IDEMPOTENT COMPLETION OF TRIANGULATED CATEGORIES

Paul Balmer and Marco Schlichting

Revised, June 14, 2000

Abstract. We show that the idempotent completion of a triangulated category has a natural structure of a triangulated category. The idempotent completion of the bounded derived category of an exact category gives the derived category of the idempotent completion. In particular, the derived category of an idempotent complete exact category is idempotent complete.

Introduction

Our article intends to close a gap in the literature by providing a proof that the idempotent completion (also called pseudo-abelian hull or karoubianisation) of a triangulated category is naturally a triangulated category (theorem 1.5). The question of whether this was possible arose in the construction of the derived category of mixed motives ([5, part I, chapter I definition 2.1.6]). In loc.cit. part II, chapter II 2.4, Levine proves our theorem for certain derived categories.

The second author's motivation for the article lies in the observation that for a ring R the unbounded derived category $D(R) = D(\mathcal{P}(R))$ in the sense of [6] of the category $\mathcal{P}(R)$ of finitely generated projective R-modules is idempotent complete iff $K_{-1}(R) = 0$. In fact, $K_{-1}(R)$ is the Grothendieck group of the idempotent completion of $D(R)$ as a triangulated category. Therefore, we need to know that the idempotent completion of a triangulated category has a natural triangulation.

An advantage of idempotent complete triangulated categories over arbitrary ones is that whenever they occur as a full triangulated subcategory of a triangulated category they are épaisse in the latter category (see Rickard's criterion [8, proposition 1.3]).

Many natural triangulated categories are idempotent complete. This holds for instance for the derived categories of perfect complexes over a quasi-separated, quasi-compact scheme (see for example [10]). We add another example by proving in theorem 2.8 that the bounded derived category of an idempotent complete exact category is idempotent complete. This has been established in the split exact (additive) case in [2]. However, the localization of an idempotent complete triangulated category with respect to an idempotent complete full triangulated (hence épaisse) subcategory is triangulated but no longer idempotent complete, in general. By theorem 1.5, the idempotent completion of the localization is still triangulated.

We would like to thank Bernhard Keller and Georges Maltsiniotis for notably improving the organization of this article.

1. Idempotent completion of arbitrary triangulated categories.

1.1. Definition. An additive category K is said to be *idempotent complete* if any idempotent $e : A \to A$, $e^2 = e$, arises from a splitting of A:

$$A = \mathrm{Im}(e) \oplus \mathrm{Ker}(e).$$

1991 *Mathematics Subject Classification.* 18E30.

Key words and phrases. Idempotent complete (karoubian, pseudo-abelian), triangulated category, derived motivic category.

1.2. Definition. Let K be an additive category. The *idempotent completion* of K is the category $\tilde{K}$ defined as follows. Objects of $\tilde{K}$ are pairs (A, e) where A is an object of K and $e : A \to A$ is an idempotent. A morphism in $\tilde{K}$ from (A, e) to (B, f) is a morphism $\alpha : A \to B$ in K such that

$$\alpha\, e = f\, \alpha = \alpha.$$

The assignment $A \mapsto (A, 1)$ defines a functor ι from K to $\tilde{K}$. The following result is well-known.

1.3. Proposition. *The category $\tilde{K}$ is additive, the functor $\iota : K \to \tilde{K}$ is additive and $\tilde{K}$ is idempotent complete. Moreover, the functor ι induces an equivalence*

$$Hom_{add}(\tilde{K}, L) \xrightarrow{\sim} Hom_{add}(K, L)$$

for each idempotent complete additive category L, where Hom_{add} denotes the (large) category of additive functors.

1.4. Remark. The functor ι is fully faithful. From now on, we think of K as a full subcategory of $\tilde{K}$. We will write "$A \in K$" to mean A is isomorphic to an object of K.

Recall that a triangulated category is an additive category K equipped with an endo-equivalence $T : K \to K$ called translation functor and a class of triangles $A \xrightarrow{a} B \xrightarrow{b} C \xrightarrow{c} T(A)$ called exact (distinguished) triangles satisfying the axioms (TR1)-(TR4) of [11, Chap. II, définition 1.1.1, pp. 93-94].

1.5. Theorem. *Let K be a triangulated category. Then its idempotent completion $\tilde{K}$ admits a unique structure of triangulated category such that the canonical functor $\iota : K \to \tilde{K}$ becomes exact. If $\tilde{K}$ is endowed with this structure, then for each idempotent complete triangulated category L, the functor ι induces an equivalence*

$$Hom_{exact}(\tilde{K}, L) \xrightarrow{\sim} Hom_{exact}(K, L)\ ,$$

where Hom_{exact} denotes the (large) category of exact functors.

The following lemmas show that the definition of the triangulated structure on $\tilde{K}$ to be given in 1.10 is the only reasonable one. We give a proof of lemma 1.6 since we didn't find it in [11].

1.6. Lemma. *Let L be a pre-triangulated category, i.e a category satisfying all but the octahedron axiom (TR4). A triangle*

$$A \oplus A' \xrightarrow{\begin{pmatrix} u & 0 \\ 0 & u' \end{pmatrix}} B \oplus B' \xrightarrow{\begin{pmatrix} v & 0 \\ 0 & v' \end{pmatrix}} C \oplus C' \xrightarrow{\begin{pmatrix} w & 0 \\ 0 & w' \end{pmatrix}} T(A) \oplus T(A')$$

is exact if and only if $A \xrightarrow{u} B \xrightarrow{v} C \xrightarrow{w} T(A)$ *and* $A' \xrightarrow{u'} B' \xrightarrow{v'} C' \xrightarrow{w'} T(A')$ *are both exact.*

1.7. Proof. If (u, v, w) and (u', v', w') are exact, then so is their direct sum. This is [11, Corollary 1.2.5], the proof of which does not use the octahedron.

