for the category of complexes of R-modules. Each of these isomorphism conjectures has already been proved for various classes of groups using a variety of different methods. See [12] or [14] for a survey of such results.

1.2. **Towards a fundamental invariant.** Our goal in this article is not to prove any of these conjectures for any particular group G. We are rather interested in the general organization and deeper properties behind this somewhat exuberant herd of conjectures. Are all these conjectures really that different? Or is there some deeper fact which should explain all of them? Intuitively, we want to comb the skein (1.1.4) from the left to isolate a fundamental functor $\mathbf{E}^{\mathsf{loc}}_{\mathsf{fund}}$

$$
\begin{tikzcd}
 & & \mathsf{Spt} \\
 & & \mathsf{Spt} \\
\mathrm{Or}(G) \arrow[r, "\mathbf{E}^{\mathsf{loc}}_{\mathsf{fund}}"] & \mathcal{M}ot^{\mathsf{loc}}_{\mathsf{dg}} \arrow[uur, "\overline{\mathbb{K}}"] \arrow[ur, "\overline{KH}"] \arrow[r, "\overline{HH}"] \arrow[dr, "\overline{HC}"] \arrow[ddr, "\overline{THH}"] \arrow[dddr, "\overline{\mathbf{E}}"] & \mathrm{Ch}(R) \\
 & & \mathrm{Ch}(R) \\
 & \vdots & \mathsf{Spt} \\
 & & \mathcal{M}
\end{tikzcd}
\tag{1.2.1}
$$

Then, we want to see if this fundamental functor should have an assembly property and whether that property would imply the same assembly property for the original functors of (1.1.4) and for any new "similar" functor that might come up in the future. As we shall see below, to construct this functor $\mathbf{E}^{\mathsf{loc}}_{\mathsf{fund}}$ and the model category $\mathcal{M}ot^{\mathsf{loc}}_{\mathsf{dg}}$, we need the theory of *non-commutative motives* as initiated in [20]. We shall then show that all the functors $\mathbf{E}$ of (1.1.4) factor via this $\mathbf{E}^{\mathsf{loc}}_{\mathsf{fund}}$ in such a way that any assembly property of $\mathbf{E}^{\mathsf{loc}}_{\mathsf{fund}}$ implies the same property for $\mathbf{E}$.

A first observation is that all the functors $\mathbf{E} : \mathrm{Or}(G) \to \mathcal{M}$ which appear in our original examples have in common that they actually factor, not only via R-linear categories, but via their associated *dg-category*:

$$
\mathrm{Or}(G) \xrightarrow{\overline{?}} \mathsf{Grp} \xrightarrow{R[-]} R\text{-}\mathsf{cat} \subset \mathsf{dgcat}\,. \tag{1.2.2}
$$

For non-connective K-theory, one can use Schlichting's construction [16, § 6.4]; for Hochschild and cyclic homology, see [7, § 5.3]; for topological Hochschild homology, see [3, § 3] or [17, § 8.1]. Let us quickly recall basic facts about this category dgcat.

1.3. **Recalling dg categories.** See Keller [7] for an introduction and survey. A *differential graded (=dg) category*, over our fixed base commutative ring R, is a category enriched over chain complexes of R-modules (morphisms sets are complexes) in such a way that composition fulfills the Leibniz rule: $d(f \circ g) = d(f) \circ g + (-1)^{\deg(f)} f \circ d(g)$. Any R-linear category can be naturally considered as a dg category with complexes of morphisms concentrated in degree zero. For instance, $\underline{R}$ denotes the dg category with one object and R as endomorphism ring. Let $\mathcal{A}$ be a small dg category. Recall from [7, § 3.1] that a *right dg $\mathcal{A}$-module* (or simply an $\mathcal{A}$-module) is a dg functor $\mathcal{A}^{\mathsf{op}} \to \mathcal{C}_{\mathsf{dg}}(R)$, with values in the dg category $\mathcal{C}_{\mathsf{dg}}(R)$ of complexes of R-modules. We denote by $\mathcal{C}(\mathcal{A})$ the category of $\mathcal{A}$-modules. The *derived category $\mathcal{D}(\mathcal{A})$ of* $\mathcal{A}$ is the localization of $\mathcal{C}(\mathcal{A})$ with respect to quasi-isomorphisms. Finally, a *derived Morita equivalence* between dg categories $\mathcal{A}$ and $\mathcal{B}$ is a dg functor $F : \mathcal{A} \to \mathcal{B}$ which induces an equivalence on the derived categories $\mathcal{D}(\mathcal{B}) \overset{\sim}{\to} \mathcal{D}(\mathcal{A})$.

