GENERALIZED TENSOR IDEMPOTENTS AND THE TELESCOPE CONJECTURE

PAUL BALMER AND GIORDANO FAVI

Abstract. We transpose Rickard's construction of idempotent representations to the general context of tensor triangulated categories. We connect our construction to the Telescope Conjecture and prove that the latter is of local nature. As an illustration, we extend the affine Telescope Theorem of Neeman to arbitrary noetherian schemes. We also develop supports for non-compact objects.

Contents

Introduction

Tensor triangular geometry is the study of tensor triangulated categories, in algebraic geometry, homotopy theory, modular representation theory and beyond; see [2, 3]. This abstract framework allows the transposition of techniques and ideas from one area to another. For instance, the basic idea of *gluing* has been abstracted from usual algebraic geometry to tensor triangular geometry in [7] and then applied to modular representation theory in [6] and [4].

In the present paper, we proceed in the opposite direction : We extend a technique from modular representation theory, namely idempotent representations in the sense of Rickard [33], to the general framework of tensor triangulated categories and give an application in algebraic geometry.

But let us first remind the reader of Rickard's insight [33].

Let G be a finite group and k a field. Consider $\mathcal{V}_G \overset{\text{def.}}{=} \operatorname{Proj}(\mathrm{H}^\bullet(G,k))$ the associated projective support variety, see [8, Vol. II, Chap. 5]. For every closed subset $W \subset \mathcal{V}_G$, Rickard tells us how to construct two possibly infinite-dimensional

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kG-modules, $E(W)$ and $F(W)$, which are *tensor idempotents*, $E(W)\otimes E(W)\simeq E(W)$ and $F(W)\otimes F(W)\simeq F(W)$, modulo projective modules. More precisely he constructs an exact triangle $E(W)\to k\to F(W)\to\Sigma E(W)$ in the stable module category $\mathrm{Stab}(kG)=kG-\mathrm{Mod}\big/kG-\mathrm{Proj}$, such that tensoring with $F(W)$ realizes localization away from W. The latter means that the functor $F(W)\otimes -:\mathrm{Stab}(kG)\longrightarrow\mathrm{Stab}(kG)$ is the Bousfield localization of $\mathrm{Stab}(kG)$ with respect to the smashing subcategory $\mathrm{Stab}_W(kG):=\langle\mathrm{stab}_W(kG)\rangle$. Here, $\langle\mathrm{stab}_W(kG)\rangle$ is the localizing subcategory of $\mathrm{Stab}(kG)$ generated by the thick $\otimes$-ideal subcategory

$$\mathrm{stab}_W(kG):=\big\{\,M\in\mathrm{stab}(kG)\,\big|\,\mathcal{V}_G(M)\subset W\,\big\}\tag{0.1}$$

of those finite dimensional representations whose projective support variety $\mathcal{V}_G(M)$ is contained in our chosen closed subset W. This interplay between the subcategory $\mathrm{stab}(kG)$ of compact objects (finite dimensional representations) and the big tensor triangulated category $\mathrm{Stab}(kG)$ is a central theme in this whole subject.

Rickard proves $E(W_1\cap W_2)\simeq E(W_1)\otimes E(W_2)$ and $F(W_1\cup W_2)\simeq F(W_1)\otimes F(W_2)$ and he constructs two Mayer-Vietoris exact triangles in $\mathrm{Stab}(kG)$

$$E(W_1\cap W_2)\longrightarrow E(W_1)\oplus E(W_2)\longrightarrow E(W_1\cup W_2)\longrightarrow\Sigma E(W_1\cap W_2)$$

$$F(W_1\cap W_2)\longrightarrow F(W_1)\oplus F(W_2)\longrightarrow F(W_1\cup W_2)\longrightarrow\Sigma\,F(W_1\cap W_2)\,.$$

All this is very useful in modular representation theory.

As Rickard already indicates, his techniques are reminiscent of the topologists' homotopy theoretic methods involved in Brown representability. Nowadays, it is well known how such techniques can be generalized to big enough triangulated categories $\mathcal{T}$, *e.g.* compactly generated ones; see Hovey-Palmieri-Strickland [22] or Neeman [31], for instance. For us, $\mathcal{T}$ will be a compactly generated tensor triangulated category, in the precise sense of Hypotheses 1.1 below. We denote by $\mathcal{T}^c\subset\mathcal{T}$ the subcategory of compact objects. For example, $(\mathrm{Stab}(kG))^c=\mathrm{stab}(kG)$.