Suppose that the direct sum of (u, v, w) and (u', v', w') is an exact triangle. Let us choose exact triangles over u and u':

$$A \xrightarrow{u} B \xrightarrow{x} D \xrightarrow{y} T(A) \qquad \text{and} \qquad A' \xrightarrow{u'} B' \xrightarrow{x'} D' \xrightarrow{y'} T(A')$$

and let us use (TR3) to construct a morphism $f : C \to D$:

$$\begin{tikzcd}[ampersand replacement=\&]
A \oplus A' \arrow[r, "\begin{pmatrix} u & 0 \\ 0 & u' \end{pmatrix}"] \arrow[d, "(1\ \ 0)"] \& B \oplus B' \arrow[r, "\begin{pmatrix} v & 0 \\ 0 & v' \end{pmatrix}"] \arrow[d, "(1\ \ 0)"] \& C \oplus C' \arrow[r, "\begin{pmatrix} w & 0 \\ 0 & w' \end{pmatrix}"] \arrow[d, dashed, "\exists\, (f\ \ g)"] \& T(A) \oplus T(A') \arrow[d, "(1\ \ 0)"] \\
A \arrow[r, "u"'] \& B \arrow[r, "x"'] \& D \arrow[r, "y"'] \& T(A)
\end{tikzcd}$$

such that $f\,v = x$ and $y\,f = w$ (just ignore g). Similarly, we construct a morphism $f' : C' \to D'$ such that $f'\,v' = x'$ and $y'\,f' = w'$. Now, the direct sum of the two exact triangles over u and u' respectively is exact (see above). Therefore we have a morphism of exact triangles in L:

$$\begin{tikzcd}
A \oplus A' \arrow[r, "\begin{pmatrix} u & 0 \\ 0 & u' \end{pmatrix}"] \arrow[d, equal] & B \oplus B' \arrow[r, "\begin{pmatrix} v & 0 \\ 0 & v' \end{pmatrix}"] \arrow[d, equal] & C \oplus C' \arrow[r, "\begin{pmatrix} w & 0 \\ 0 & w' \end{pmatrix}"] \arrow[d, "\begin{pmatrix} f & 0 \\ 0 & f' \end{pmatrix}"] & T(A) \oplus T(A') \arrow[d, equal] \\
A \oplus A' \arrow[r, "\begin{pmatrix} u & 0 \\ 0 & u' \end{pmatrix}"'] & B \oplus B' \arrow[r, "\begin{pmatrix} x & 0 \\ 0 & x' \end{pmatrix}"'] & D \oplus D' \arrow[r, "\begin{pmatrix} w & 0 \\ 0 & w' \end{pmatrix}"'] & T(A) \oplus T(A').
\end{tikzcd}$$

The "5-lemma" holds as soon as we have a pre-triangulated category and therefore f and f' are isomorphisms. But then, the candidate triangles of the lemma are isomorphic to the ones we chose at the beginning of the proof, respectively with the isomorphisms $(\mathrm{Id}_A, \mathrm{Id}_B, f)$ and $(\mathrm{Id}_{A'}, \mathrm{Id}_{B'}, f')$. ☐

1.8. Lemma. *Let $s : Z \to Y$ be a morphism in a pre-triangulated category L which admits a retraction $v : Y \to Z$, that is $v \circ s = 1_Z$. Then the idempotent $s \circ v : Y \to Y$ splits in L:*

$$Y \cong Z \oplus Im(1 - s \circ v).$$

In particular, if we have an isomorphism $A \oplus B \simeq C$ in $\widetilde{L}$ and if $A, C \in L$ then $B \in L$.

1.9. Proof. Use (TR1) to write down an exact triangle (u, v, w) containing v. We have $w = w \circ v \circ s = 0$ since $v \circ s = 1_Z$ and $w \circ v = 0$ as they are two consecutive arrows of a triangle ([11, Chap. II, Corollaire 1.2.2, p. 97]). Now apply [11, Chap. II, Proposition 1.2.6, p. 100] whose proof doesn't use (TR4). The second part is straigthforward. ☐

1.10. Definition. Let K be a triangulated category. Let us denote by $T : K \to K$ its translation functor. Define $T : \tilde{K} \to \tilde{K}$ by $T(A, e) = \big(T(A), T(e)\big)$. Clearly $T \circ \iota = \iota \circ T$.

Define a triangle in $\tilde{K}$

$$(\Delta) \qquad A \xrightarrow{\alpha} B \xrightarrow{\beta} C \xrightarrow{\gamma} T(A)$$

to be *exact* when it is a direct factor of an exact triangle of K. That is when there is an exact triangle Δ' of K and triangle maps $s : \Delta \to \Delta'$ and $r : \Delta' \to \Delta$ with $r \circ s = 1_\Delta$. Equivalently, when there is a triangle Δ'' in $\tilde{K}$ such that $\Delta \oplus \Delta''$ is isomorphic to an exact triangle in K.