1.4. **Localizing functors.** Returning to our general discussion, an important gain of introducing dg categories in (1.2.2) is that we now have a (cofibrantly generated) *Quillen model category structure* on the category dgcat of small dg categories, by [19, Thm. 5.3]. In other words, we can do homotopy theory of dg categories and speak of homotopy colimits thereof. The weak equivalences are the derived Morita equivalences. We could therefore already consider isomorphism conjectures for the basic functor $\mathrm{Or}(G) \to \mathsf{dgcat}$ of (1.2.2), à la Davis-Lück.

However this would be way too naive. Instead, let us return to the functors $\mathbf{E} : \mathrm{Or}(G) \to \mathcal{M}$ that we want to study in the first place, or by the above discussion, let us consider the functors $E : \mathsf{dgcat} \to \mathcal{M}$ that we want to compose with the basic functor (1.2.2). Another common property shared by our examples is that they are *localizing* in the sense of [18, §10]. This means that the functor $E : \mathsf{dgcat} \to \mathcal{M}$ preserves filtered homotopy colimits and final object and that E is such that every (Drinfeld) short exact sequence of dg categories

$$\big(\, \mathcal{A} \overset{I}{\rightarrowtail} \mathcal{B} \overset{P}{\twoheadrightarrow} \mathcal{C} \,\big) \qquad \overset{E}{\longmapsto} \qquad \big(\, E(\mathcal{A}) \overset{E(I)}{\longrightarrow} E(\mathcal{B}) \overset{E(P)}{\longrightarrow} E(\mathcal{C}) \longrightarrow E(\mathcal{A})[1] \,\big)$$

is mapped to a distinguished triangle in the homotopy category $\mathsf{Ho}(\mathcal{M})$. Thanks to the work of Waldhausen [22] and Schlichting [16], of Weibel [23], of Keller [8] and of Blumberg-Mandell [3] (see also [17]), all the above classical theories are localizing invariants. It will be convenient to have a short name for the induced functors on the orbit category.

1.4.1. *Definition.* Let $\mathcal{M}$ be a stable model category and $\mathbf{E} : \mathrm{Or}(G) \to \mathcal{M}$ a functor. We say that $\mathbf{E}$ is a *localizing invariant* if it factors through $\mathrm{Or}(G) \to \mathsf{dgcat}$ as in (1.2.2) followed by a localizing functor $E : \mathsf{dgcat} \to \mathcal{M}$ in the above sense.

Now comes an important step in our discussion. Namely, one of the main results of [18] is the existence of a *universal localizing functor* on dg categories

$$\mathcal{U}^{\mathsf{loc}}_{\mathsf{dg}} : \mathsf{dgcat} \longrightarrow \mathcal{M}ot^{\mathsf{loc}}_{\mathsf{dg}}\,,$$

that is a localizing functor from dgcat to a stable model category $\mathcal{M}ot^{\mathsf{loc}}_{\mathsf{dg}}$ through which all other localizing functors will factor uniquely. Because of this universal property, which is reminiscent of the theory of motives, the homotopy category $\mathsf{Ho}(\mathcal{M}ot^{\mathsf{loc}}_{\mathsf{dg}})$ of this new model category $\mathcal{M}ot^{\mathsf{loc}}_{\mathsf{dg}}$ is sometimes called the *triangulated category of non-commutative motives*. To be precise, the uniqueness involves the language of derivators, see [18, Thm. 10.5] but this is not absolutely essential for the present article. Before stating that variant of universality, let us fix the notation:

1.4.2. *Definition.* Composing the basic functor of (1.2.2) with $\mathcal{U}^{\mathsf{loc}}_{\mathsf{dg}}$, we obtain what we call the *fundamental localizing invariant*

$$\mathbf{E}^{\mathsf{loc}}_{\mathsf{fund}} : \ \mathrm{Or}(G) \xrightarrow{\overline{?}} \mathsf{Grp} \xrightarrow{R[-]} R\text{-}\mathsf{cat} \subset \mathsf{dgcat} \xrightarrow{\mathcal{U}^{\mathsf{loc}}_{\mathsf{dg}}} \mathcal{M}ot^{\mathsf{loc}}_{\mathsf{dg}}$$

on the orbit category of any group G.