The above is classical by now. Nonetheless, until recently, one essential ingredient in Rickard's work did not have a clear generalization to any compactly generated tensor triangulated category $\mathcal{T}$: It is the projective support variety $\mathcal{V}_G$. This is a fundamental question since $\mathcal{V}_G$ contains the closed subsets W which parametrize Rickard's idempotents. Similarly, the above Mayer-Vietoris results involve some ambient space in which to take union and intersection of W_1 and W_2. In fact, in the above construction, the only place where $\mathcal{V}_G$ plays an essential role is in the definition of the thick ideal $\mathrm{stab}_W(kG)$ of compact objects, as in (0.1) above. In this definition, one could actually use any "support datum" (Rem. 5.3) instead of $\mathcal{V}_G(-)$ and still obtain thick ideals. See more on this in Remark 5.26 below. However, assuming we want the *best* theory right away, we should try to use a notion of support which catches *all* thick ideals of compact objects immediately. This is exactly what the *spectrum* of [2] does for us.

Recall from [2] that for a tensor triangulated category $\mathcal{K}$, like here $\mathcal{K}=\mathcal{T}^c$, the spectrum $\mathrm{Spc}(\mathcal{K})$ is a topological space in which every object $x\in\mathcal{K}$ has a support $\mathrm{supp}(x)\subset\mathrm{Spc}(\mathcal{K})$ and such that one can parametrize all thick ideals in $\mathcal{K}$ via $W\mapsto\mathcal{K}_W:=\big\{\,x\in\mathcal{K}\,\big|\,\mathrm{supp}(x)\subset W\,\big\}$, where $W\subset\mathrm{Spc}(\mathcal{K})$ runs through so-called *Thomason subsets* (Def. 5.7). In the example of $\mathcal{T}=\mathrm{Stab}(kG)$, we have $\mathcal{K}=\mathcal{T}^c=\mathrm{stab}(kG)$ and its spectrum coincides with $\mathcal{V}_G$ in such a way that $\mathrm{supp}(M)=\mathcal{V}_G(M)$, see [2, Cor. 5.10]. (And $\mathcal{V}_G$ is noetherian, so every closed subset is Thomason.)

In short, a conceptual generalization of Rickard's framework to all possible $\mathcal{T}$ is provided by tensor triangular geometry of $\mathcal{T}^c$, following [2].

Using our approach, we then prove the complete generalization of Rickard's results to any compactly generated tensor triangulated category $\mathcal{T}$ (as in Hyp. 1.1): For every Thomason subset $W \subset \mathrm{Spc}(\mathcal{T}^c)$ we construct two $\otimes$-idempotents $e(W) \simeq e(W) \otimes e(W)$ and $f(W) \simeq f(W) \otimes f(W)$ in $\mathcal{T}$, together with an exact triangle $e(W) \to \mathbb{1} \to f(W) \to \Sigma e(W)$, such that $f(W) \otimes - : \mathcal{T} \longrightarrow \mathcal{T}$ realizes Bousfield localization with respect to $\mathcal{T}_W := \langle (\mathcal{T}^c)_W \rangle$, *i.e.* localization away from W. (Here $\mathbb{1} \in \mathcal{T}$ is the unit of the tensor structure on $\mathcal{T}$.) This is done in Section 5. We establish generalized Mayer-Vietoris triangles in Theorem 5.18.

There are as many applications of this construction as there are compactly generated tensor triangulated categories $\mathcal{T}$. This goes even beyond the areas mentioned above and does apply to motivic theory or noncommutative topology (equivariant KK-theory of C^*-algebras) for instance; see Dell'Ambrogio [19]. We do not pursue this direction here. Instead, we connect our construction to the so-called Telescope Conjecture, which is the *property* that every smashing subcategory of $\mathcal{T}$ be generated from its compact part; see Def. 4.2. Some categories $\mathcal{T}$ have this property, some don't. It is notoriously unclear whether the stable homotopy category $\mathcal{T} = \mathrm{SH}$ of topological spectra has the Telescope property or not. See Remark 4.3.