1.11. Remark. In view of lemma 1.6 and lemma 1.8, a triangle in K is exact iff it is exact in $\tilde{K}$. Suppose a triangle Δ of K becomes exact in $\tilde{K}$. By definition, there is a triangle Δ' such that $\Delta \oplus \Delta'$ is isomorphic to an exact triangle of K. By lemma 1.8, Δ' is isomorphic to a triangle in K, and by lemma 1.6, Δ and Δ' are exact in K.

1.12. Theorem. *With the collection of exact triangles given by 1.10, $\tilde{K}$ is a triangulated category.*

Before we start proving the theorem, we need a preliminary lemma.

1.13. Lemma. *Given a commutative diagram in a pre-triangulated category L*

$$\begin{tikzcd}
A \arrow[r, "u"] \arrow[d, "p"] & B \arrow[r, "v"] \arrow[d, "q"] & C \arrow[r, "w"] & T(A) \arrow[d, "T(p)"] \\
A \arrow[r, "u"'] & B \arrow[r, "v"'] & C \arrow[r, "w"'] & T(A)
\end{tikzcd}$$

in which the rows are exact triangles. Suppose $p = p^2$ and $q = q^2$ are idempotents. Then there is an idempotent $r = r^2 : C \to C$ such that the diagram

$$
(*) \qquad \begin{tikzcd}
A \arrow[r, "u"] \arrow[d, "p"] & B \arrow[r, "v"] \arrow[d, "q"] & C \arrow[r, "w"] \arrow[d, "r"] & T(A) \arrow[d, "T(p)"] \\
A \arrow[r, "u"'] & B \arrow[r, "v"'] & C \arrow[r, "w"'] & T(A)
\end{tikzcd}
$$

commutes.

1.14. Proof. By (TR3) there is a $c : C \to C$ making $(*)$ commutative with c instead of r. Of course, c^2 also makes the above diagram commute and so the difference $h := c^2 - c$ has trivial square:

$$h^2 = 0.$$

This is quite classical but let us remind the reader of the proof: from $h\,v = (c^2 - c)\,v = 0$, we can factor h through w, *i.e.* there exists $\bar{h} : T(A) \to C$ such that $h = \bar{h}\,w$ and then $h^2 = \bar{h}\,w\,h = 0$ since $w\,h = w\,(c^2 - c) = 0$.

Applying the trick of lifting idempotents, we set

$$r = c + h - 2\,c\,h.$$

Observe that c and h commute. From $h^2 = 0$, we get $r^2 = c^2 + 2\,c\,h - 4\,c^2\,h$ and then replacing c^2 by $c + h$, we have $r^2 = c + h + 2\,c\,h - 4\,c\,h = r$, using again $h^2 = 0$. Clearly, r can replace c in the above diagram, since $h\,v = 0$ and $w\,h = 0$. By our computation, r is an idempotent. $\square$

1.15. Proof of theorem 1.12. We have to check the four axioms of [11, Chap. II, Définition 1.1.1, pp. 93-94].

(TR1). Any triangle isomorphic to an exact triangle is exact. This follows directly from the definition. If A is an object of $\tilde{K}$, there exists A' such that $A \oplus A' \in K$ $\big($namely, if $A = (B, e)$ take $A' = (B, 1 - e)$ and check that $A \oplus A' \cong \iota(B)\big)$. Then exactness of $A \oplus A' \xrightarrow{1} A \oplus A' \longrightarrow 0 \longrightarrow T(A) \oplus T(A')$ in K insures exactness of $A \xrightarrow{1} A \longrightarrow 0 \longrightarrow T(A)$ in $\tilde{K}$, by definition. We still have to check that any morphism fits into an exact triangle.

Let $\alpha : A \to B$ be a morphism in $\tilde{K}$. Let A' and B' be such that $A \oplus A' \in K$ and $B \oplus B' \in K$. Let

$$
(1) \qquad A \oplus A' \xrightarrow{a} B \oplus B' \xrightarrow{a_1} D \xrightarrow{a_2} T(A \oplus A')
$$

be an exact triangle in K, where $a = \begin{pmatrix} \alpha & 0 \\ 0 & 0 \end{pmatrix}$. Now, using lemma 1.13 in K, we complete the following left commutative square into a morphism of exact triangles in K such that $p = p^2 : D \to D$ is an idempotent:

$$
\begin{tikzcd}
A \oplus A' \arrow[r, "a"] \arrow[d, "\begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}"'] & B \oplus B' \arrow[r, "a_1"] \arrow[d, "\begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}"'] & D \arrow[r, "a_2"] \arrow[d, "p"'] & T(A \oplus A') \arrow[d, "\begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}"] \\
A \oplus A' \arrow[r, "a"'] & B \oplus B' \arrow[r, "a_1"'] & D \arrow[r, "a_2"'] & T(A \oplus A').
\end{tikzcd}
$$

Then $A \xrightarrow{a} B \xrightarrow{p\,a_1} Im(p) \xrightarrow{a_2\,p} T(A)$ is an exact triangle as it is a direct factor of (1).

(TR2) is direct from the definition.