1.4.3. **Theorem.** *Let G be a group. For any localizing invariant $\mathbf{E} : \mathrm{Or}(G) \to \mathcal{M}$ on its orbit category (Definition 1.4.1), there exists a functor $\overline{E} : \mathsf{Ho}(\mathcal{M}ot^{\mathsf{loc}}_{\mathsf{dg}}) \to \mathsf{Ho}(\mathcal{M})$ on homotopy categories, which has the following two properties:*

(1) The following diagram commutes up to isomorphism

(1.4.4)
$$\begin{tikzcd} \mathrm{Or}(G) \arrow[r, "\mathbf{E}^{\mathsf{loc}}_{\mathsf{fund}}"] \arrow[dr, "\mathbf{E}"'] & \mathcal{M}ot^{\mathsf{loc}}_{\mathsf{dg}} \arrow[r] & \mathsf{Ho}(\mathcal{M}ot^{\mathsf{loc}}_{\mathsf{dg}}) \arrow[d, "\overline{E}"] \\ & \mathcal{M} \arrow[r] & \mathsf{Ho}(\mathcal{M})\,. \end{tikzcd}$$

(2) The functor $\overline{E}$ preserves homotopy colimits.

Proof. By Definition 1.4.1 there exits a localizing functor $E : \mathsf{dgcat} \to \mathcal{M}$ making the following diagram commute

(1.4.5)
$$\begin{tikzcd} \mathrm{Or}(G) \arrow[r, "\overline{?}"] \arrow[drr, "\mathbf{E}"'] & \mathsf{Grp} \arrow[r, "R[-]"] & R\text{-}\mathsf{cat} \subset \mathsf{dgcat} \arrow[d, "E"] \\ & & \mathcal{M}\,. \end{tikzcd}$$

Since E is localizing, [18, Thm. 10.5] yields a well-defined homotopy colimit preserving functor $\overline{E} : \mathsf{Ho}(\mathcal{M}ot^{\mathsf{loc}}_{\mathsf{dg}}) \to \mathsf{Ho}(\mathcal{M})$ making the following triangle commute

(1.4.6)
$$\begin{tikzcd} \mathsf{Ho}(\mathsf{dgcat}) \arrow[r] \arrow[d, "E"'] & \mathsf{Ho}(\mathcal{M}ot^{\mathsf{loc}}_{\mathsf{dg}}) \arrow[dl, "\overline{E}"] \\ \mathsf{Ho}(\mathcal{M})\,. & \end{tikzcd}$$

Hence, by combining (1.4.5) with (1.4.6) and with the definition of $\mathbf{E}^{\mathsf{loc}}_{\mathsf{fund}}$ we obtain the above commutative diagram (1.4.4). Item (2) is now clear. $\square$

1.4.7. *Remark.* Although this plays no role for the sequel, it is legitimate to wonder how unique the functor $\overline{E}$ of Theorem 1.4.3 is. We do not know of any uniqueness result for two reasons. First, the functor $E : \mathsf{dgcat} \to \mathcal{M}$ in Definition 1.4.1 need not be controlled by its value on the very special dg-categories $R[\overline{G/H}]$. Secondly, although [18, Thm. 10.5] does provide some uniqueness statement for the factorization in (1.4.6), this uniqueness can only be expressed in terms of derivators.