Here, we prove that the Telescope property is of local nature relatively to open covers of $\mathrm{Spc}(\mathcal{T}^c)$; see a precise statement in Theorem 6.6. As a corollary, we give a quick proof that the Telescope property holds for the derived category of a noetherian scheme (Alonso *et al.* [1, Thm. 5.8]) by simply reducing it to the affine case proven by Neeman [30]; see Corollary 6.8.

Another thing one can do with tensor idempotents is create *residue objects* $\kappa(\mathcal{P}) \in \mathcal{T}$ for $\mathcal{P} \in \mathrm{Spc}(\mathcal{T}^c)$ in the spectrum of the compact part of $\mathcal{T}$. This is done in Section 7 and allows the construction of support for big objects, *i.e.* subsets $\mathrm{Supp}(t) \subset \mathrm{Spc}(\mathcal{T}^c)$ for any $t \in \mathcal{T}$. This generalizes the first definition of supports $\mathcal{V}_G(M)$ for possibly non finite-dimensional representations M, given in [9].

The structure of the paper should now be clear from the table of contents.

1. All hypotheses on the table

1.1. **Hypotheses.** Throughout the paper, $\mathcal{T}$ stands for a *(rigidly-) compactly generated tensor triangulated category.* The poetically inclined reader might prefer to call $\mathcal{T}$ a "unital algebraic stable homotopy category", as in [22][1]. We prefer the above somewhat more explicit terminology. We denote by

$$\mathcal{K} = \mathcal{T}^c$$

the subcategory of rigid and compact objects. We now unfold the details.

We assume familiarity with the notion of triangulated category and we denote by $\Sigma : \mathcal{T} \to \mathcal{T}$ the suspension functor. We assume that $\mathcal{T}$ has arbitrary (small) coproducts $\sqcup_{i \in I} t_i$. An object $c \in \mathcal{T}$ is *compact* if, for every set $\{t_i\}_{i \in I}$ of objects of $\mathcal{T}$, the natural map $\oplus_{i \in I} \mathrm{Hom}(c, t_i) \to \mathrm{Hom}(c, \sqcup_{i \in I} t_i)$ is an isomorphism. In [22], compact objects are called *small.* We assume that $\mathcal{T}$ is *compactly generated*, that is,

[1]The Brown representability hypothesis [22, Def. 1.1.4. (e)] is redundant by [31, Thm. 8.3.3].

the subcategory of compact objects $\mathcal{K} = \mathcal{T}^c$ is essentially small (has a *set* of isomorphism classes) and, most importantly, if an object $t \in \mathcal{T}$ is such that $\mathrm{Hom}(c, t) = 0$ for every $c \in \mathcal{K}$ then $t = 0$. See also Remark 2.3 below.

Finally, let us discuss tensor and rigidity. We assume that $\mathcal{T}$ admits a closed symmetric monoidal structure (see for instance [22, App. A])

$$\otimes : \mathcal{T} \times \mathcal{T} \longrightarrow \mathcal{T}\,.$$

Again, we remind the reader. We have natural isomorphisms $x \otimes y \cong y \otimes x$, which for our present purposes we may treat as identities, and a unit $\mathbb{1}$ for the tensor product: $\mathbb{1} \otimes x \cong x$. Moreover, there exists an adjoint $\underline{\mathrm{hom}} : \mathcal{T}^{\mathrm{op}} \times \mathcal{T} \longrightarrow \mathcal{T}$ such that $\mathrm{Hom}(x \otimes y, z) \cong \mathrm{Hom}(x, \underline{\mathrm{hom}}(y, z))$. Both functors, $\otimes$ and $\underline{\mathrm{hom}}$, are also assumed exact in each variable. An object $x \in \mathcal{T}$ is called *rigid* (or *strongly dualizable*) if for every $y \in \mathcal{T}$, the natural map $\underline{\mathrm{hom}}(x, \mathbb{1}) \otimes y \to \underline{\mathrm{hom}}(x, y)$ is an isomorphism. We also require that $\mathbb{1}$ is compact and that every compact object is rigid, which is equivalent to say that compact objects coincide with rigid objects; see [22, Thm. 2.1.3 (d)]. So, the subcategory $\mathcal{K}$ of rigid-compact objects is essentially small, rigid (meaning that all its objects are so) and $\otimes$-triangulated.