(TR3). Consider a partial map $(\alpha,\beta):\Delta\to\Gamma$ of exact triangles in $\tilde{K}$ (the left square commutes):

$$\begin{tikzcd} \Delta \arrow[d, "(\alpha,\beta)"'] & A \arrow[r, "a"] \arrow[d, "\alpha"] & B \arrow[r, "b"] \arrow[d, "\beta"] & C \arrow[r, "c"] & T(A) \arrow[d, "T(\alpha)"] \\ \Gamma & X \arrow[r, "x"'] & Y \arrow[r, "y"'] & Z \arrow[r, "z"'] & T(X). \end{tikzcd}$$

By definition, there are maps of triangles $i:\Delta\to\Delta'$, $p:\Delta'\to\Delta$, $j:\Gamma\to\Gamma'$, $q:\Gamma'\to\Gamma$ such that $p\,i=1_\Delta$, $q\,j=1_\Gamma$ and such that Δ' and Γ' are exact triangles in K. The partial map of triangles (α,β) induces a partial map of triangles $j\circ(\alpha,\beta)\circ p$ from Δ' to Γ'. Since the two latter triangles are exact in K, we can apply (TR3) to extend the partial map of triangles $j\circ(\alpha,\beta)\circ p$ to a real map $a:\Delta'\to\Gamma'$ of triangles. Then $q\circ a\circ i:\Delta\to\Gamma$ is a map of triangles extending (α,β).

So far, we have established that $\tilde{K}$ is a pre-triangulated category, in the sense that it satisfies all the axioms but the octahedron axiom (TR4).

(TR4) - Octahedron. Let $u:X\to Y$ and $v:Y\to Z$ be two composable morphisms. Let $w=v\circ u$ and choose exact triangles on u, v and w in $\tilde{K}$:

$$(1)\qquad X\xrightarrow{u}Y\xrightarrow{u_1}U\xrightarrow{u_2}T(X)$$

$$(2)\qquad Y\xrightarrow{v}Z\xrightarrow{v_1}V\xrightarrow{v_2}T(Y)$$

$$(3)\qquad X\xrightarrow{w}Z\xrightarrow{w_1}W\xrightarrow{w_2}T(X).$$

Choose A,B and C in $\tilde{K}$ such that $X\oplus A\in K$, $Y\oplus B\in K$ and $Z\oplus C\in K$. Add to (1) the trivial triangles $A\longrightarrow 0\longrightarrow T(A)\xrightarrow{1}T(A)$ and $0\longrightarrow B\xrightarrow{1}B\longrightarrow 0$ to obtain the following triangle which is exact in $\tilde{K}$ (cf. lemma 1.6):

$$(4)\qquad X\oplus A\xrightarrow{\begin{pmatrix}u&0\\0&0\end{pmatrix}}Y\oplus B\xrightarrow{\begin{pmatrix}u_1&0\\0&1\\0&0\end{pmatrix}}U\oplus B\oplus T(A)\xrightarrow{\begin{pmatrix}u_2&0&0\\0&0&1\end{pmatrix}}T(X)\oplus T(A).$$

Observe that the first morphism of (4) is in K and therefore fits into an exact triangle of K which is, via ι, an exact triangle of $\tilde{K}$. Those two triangles are isomorphic since $\tilde{K}$ is pre-triangulated. Therefore, (4) is isomorphic to an exact triangle of K.

Similarly, the two following triangles are isomorphic to exact triangles of K:

$$(5)\qquad Y\oplus B\xrightarrow{\begin{pmatrix}v&0\\0&0\end{pmatrix}}Z\oplus C\xrightarrow{\begin{pmatrix}v_1&0\\0&1\\0&0\end{pmatrix}}V\oplus C\oplus T(B)\xrightarrow{\begin{pmatrix}v_2&0&0\\0&0&1\end{pmatrix}}T(Y)\oplus T(B)$$

$$(6)\qquad X\oplus A\xrightarrow{\begin{pmatrix}w&0\\0&0\end{pmatrix}}Z\oplus C\xrightarrow{\begin{pmatrix}w_1&0\\0&1\\0&0\end{pmatrix}}W\oplus C\oplus T(A)\xrightarrow{\begin{pmatrix}w_2&0&0\\0&0&1\end{pmatrix}}T(X)\oplus T(A).$$

Let us put them into an octahedron (cf. figure (7)). The octahedron exists because it is isomorphic to an octahedron in K. In particular, we find $f:U\oplus B\oplus T(A)\to W\oplus C\oplus T(A)$ and $g:W\oplus C\oplus T(A)\to V\oplus C\oplus T(B)$ which fit into the diagram of figure (7). The 0's and 1's appearing in f and g come from the commutativities required by the octahedron axiom. Moreover, we have:

(8) $g_1\,w_1=v_1$
(9) $w_2\,f_1=u_2$
(10) $f_1\,u_1=w_1\,v$
(11) $v_2\,g_1=T(u)\,w_2$.

From the relation $g\,f = 0$ we obtain:

(12) $g_1\,f_2 + g_2 = 0$
(13) $g_3\,f_1 + f_3 = 0$
(14) $g_3\,f_2 + f_4 + g_4 = 0$.