The notation $\overline{E}$ above explains the functors $\mathbb{K}$, $\overline{KH}$, $\overline{HH}$, $\overline{KC}$, and $\overline{THH}$, which appear in (1.2.1). It is quite remarkable that $\overline{E}$ now *preserves homotopy colimits* (not only filtered ones)! Hence $\overline{E}$ will preserve *any* assembly property that $\mathbf{E}^{\mathsf{loc}}_{\mathsf{fund}}$ might enjoy. Indeed, we then obtain the main motivation for the Mamma Conjecture:

1.4.8. **Corollary.** *Let G be a group and $\mathcal{F}$ a family of subgroups. If the fundamental additive functor $\mathbf{E}^{\mathsf{loc}}_{\mathsf{fund}}$ has the $\mathcal{F}$-assembly property, then all localizing invariants on $\mathrm{Or}(G)$ have the same $\mathcal{F}$-assembly property.*

Proof. The assumption is that the $(\mathbf{E}^{\mathsf{loc}}_{\mathsf{fund}}, \mathcal{F}, G)$-assembly map (1.1.1)

$$\operatorname*{hocolim}_{\mathrm{Or}(G,\mathcal{F})} \mathbf{E}^{\mathsf{loc}}_{\mathsf{fund}} \longrightarrow \mathbf{E}^{\mathsf{loc}}_{\mathsf{fund}}(G)\,,$$

is a weak equivalence, *i.e.* an isomorphism in $\mathsf{Ho}(\mathcal{M}ot^{\mathsf{loc}}_{\mathsf{dg}})$. For every localizing invariant $\mathbf{E}$, consider $\overline{E}$ as in Theorem 1.4.3. Applying the functor $\overline{E}$ to the above weak equivalence and using that $\overline{E}$ commutes with homotopy colimits gives precisely the $(\mathbf{E}, \mathcal{F}, G)$-assembly property. $\square$

The reader should now be very tempted to state the following:

1.4.9. **Mamma Conjecture.** *Given a group G, the fundamental localizing invariant $\mathbf{E}^{\mathsf{loc}}_{\mathsf{fund}} : \mathrm{Or}(G) \to \mathcal{M}ot^{\mathsf{loc}}_{\mathsf{dg}}$ has the $\mathcal{VC}$-assembly property.*

Corollary 1.4.8 says that the Mamma Conjecture implies *all* isomorphism conjectures for localizing invariants that can be found on the market. Note that our choice of the family $\mathcal{VC}$ of virtually cyclic groups is merely borrowed from Farrell-Jones and another family $\mathcal{F}$ might be preferable. In any case, the result is that once this is achieved for some family $\mathcal{F}$, then all localizing invariants will automatically inherit the same $\mathcal{F}$-assembly property.

The motivation for focusing on the fundamental localizing invariant $\mathbf{E}^{\mathsf{loc}}_{\mathsf{fund}} : \mathrm{Or}(G) \to \mathcal{M}ot^{\mathsf{loc}}_{\mathsf{dg}}$ came from the common property shared by most invariants for which isomorphism conjectures already exist, namely to be localizing. However, these examples might in fact enjoy some other common property, stronger than being localizing. If such a property is isolated in the future, one should construct the corresponding fundamental invariant and modify (*i.e.* weaken) the Mamma Conjecture accordingly.

2. Reducing the Mamma to K-theory

In this section, we give further justification why the Mamma Conjecture 1.4.9 could actually hold. In a nutshell, we shall reduce it to some *sophisticated* form of K-theory.

2.1. **More on non-commutative motives.** The first approach to understanding the special role K-theory plays in this story is by studying the stable model category $\mathcal{M}ot^{\mathsf{loc}}_{\mathsf{dg}}$, in which our fundamental invariant $\mathbf{E}^{\mathsf{loc}}_{\mathsf{fund}}$ takes values (which models the triangulated category of non-commutative motives). Recall from Kontsevich [9, 10] that a dg category $\mathcal{A}$ is called *smooth* if it is perfect as a bimodule over itself and *proper* if every morphism complex $\mathcal{A}(x, y)$ is a perfect complex of R-modules. In order to simplify the exposition we shall denote by the same letter $\mathcal{A}$ a dg category in dgcat and its image $\mathcal{U}^{\mathsf{loc}}_{\mathsf{dg}}(\mathcal{A})$ in $\mathcal{M}ot^{\mathsf{loc}}_{\mathsf{dg}}$. A remarkable fact about non-commutative motives is that the mapping spectrum in $\mathcal{M}ot^{\mathsf{loc}}_{\mathsf{dg}}$ is given by non-connective K-theory, at least in the smooth case:

2.1.1. **Theorem** ([4, Thm. 8.2]). *Given dg categories $\mathcal{A}$ and $\mathcal{B}$, with $\mathcal{A}$ smooth and proper, there is a canonical stable weak equivalence of spectra*

$$\mathsf{Map}_{\mathcal{M}ot^{\mathsf{loc}}_{\mathsf{dg}}}(\mathcal{A},\mathcal{B}) \cong \mathbb{K}(\mathcal{A}^{\mathsf{op}} \otimes^{\mathbb{L}} \mathcal{B})\,, \tag{2.1.2}$$

where $\mathcal{A}^{\mathsf{op}}$ stands for the opposite dg category and $\mathcal{A}^{\mathsf{op}} \otimes^{\mathbb{L}} \mathcal{B}$ for the derived tensor product in the homotopy category $\mathsf{Ho}(\mathsf{dgcat})$.

2.2. **Revisiting the Mamma.** In order to rephrase the Mamma Conjecture 1.4.9 we now recall a "folklore" conjecture [21] in non-commutative algebraic geometry, which is wide open and of independent interest.

2.2.1. **Conjecture (non-commutative resolution of singularities).** *For a given base ring R, the triangulated category $\mathsf{Ho}(\mathcal{M}ot^{\mathsf{loc}}_{\mathsf{dg}})$ is generated by the smooth and proper dg categories.*

Intuitively speaking, every dg category can be "resolved" in $\mathsf{Ho}(\mathcal{M}ot^{\mathsf{loc}}_{\mathsf{dg}})$ by smooth and proper ones.

2.2.2. **Theorem.** *Assume non-commutative resolution of singularities for R (2.2.1). Let G be a group and $\mathcal{F}$ be a family of subgroups (for instance $\mathcal{F} = \mathcal{VC}$ the virtually cyclic subgroups). Then the following conditions are equivalent:*

(1) *The fundamental localizing functor $\mathbf{E}^{\mathsf{loc}}_{\mathsf{fund}} : \mathrm{Or}(G) \to \mathcal{M}ot^{\mathsf{loc}}_{\mathsf{dg}}$ has the $\mathcal{F}$-assembly property for G.*

(2) *Every localizing invariant $\mathbf{E} : \mathrm{Or}(G) \to \mathcal{M}$ has the $\mathcal{F}$-assembly property for G.*

(3) *The localizing invariant $\mathbb{K}(\mathcal{A} \otimes^{\mathbb{L}} R[\overline{?}])) : \mathrm{Or}(G) \to \mathsf{Spt}$ has the $\mathcal{F}$-assembly property for G, for every smooth and proper dg category $\mathcal{A}$.*

Proof. The implication (1) $\Rightarrow$ (2) is the content of Corollary 1.4.8. Implication (2) $\Rightarrow$ (3) is clear. In order to prove the implication (3) $\Rightarrow$ (1) one needs to show that the $(\mathbf{E}^{\mathsf{loc}}_{\mathsf{fund}}, \mathcal{F}, G)$-assembly map (1.1.1)

$$\operatorname*{hocolim}_{\mathrm{Or}(G,\mathcal{F})} \mathbf{E}^{\mathsf{loc}}_{\mathsf{fund}} \longrightarrow \mathbf{E}^{\mathsf{loc}}_{\mathsf{fund}}(G) \tag{2.2.3}$$

is an isomorphism in $\mathsf{Ho}(\mathcal{M}ot^{\mathsf{loc}}_{\mathsf{dg}})$. Since by hypothesis the triangulated category $\mathsf{Ho}(\mathcal{M}ot^{\mathsf{loc}}_{\mathsf{dg}})$ is generated by the smooth and proper dg categories $\mathcal{A}$, it suffices to show that the image of (2.2.3) under the functors $\mathsf{Map}_{\mathcal{M}ot^{\mathsf{loc}}_{\mathsf{dg}}}(\mathcal{A},-)$ is a weak equivalence of spectra. As explained in [4, Remark 3.2], every smooth and proper dg category $\mathcal{A}$ is compact in $\mathsf{Ho}(\mathcal{M}ot^{\mathsf{loc}}_{\mathsf{dg}})$. This allows us to commute $\mathsf{Map}_{\mathcal{M}ot^{\mathsf{loc}}_{\mathsf{dg}}}(\mathcal{A},-)$ with the homotopy colimit. On the other hand, Equation (2.1.2) together with the definition of $\mathbf{E}^{\mathsf{loc}}_{\mathsf{fund}}$ give us that $\mathsf{Map}_{\mathcal{M}ot^{\mathsf{loc}}_{\mathsf{dg}}}(\mathcal{A},\mathbf{E}^{\mathsf{loc}}_{\mathsf{fund}}(-)) \cong \mathbb{K}(\mathcal{A}^{\mathsf{op}} \otimes^{\mathbb{L}} R[\overline{?}]))$ and we are reduced to precisely the content of (3), since $\mathcal{A}^{\mathsf{op}}$ is smooth and proper exactly when $\mathcal{A}$ is. ☐