1.2. **Examples.** We refer to [22, Ex. 1.2.3] for justifications.

(1) $\mathcal{T} = \mathrm{SH}$, the stable homotopy category of topological spectra. Here $\mathcal{K} = \mathrm{SH}^{\mathrm{fin}}$, the stable homotopy category of *finite* spectra. See [28] for details.

(2) $\mathcal{T} = \mathrm{D}(R\text{−Mod})$, the unbounded derived category of a commutative ring R (see [26, Example 5.8]). Here $\mathcal{K} = \mathrm{K^b}(R\text{−proj})$. More generally $\mathcal{T} = \mathrm{D}(X) := \mathrm{D}_{\mathrm{Qcoh}}(\mathcal{O}_X)$, the derived category of unbounded complexes of $\mathcal{O}_X$-modules with quasi-coherent homology over a quasi-compact and quasi-separated scheme X. (Quasi-separated means that quasi-compact open subsets are stable under pairwise intersection, *i.e.* form an open basis. This is equivalent to say that X is spectral, see Rem. 5.11.) By Bondal and van den Bergh [14, Thm. 3.1.1], $\mathcal{K} = \mathcal{T}^c$ is the category $\mathrm{D}^{\mathrm{perf}}(X)$ of *perfect* complexes over X, *i.e.* complexes which are locally quasi-isomorphic to bounded complexes of vector bundles; see SGA 6 [12] or [36]. It is shown in [13, Cor. 5.5] that the natural functor $\mathrm{D}(\mathrm{Qcoh}(X)) \to \mathrm{D}(X)$ is an equivalence when X is moreover separated, which the reader can assume if $\mathrm{D}(\mathrm{Qcoh}(X))$ sounds more familiar. The tensor structure is standard; detailed references are collected in [17, § 1] for instance.

(3) $\mathcal{T} = \mathrm{Stab}(kG)$, the stable category of all kG-modules, for G a finite group (scheme). Here $\mathcal{K} = \mathrm{stab}(kG)$, the stable category of *finite dimensional* kG-modules.

2. Bousfield localization and smashing ideals

2.1. **Definition.** Recall that a triangulated subcategory $\mathcal{S} \subset \mathcal{T}$ is called *thick* if it is stable under taking direct summands. It is called *localizing* if it is stable under arbitrary coproducts. It is colocalizing if it is stable under arbitrary products. (Co)localizing subcategories are thick. Finally, $\mathcal{S}$ is called $\otimes$-*ideal* if $\mathcal{T} \otimes \mathcal{S} \subset \mathcal{S}$.

2.2. **Notation.** If $\mathcal{E}$ is a class of objects of $\mathcal{T}$, we will denote by $\langle \mathcal{E} \rangle$ the smallest localizing triangulated subcategory of $\mathcal{T}$ which contains all objects of $\mathcal{E}$. Similarly $\langle \mathcal{E} \rangle^{\otimes}$ for the smallest localizing $\otimes$-ideal.

2.3. *Remark.* In Hypotheses 1.1, the fact that $\mathcal{T}$ is compactly generated can be expressed as $\langle\mathcal{K}\rangle = \mathcal{T}$. Consequently, if $\mathcal{E} \subset \mathcal{T}$ satisfies $\mathcal{K}\otimes\mathcal{E} \subset \langle\mathcal{E}\rangle$ (typically $\mathcal{E} \subset \mathcal{K}$ $\otimes$-ideal inside $\mathcal{K}$, that is $\mathcal{K}\otimes\mathcal{E} \subset \mathcal{E}$), then $\langle\mathcal{E}\rangle$ is $\otimes$-ideal in $\mathcal{T}$ and $\langle\mathcal{E}\rangle = \langle\mathcal{E}\rangle^{\otimes}$.

We now recall standard results about localization of triangulated categories. Most of this material can be found in the now classical reference [31] or in the monograph [22]. For a recent comparative survey, see Krause [26].