We shall now use the following endomorphism of $W \oplus C \oplus T(A)$

$$\sigma := \begin{pmatrix} 1 & 0 & -f_2 \\ g_3 & 1 & g_4 \\ 0 & 0 & 1 \end{pmatrix}$$

as presented in figure (7), in order to modify our octahedron. Direct computation gives:

$$\begin{pmatrix} 1 & 0 & f_2 \\ -g_3 & 1 & f_4 \\ 0 & 0 & 1 \end{pmatrix} \cdot \sigma = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & g_3\,f_2 + g_4 + f_4 \\ 0 & 0 & 1 \end{pmatrix} \overset{(14)}{=} \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}$$

and similarly

$$\sigma \cdot \begin{pmatrix} 1 & 0 & f_2 \\ -g_3 & 1 & f_4 \\ 0 & 0 & 1 \end{pmatrix} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & g_3\,f_2 + f_4 + g_4 \\ 0 & 0 & 1 \end{pmatrix} \overset{(14)}{=} \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}$$

which implies that σ is an automorphism with inverse:

$$\sigma^{-1} = \begin{pmatrix} 1 & 0 & f_2 \\ -g_3 & 1 & f_4 \\ 0 & 0 & 1 \end{pmatrix}. \tag{15}$$

Let us modify up to isomorphism the foreground triangle of figure (7) by using this automorphism σ to obtain the following candidate triangle:

(16)
$$U \oplus B \oplus T(A) \xrightarrow{\sigma\,f} W \oplus C \oplus T(A) \xrightarrow{g\,\sigma^{-1}} V \oplus C \oplus T(B) \xrightarrow{\begin{pmatrix} T(u_1)\,v_2 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{pmatrix}} T(U) \oplus T(B) \oplus T^2(A)$$

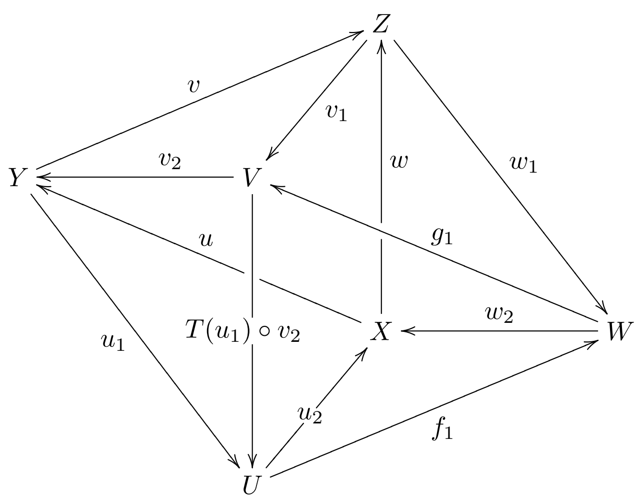
which by its construction is isomorphic to an exact triangle of K. We compute directly

$$\sigma\,f = \begin{pmatrix} 1 & 0 & -f_2 \\ g_3 & 1 & g_4 \\ 0 & 0 & 1 \end{pmatrix} \cdot \begin{pmatrix} f_1 & 0 & f_2 \\ f_3 & 0 & f_4 \\ 0 & 0 & 1 \end{pmatrix} = \begin{pmatrix} f_1 & 0 & 0 \\ g_3\,f_1 + f_3 & 0 & g_3\,f_2 + f_4 + g_4 \\ 0 & 0 & 1 \end{pmatrix} = \begin{pmatrix} f_1 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 1 \end{pmatrix}$$

by (13) and (14). Similarly, we have:

$$g\,\sigma^{-1} \overset{(15)}{=} \begin{pmatrix} g_1 & 0 & g_2 \\ g_3 & 1 & g_4 \\ 0 & 0 & 0 \end{pmatrix} \cdot \begin{pmatrix} 1 & 0 & f_2 \\ -g_3 & 1 & f_4 \\ 0 & 0 & 1 \end{pmatrix} = \begin{pmatrix} g_1 & 0 & g_1\,f_2 + g_2 \\ 0 & 1 & g_3\,f_2 + f_4 + g_4 \\ 0 & 0 & 0 \end{pmatrix} = \begin{pmatrix} g_1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{pmatrix}$$

by (12) and (14). Putting all this together, we obtain the following picture in $\tilde{K}$:



in which all commutativities to be an octahedron are satisfied (use relations 8,9,10,11). The only point is to check that the triangle

$$U \xrightarrow{f_1} W \xrightarrow{g_1} V \xrightarrow{T(u_1)\, v_2} T(U)$$

is exact in $\tilde{K}$. But this is immediate from the exact triangle (16), the explicit computations of σf and $g\,\sigma^{-1}$ and from definition 1.10. $\square$

1.16. Remark. One can avoid the calculations involving σ using the following idea due to Bernhard Keller. Observe that the front triangle of the octrahedron (cf. figure (7)) admits a filtration by subtriangles

$$\begin{tikzcd}
B \arrow[r, "0"] \arrow[d] & C \arrow[r, "j"] \arrow[d] & C \oplus T(B) \arrow[r, "q"] \arrow[d] & T(B) \arrow[d] \\
U \oplus B \arrow[r] \arrow[d] & W \oplus C \arrow[r] \arrow[d] & V \oplus C \oplus T(B) \arrow[r] \arrow[d] & T(U) \oplus T(B) \arrow[d] \\
U \oplus B \oplus T(A) \arrow[r, "f"] & W \oplus C \oplus T(A) \arrow[r, "g"] & V \oplus C \oplus T(B) \arrow[r] & T(U) \oplus T(B) \oplus T^2(A).
\end{tikzcd}$$

The inclusions are the canonical inclusions into the corresponding factors and

$$j = \begin{pmatrix} id_C \\ 0 \end{pmatrix}, \quad q = (\,0 \quad id_{T(B)}\,).$$

Call a triangle trivial if it is isomorphic to $(id, 0, 0)$ or one of its translations. Then the successive subquotients of the above filtration are the triangle $(0, j, q)$, the triangle $(f_1, g_1, T(u_1)v_2)$, and the trivial triangle $(id_{T(A)}, 0, 0)$. Remark that $(0, j, q)$ is the sum of two trivial triangles. By the lemma below, the front triangle is isomorphic to the sum of its three subquotients and hence $(f_1, g_1, T(u_1)v_2)$ is an exact triangle of $\tilde{K}$.