2.2.4. *Remark.* Assume 2.2.1. Then Theorem 2.2.2 shows that the Mamma Conjecture is equivalent to the functors $\mathbb{K}(\mathcal{A} \otimes^{\mathbb{L}} R[\overline{?}]) : \mathrm{Or}(G) \to \mathsf{Spt}$ having the $\mathcal{VC}$-assembly property, for every smooth and proper dg category $\mathcal{A}$. Among the latter is $\mathcal{A} = \underline{R}$, in which case the functor in question is the one $\mathbb{K}(R[\overline{?}])$ used by Davis and Lück in the reformulation of the Farrell-Jones Conjecture. Hence all isomorphism conjectures for localizing invariants boil down to versions of the K-theoretic Farrell-Jones Isomorphism Conjecture with "coefficients" in smooth and proper dg categories $\mathcal{A}$.

2.3. **Additive version.** Recall from [18, §15] that the notion of localizing functor admits an additive analogue. These are the functors $E : \mathsf{dgcat} \to \mathcal{M}$ that preserve filtered homotopy colimits and the final object and that map *split* short exact sequences

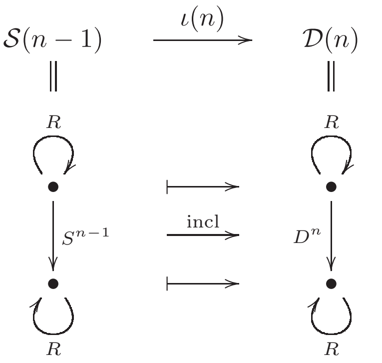
$$\left(\begin{tikzcd} \mathcal{A} \arrow[r, tail, "I"'] & \mathcal{B} \arrow[l, bend right] \arrow[r, two heads, "P"'] & \mathcal{C} \arrow[l, bend right] \end{tikzcd} \right) \quad \overset{E}{\longmapsto} \quad \left(E(\mathcal{A}) \oplus E(\mathcal{C}) \overset{\sim}{\to} E(\mathcal{B}) \right)$$

to direct sums in $\mathsf{Ho}(\mathcal{M})$. As explained in *loc. cit.* every localizing functor is additive but the converse is not true. Connective K-theory (K) is an example of an additive functor which is *not* localizing. We have also the analogue *universal additive functor*

$$\mathcal{U}_{\mathsf{dg}}^{\mathsf{add}} : \mathsf{dgcat} \longrightarrow \mathcal{M}ot_{\mathsf{dg}}^{\mathsf{add}}$$

which induces the *fundamental additive invariant* $\mathbf{E}_{\mathsf{fund}}^{\mathsf{add}} : \mathrm{Or}(G) \to \mathcal{M}ot_{\mathsf{dg}}^{\mathsf{add}}$ by precomposition with (1.2.2) as before. Using the universal property of $\mathcal{U}_{\mathsf{dg}}^{\mathsf{add}}$ we obtain also the additive analogues of Theorem 1.4.3 and Corollary 1.4.8. It is then tempting, although probably too optimistic, to state an additive stronger version of the Mamma Conjecture (1.4.9); see Remark 2.3.3 below.