Recall the *Verdier localization* $q : \mathcal{T} \longrightarrow \mathcal{T}/\mathcal{S}$ of a triangulated category $\mathcal{T}$ by a thick subcategory $\mathcal{S}$, see [31, § 2.1]. One pitfall with Verdier's construction is that the quotient $\mathcal{T}/\mathcal{S}$, in which morphisms are equivalence classes of fractions, might have proper classes of morphisms, not sets, *i.e.* $\mathcal{T}/\mathcal{S}$ might be a "large" category. Sometimes, the functor $q : \mathcal{T} \longrightarrow \mathcal{T}/\mathcal{S}$ has an adjoint, which is automatically faithful, hence realizes $\mathcal{T}/\mathcal{S}$ as a subcategory of $\mathcal{T}$, avoiding the above pitfall. This happens in Bousfield localization, as we now recall.

2.4. **Definition.** A *Bousfield localization functor* on $\mathcal{T}$ is a pair (L, λ) where $L : \mathcal{T} \to \mathcal{T}$ is an exact functor and $\lambda : \mathrm{Id}_{\mathcal{T}} \to L$ is a natural transformation such that $L\lambda : L \to L^2$ is an isomorphism and $L\lambda = \lambda L$. A *morphism* of localization functors $(L, \lambda) \to (L', \lambda')$ on the same category $\mathcal{T}$ is a natural transformation $\ell : L \to L'$ such that $\ell \circ \lambda = \lambda'$. For a localization functor (L, λ), we have $\mathrm{Im}(L) = \{t \in \mathcal{T} \,|\, \lambda_t \text{ is an isomorphism}\}$. (For a functor F, we denote by $\mathrm{Im}(F)$ its essential image.) These are often called *L-local objects.*

The dual notion of *colocalization functor* is also important for us: (Γ, γ) is a colocalization functor on $\mathcal{T}$ if $(\Gamma^{\mathrm{op}}, \gamma^{\mathrm{op}})$ is a localization functor on $\mathcal{T}^{\mathrm{op}}$, etc.

2.5. **Definition.** If $\mathcal{A}, \mathcal{B}$ are classes of objects in $\mathcal{T}$ we write $\mathcal{A}\perp\mathcal{B}$ if $\mathrm{Hom}(a, b) = 0$ for all $a \in \mathcal{A}$ and $b \in \mathcal{B}$. For a class $\mathcal{C}$ of objects of $\mathcal{T}$ we consider its *right orthogonal* $\mathcal{C}^{\perp} = \{t \in \mathcal{T} \,|\, \mathrm{Hom}(c, t) = 0 \text{ for all } c \in \mathcal{C}\}$ and its *left orthogonal* ${}^{\perp}\mathcal{C} = \{t \in \mathcal{T} \,|\, \mathrm{Hom}(t, c) = 0 \text{ for all } c \in \mathcal{C}\}$. Both ${}^{\perp}\mathcal{C}$ and $\mathcal{C}^{\perp}$ are thick subcategories of $\mathcal{T}$. Moreover, ${}^{\perp}\mathcal{C}$ is always localizing and $\mathcal{C}^{\perp}$ is always colocalizing.

2.6. **Theorem** (Bousfield localization)**.** *Let $\mathcal{T}$ be a triangulated category and $\mathcal{S}$ be a thick subcategory. Then the following statements are equivalent:*

(i) There exists a localization functor $(L, \lambda) : \mathcal{T} \to \mathcal{T}$ with $\mathcal{S} = \mathrm{Ker}(L)$.
(ii) There exists a colocalization functor $(\Gamma, \gamma) : \mathcal{T} \to \mathcal{T}$ with $\mathcal{S} = \mathrm{Im}(\Gamma)$.
(iii) Each $t \in \mathcal{T}$ fits in an exact triangle $t' \to t \to t'' \to \Sigma t'$ with $t' \in \mathcal{S}$ and $t'' \in \mathcal{S}^{\perp}$.
(iv) The Verdier quotient functor $q : \mathcal{T} \to \mathcal{T}/\mathcal{S}$ exists and has a right adjoint.
(v) The composition $\mathcal{S}^{\perp} \to \mathcal{T} \to \mathcal{T}/\mathcal{S}$ is an equivalence.