Lemma. *Suppose that*

$$(*) \qquad A \oplus X \xrightarrow{\begin{pmatrix} a & u \\ 0 & x \end{pmatrix}} B \oplus Y \xrightarrow{\begin{pmatrix} b & v \\ 0 & y \end{pmatrix}} C \oplus Z \xrightarrow{\begin{pmatrix} c & w \\ 0 & z \end{pmatrix}} TA \oplus TX$$

is a triangle in a pre-triangulated category. Suppose that (a, b, c) or (x, y, z) is a sum of trivial triangles. Then $(*)$ is isomorphic to the sum of (a, b, c) with (x, y, z).

To prove the lemma, let us suppose that (a, b, c) is a sum of trivial triangles. Then (a, b, c) is contractible, i.e. there exist morphisms

$$A \xleftarrow{a'} B \xleftarrow{b'} C \xleftarrow{c'} T(A)$$

such that $id_A = a'a + (T^{-1}c)(T^{-1}c')$, $id_B = aa' + b'b$, and $id_C = bb' + c'c$. Then it is easy to check that the three morphisms

$$\begin{pmatrix} id_A & a'\,u \\ 0 & id_X \end{pmatrix} \quad \begin{pmatrix} id_B & b'\,v \\ 0 & id_Y \end{pmatrix} \quad \begin{pmatrix} id_C & c'\,w \\ 0 & id_Z \end{pmatrix}$$

yield an isomorphism of $(*)$ with the sum of (a, b, c) and (x, y, z).

1.17. Remark. It is easy to check that if K satisfies the enriched version of the octahedron axiom described in [1, Remark 1.1.13, p. 25-26], then so does $\tilde{K}$. This is left to the reader.

1.18. Proof of theorem 1.5. Clearly, by construction of the triangulation on $\tilde{K}$, the functor $\iota : K \to \tilde{K}$ is exact (see remark 1.11). By lemma 1.6, any triangulation on $\tilde{K}$ has to contain the class of exact triangles given in definition 1.10. It is a well known fact that there cannot exist two different triangulated structures on an additive category such that one of them contains the other (easy consequence of TR1-TR3). This proves uniqueness of the triangulated structure. The rest follows from proposition 1.3 once we have shown that any additive functor $f : \tilde{K} \to L$ is exact as soon as $f \circ \iota$ is exact. But this is an immediate consequence of lemma 1.6 and definition 1.10. □

2. Derived categories of idempotent complete exact categories.

The fact that the localization of certain triangulated categories (e.g. perfect complexes) are not idempotent complete forced Thomason in [10] to introduce negative K-theory of schemes. It is therefore an important problem to decide whether a given triangulated category is idempotent complete or not (see also our introduction). The content of this section is to prove that the bounded derived category of an idempotent complete exact category is idempotent complete. For instance, the bounded derived category of an abelian category is idempotent complete (See Corollary 2.10).

2.1. Background. For the basic notion of exact categories and their derived categories, the reader is referred to Keller [3] or to Neeman [6]. Let us recall shortly what we will need hereafter.

(1) An *exact category* is an additive category $\mathcal{E}$ with a collection of *exact sequences* $\{E \rightarrowtail F \twoheadrightarrow G\}$ where the first morphisms $E \rightarrowtail F$ appearing in those exact sequences are called *admissible monomorphisms* and the second ones *admissible epimorphisms*. They have to satisfy a couple of very natural axioms:
   (1) If $E \rightarrowtail F \twoheadrightarrow G$ is an exact sequence of $\mathcal{E}$, then $E \rightarrowtail F$ is a kernel of $F \twoheadrightarrow G$ and $F \twoheadrightarrow G$ is a cokernel of $E \rightarrowtail F$.
   (2) Any split sequence $E \to E \oplus F \to F$ (with usual maps) is exact. Any sequence isomorphic to an exact sequence is exact.
   (3) Admissible monomorphisms are closed under composition and push-out along any morphism. Admissible epimorphisms are closed under composition and pullback along any morphism.

   The additional "obscure" axiom invoked by Quillen (cf. [7, axiom c), p. 99]) is known to be superfluous (cf. [4, appendix A]). In order to avoid set theoretical difficulties we suppose our exact categories to be small.
(2) Any exact category $\mathcal{E}$ can be embedded as a full subcategory in an abelian category $\mathcal{A}$ in such a way that a sequence in $\mathcal{E}$ is exact iff it is exact in $\mathcal{A}$. If $\mathcal{E}$ is idempotent complete (or even less), this embedding can be chosen in a way that any map in $\mathcal{E}$ which becomes an epimorphism in $\mathcal{A}$ was already an admissible epimorphism in $\mathcal{E}$. For further details we refer the reader to appendix A of [10].
(3) For an exact category $\mathcal{E}$, we let $\mathcal{K}_b(\mathcal{E})$ resp. $\mathcal{K}_+(\mathcal{E})$ be the category whose objects are bounded resp. bounded below complexes with homological indexing and whose morphisms are chain maps up to chain homotopy. These are triangulated categories. An *acyclic complex* is a complex E_* whose differentials decompose as