An interest of this additive approach, is that the category $\mathsf{Ho}(\mathcal{M}ot_{\mathsf{dg}}^{\mathsf{add}})$ is generated by a class of homotopically finitely presented objects coming from dgcat, more precisely the so-called *strictly finite dg cells*. In heuristic terms, they are the dg-category analogues of finite CW-complexes. More precisely, these are the dg categories obtained out of $\varnothing$ by finitely many push-outs along $\varnothing \to \underline{R}$ and along the following dg-analogues $\mathcal{S}(n-1) \to \mathcal{D}(n)$ of the topological inclusions $S^{n-1} \hookrightarrow D^n$:



where

$$\begin{tikzcd} S^{n-1} \arrow[r, "\mathrm{incl}"] \arrow[d, equal] & D^n \arrow[d, equal] & \\ \vdots & \vdots & \\ 0 \arrow[r] \arrow[d] & 0 \arrow[d] & \\ 0 \arrow[r] \arrow[d] & R \arrow[d, "\mathrm{id}"] & \\ R \arrow[r, "\mathrm{id}"] \arrow[d] & R \arrow[d] & \text{(degree } n{-}1) \\ 0 \arrow[r] & 0 & \\ \vdots & \vdots & \end{tikzcd}$$

See more in [18, §15] and [4, §2.1]. The additive analogue of Theorem 2.1.1 has then the following cleaner form, without smoothness assumptions:

2.3.1. **Theorem** ([18, Thm. 15.10])**.** *Given dg categories $\mathcal{A}$ and $\mathcal{B}$, with $\mathcal{A}$ a strictly finite dg cell, there is a canonical stable weak equivalence of spectra*

$$\mathsf{Map}_{\mathcal{M}ot_{\mathsf{dg}}^{\mathsf{add}}}(\mathcal{A}, \mathcal{B}) \cong K(\mathsf{rep}_{\mathsf{dg}}(\mathcal{A}, \mathcal{B}))\,,$$

where $\mathsf{rep}_{\mathsf{dg}}(\mathcal{A}, \mathcal{B})$ stands for the internal-Hom of the homotopy category $\mathsf{Ho}(\mathsf{dgcat})$ and K is connective K-theory.

Theorem 2.2.2 has the following additive version, which does not depend on resolution of singularities:

2.3.2. **Theorem.** *Let G be a group and $\mathcal{F}$ be a family of subgroups. Then the following conditions are equivalent:*

(1) *The fundamental additive functor $\mathbf{E}^{\mathsf{add}}_{\mathsf{fund}}$ has the $\mathcal{F}$-assembly property for G.*
(2) *Every additive invariant $\mathbf{E}$ has the $\mathcal{F}$-assembly property for G.*
(3) *The additive invariant $K(\mathsf{rep}_{\mathsf{dg}}(\mathcal{A}, [\,\overline{?}\,])) : \mathrm{Or}(G) \to \mathsf{Spt}$ has the $\mathcal{F}$-assembly property for G, for every strictly finite dg cell $\mathcal{A}$.*

Proof. The proof is similar to the one of Theorem 2.2.2. In the proof of (1) $\Rightarrow$ (2) use the additive analogue of Corollary 1.4.8 and in the proof of (3) $\Rightarrow$ (1) use Theorem 2.3.1 instead of Theorem 2.1.1 and the fact that by construction of $\mathcal{M}ot^{\mathsf{add}}_{\mathsf{dg}}$ the strictly finite dg cells form a set of compact generators of the homotopy category $\mathsf{Ho}(\mathcal{M}ot^{\mathsf{add}}_{\mathsf{dg}})$; see [18, §15]. □

2.3.3. *Remark.* The $\mathcal{F}$-assembly property for $\mathbf{E}^{\mathsf{add}}_{\mathsf{fund}}$ has few chances to hold for random choices of G, $\mathcal{F}$ and R. For instance, if $\mathcal{F} = \mathcal{V}C$ this property would imply the $(K, \mathcal{V}C, G)$-isomorphism conjecture for $R = \mathbb{Z}$ and this is known to fail; see [14, Rem. 15]. However, if R is a *regular* ring (*i.e.* noetherian and of finite projective dimension) in which the orders of all finite subgroups of G are invertible, then this obstruction vanishes because the $(K, \mathcal{V}C, G)$-isomorphism conjecture follows from the Farrell-Jones Conjecture; see [14, Prop. 70]. A large class of examples is given by taking $R = \mathbb{Q}$ or $\mathbb{C}$ and G arbitrary. Another large class of examples is given by taking $R = \mathbb{Z}$ and G torsion-free.

2.3.4. *Remark.* The additive case just discussed in § 2.3 was the original focus in the longer preprint form of our article, see [1], before we improved the results by considering localizing invariants.

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