$$\begin{tikzcd} \cdots \arrow[r] & E_n \arrow[rr, "d_n"] \arrow[dr, "a_n"', two heads] & & E_{n-1} \arrow[r] & \cdots \\ & & F_n \arrow[ur, "b_n"', tail] & & \end{tikzcd}$$

   in such a way that $F_n \overset{b_n}{\rightarrowtail} E_{n-1} \overset{a_{n-1}}{\twoheadrightarrow} F_{n-1}$ is an exact sequence of $\mathcal{E}$ for all $n \in \mathbb{Z}$. A map of chain complexes is a *quasi-isomorphism* if its cone is homotopy equivalent to an acyclic complex. The full subcategory of $\mathcal{K}_b(\mathcal{E})$ resp. $\mathcal{K}_+(\mathcal{E})$ consisting of acyclic complexes is a full triangulated subcategory of $\mathcal{K}_b(\mathcal{E})$ resp. $\mathcal{K}_+(\mathcal{E})$ (see [6, lemma 1.1]). The *bounded derived category* $\mathrm{D_b}(\mathcal{E})$ resp.

bounded below derived category $\mathrm{D}_+(\mathcal{E})$ of $\mathcal{E}$ is the quotient of the triangulated category $\mathcal{K}_b(\mathcal{E})$ resp. $\mathcal{K}_+(\mathcal{E})$ by the full triangulated subcategory of acyclic complexes, i.e. it is the localization of $\mathcal{K}_b(\mathcal{E})$ resp. of $\mathcal{K}_+(\mathcal{E})$ with respect to quasi-isomorphisms. The respective localizations are obtained by a calculus of fractions. Furthermore, $\mathrm{D}_\mathrm{b}(\mathcal{E})$ is a full triangulated subcategory of $\mathrm{D}_+(\mathcal{E})$.

If the exact category $\mathcal{E}$ is idempotent complete then a bounded below complex E_* is acyclic iff it has trivial homology computed in the ambient abelian category of 2.1 (2), a quasi-isomorphism is a chain map whose cone is acyclic, the triangulated subcategories of acyclic complexes are isomorphism closed and épaisses (see [6, lemma 1.2]), and the set of quasi-isomorphisms is saturated.

2.2. Lemma. *Let K be a small triangulated category. If $K_0(\tilde{K}) = 0$ then K is idempotent complete.*

2.3. Proof. This follows easily from Landsburg's criterion ([9, lemma 2.4]) identifying the objects of a triangulated category which give the same class in K_0, see also [9, theorem 2.1]. □

2.4. Lemma. *Let $\mathcal{E}$ be an exact category. The derived category of bounded below complexes $\mathrm{D}_+(\mathcal{E})$ is idempotent complete.*

2.5. Proof. By the previous lemma, we only have to check that $K_0(\widetilde{\mathrm{D}_+(\mathcal{E})}) = 0$. The functor

$$S = \bigoplus_{k\geq 0} T^{2k} : \mathrm{D}_+(\mathcal{E}) \to \mathrm{D}_+(\mathcal{E})$$

is well defined on bounded below complexes and chain maps. It passes to the derived category and prolongs to a functor $S : \widetilde{\mathrm{D}_+(\mathcal{E})} \to \widetilde{\mathrm{D}_+(\mathcal{E})}$. There is a natural equivalence $(T^2 \circ S) \oplus id \simeq S$. Since the functor T^2 induces the identity on K_0, it follows that $K_0(S) + K_0(id) = K_0(S)$, so $K_0(id) = 0$, hence $K_0(\widetilde{\mathrm{D}_+(\mathcal{E})}) = 0$. This is a variant of the usual "Eilenberg swindle". □

2.6. Lemma. *Let $\mathcal{E}$ be an idempotent complete exact category.*

(1) *Let $M = \cdots \to E_n \xrightarrow{d_n} E_{n-1} \longrightarrow \cdots$ be a (not necessarily bounded) acyclic complex. Then d_n has a kernel in $\mathcal{E}$ for all $n \in \mathbb{Z}$ and the truncation*

$$\sigma_{\leq n}(M) = \cdots \longrightarrow 0 \longrightarrow \ker(d_n) \longrightarrow E_n \longrightarrow E_{n-1} \longrightarrow \cdots$$

is a complex in $\mathcal{E}$ which is quasi-isomorphic to M by the natural map $M \to \sigma_{\leq n}(M)$.

(2) *Let $s : L \to N$ be a morphism of complexes of $\mathcal{E}$. Suppose that s is a quasi-isomorphism. Suppose further that one of L or N is bounded above. Call the other one X, then X has the following property: for some $n_0 \in \mathbb{Z}$ all its boundary maps d_n have a kernel in $\mathcal{E}$ for $n \geq n_0$. In particular, $\sigma_{\leq n_0} X$ is a complex in $\mathcal{E}$ and the natural chain map $X \to \sigma_{\leq n_0} X$ is a quasi-isomorphism.*

2.7. Proof. (1). This follows from the definition of M acyclic, see 2.1 (3).

(2). Apply (1) to the mapping cone of s which is acyclic by idempotent completeness of $\mathcal{E}$ (see 2.1 (3)) and observe that for n big enough the boundary of this mapping cone is the boundary of the one of the two complexes which is not supposed bounded above. For the last statement use the description of quasi-isomorphisms for idempotent complete exact categories given in 2.1 (3). □

2.8. Theorem. *Let $\mathcal{E}$ be an idempotent complete exact category. Then $\mathrm{D}_\mathrm{b}(\mathcal{E})$ is idempotent complete.*

2.9. Proof. Since $\mathcal{E}$ is idempotent complete, there exists an embedding $\mathcal{E} \hookrightarrow \mathcal{A}$ as described in point 2.1, part (2).

Let (M, p) be a bounded complex equipped with an idempotent $p = p^2 : M \to M$ in $\mathrm{D}_\mathrm{b}(\mathcal{E})$. By lemma 2.2, there exist two bounded below complexes $P, Q \in \mathrm{D}_+(\mathcal{E})$ and an isomorphism in $\mathrm{D}_+(\mathcal{E})$

$$f : M \xrightarrow{\sim} P \oplus Q$$

such that p becomes $\begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}$ on $P \oplus Q$. The isomorphism f can be represented by a fraction of quasi-isomorphisms:

$$M \xrightarrow{\ s\ } R \xleftarrow{\ t\ } P \oplus Q$$

for some bounded below complex $R = \cdots \longrightarrow R_n \xrightarrow{d_n} R_{n-1} \longrightarrow \cdots$. Here we use that the set of quasi-isomorphisms is saturated (see 2.1 (3)). Applying lemma 2.6 (2) to M, R and s we have that $\ker d_n \in \mathcal{E}$ for n greater or equal than some $n_0 \in \mathbb{Z}$. Now apply the same lemma to $P \oplus Q$, $\sigma_{\leq n_0}(R)$ and the quasi-isomorphism $P \oplus Q \xrightarrow{\sim} R \xrightarrow{\sim} \sigma_{\leq n_0}(R)$. This proves that for n big enough, the kernel of the boundary map of $P \oplus Q$ is in $\mathcal{E}$. But in the abelian category $\mathcal{A}$, this kernel is the sum of the corresponding kernels in P and Q. Since $\mathcal{E}$ is idempotent complete, it forces both of them to be in $\mathcal{E}$.

Put $\tilde{P}$ and $\tilde{Q}$ to be the corresponding truncations. Then we have an isomorphism

$$M \simeq \tilde{P} \oplus \tilde{Q}$$

in $\mathrm{D_b}(\mathcal{E})$ which carries the idempotent p to $\begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}$ on $\tilde{P} \oplus \tilde{Q}$, due to the functoriality of the truncations described above. $\square$

2.10. Corollary. *Let $\mathcal{A}$ be an abelian category. Then $\mathrm{D_b}(\mathcal{A})$ is idempotent complete.*

2.11. Proof. An abelian category is idempotent complete. $\square$

2.12. Corollary. *Let $\mathcal{E}$ be an exact category. Then* $\widetilde{\mathrm{D_b}(\mathcal{E})} \cong \mathrm{D_b}(\widetilde{\mathcal{E}})$.

2.13. Proof. The inclusion of exact categories $\mathcal{E} \subset \widetilde{\mathcal{E}}$ induces a full inclusion of triangulated categories $\mathrm{D_b}(\mathcal{E}) \subset \mathrm{D_b}(\widetilde{\mathcal{E}})$. It is easily seen that every object of $\mathrm{D_b}(\widetilde{\mathcal{E}})$ is a direct summand of an object of $\mathrm{D_b}(\mathcal{E})$. The corollary then follows from theorem 2.8. $\square$

References

[1] A. BEILINSON, J. BERNSTEIN, P. DELIGNE, *Faisceaux Pervers*, Astérisque 100, Société Mathématique de France (1982).

[2] M. BÖKSTEDT, A. NEEMAN, *Homotopy limits in triangulated categories*, Compositio Math. 86 (1993), no. 2, 209–234.

[3] Bernhard KELLER, *Derived categories and their uses*, In: Handbook of algebra, volume I, M. Hazewinkel (ed.), Elsevier, Amsterdam (1996), pp. 671-701.

[4] Bernhard KELLER, *Chain complexes and stable categories*, Manuscripta Math. 67 (1990), no. 4, pp. 379–417.

[5] Marc LEVINE, *Mixed motives*, Math. surveys and monographs 57, American Mathematical Society, Providence, RI (1998) x+515 pp.

[6] Amnon NEEMAN, *The derived category of an exact category*, J. of Algebra 135, pp. 388-394 (1990).

[7] Daniel QUILLEN, *Higher algebraic K-theory I*, Algebraic K-theory, I: Higher K-theories (Proc. Conf., Battelle Memorial Inst., Seattle, Wash., 1972), pp. 85–147. Lecture Notes in Math., Vol. 341, Springer, Berlin 1973.

[8] Jeremy RICKARD, *Derived categories and stable equivalence*, J. Pure Applied Algebra 61 (1989), no. 3, pp. 303–317.

[9] R. W. THOMASON, *The classification of triangulated subcategories*, Compositio Math. 105 (1997), no. 1, 1–27.

[10] R. W. THOMASON, T. TROBAUGH, *Higher algebraic K-theory of schemes and of derived categories*, In: The Grothendieck Festschrift, volume III, P. Cartier (ed.), Progress in mathematics volume 88, pp. 247-435, Birkäuser (1990).

[11] Jean-Louis VERDIER, *Catégories dérivées des catégories abéliennes*, Astérisque 239, Société mathématique de France (1996).