Assume that (i)–(v) hold. Then $\mathcal{S}^{\perp} = \mathrm{Im}(L) = \mathrm{Ker}(\Gamma)$ and ${}^{\perp}(\mathcal{S}^{\perp}) = \mathcal{S}$. Moreover, the triangle in (iii) is functorial in t and is the unique one with these properties (up to unique isomorphism of triangles being the identity on t). We denote it by

$$\Delta_{\mathcal{S}}(t) := \Big(\Gamma_{\mathcal{S}}(t) \to t \to L_{\mathcal{S}}(t) \to \Sigma\Gamma_{\mathcal{S}}(t)\Big). \tag{2.7}$$

Proof. See [31, Chap. 9] or [22, Chap. 3] or [26, § 4] (for this and even more). □

2.8. **Definition.** A localizing subcategory $\mathcal{S}$ satisfying the equivalent conditions of Thm. 2.6 is called a *Bousfield subcategory of* $\mathcal{T}$. The *localization exact triangle* for $\mathcal{S}$ will refer to $\Delta_{\mathcal{S}}$ of (2.7) above, featuring the *(co)localization functors* $\Gamma_{\mathcal{S}}$ and $L_{\mathcal{S}}$.

2.9. *Remark.* If $\mathcal{S} \subset \mathcal{S}'$ are Bousfield subcategories of $\mathcal{T}$ then there is a unique morphism of triangles $(\epsilon, \mathrm{id}, \varphi) : \Delta_{\mathcal{S}} \to \Delta_{\mathcal{S}'}$. In the following (solid) diagram

$$\begin{tikzcd}
\Gamma_{\mathcal{S}}(t) \arrow[r, "\gamma"] \arrow[d, dotted, "\epsilon"'] & t \arrow[r, "\lambda"] \arrow[d, equal] & L_{\mathcal{S}}(t) \arrow[r] \arrow[d, dotted, "\varphi"] & \Sigma\Gamma_{\mathcal{S}}(t) \arrow[d, dotted, "\Sigma\epsilon"] \\
\Gamma_{\mathcal{S}'}(t) \arrow[r, "\gamma'"] & t \arrow[r, "\lambda'"] & L_{\mathcal{S}'}(t) \arrow[r] & \Sigma\Gamma_{\mathcal{S}'}(t),
\end{tikzcd}$$

we have $\lambda' \circ \gamma \in \mathrm{Hom}(\Gamma_{\mathcal{S}}(t), L_{\mathcal{S}'}(t)) = 0$ since $\Gamma_{\mathcal{S}}(t) \in \mathcal{S} \subset \mathcal{S}' \perp (\mathcal{S}')^{\perp} \ni L_{\mathcal{S}'}(t)$. Using standard properties of exact triangles, we get the existence of ϵ such that $\gamma'\epsilon = \gamma$, and then of φ making the diagram commute. These ϵ and φ are unique since another choice would differ by a map in $\mathrm{Hom}(\Sigma\Gamma_{\mathcal{S}}(t), L_{\mathcal{S}'}(t)) = 0$. This yields a unique morphism of localization functors $\varphi : L_{\mathcal{S}} \to L_{\mathcal{S}'}$.

The equivalence $\mathcal{T}/\mathcal{S} \cong \mathcal{S}^{\perp}$ of Thm. 2.6 (v) allows us to discuss $\mathcal{T}/\mathcal{S}$ inside $\mathcal{T}$, without calculus of fractions. As an interlude, we illustrate this with the *gluing* technique, originally treated without Bousfield localization in [7].

2.10. **Definition** (See [7, Def. 2.1])**.** We say that two thick subcategories $\mathcal{S}_1, \mathcal{S}_2$ of a triangulated category $\mathcal{T}$ are in *formal Mayer-Vietoris situation* if $\mathcal{S}_1 \perp \mathcal{S}_2$ and $\mathcal{S}_2 \perp \mathcal{S}_1$, *i.e.* $\mathrm{Hom}(\mathcal{S}_1, \mathcal{S}_2) = \mathrm{Hom}(\mathcal{S}_2, \mathcal{S}_1) = 0$. Then $\mathcal{S}_1 \cap \mathcal{S}_2 = 0$. We denote by $\mathcal{S}_1 \oplus \mathcal{S}_2$ the thick subcategory whose objects are $\{t \in \mathcal{T} \mid t \simeq s_1 \oplus s_2 \text{ for some } s_1 \in \mathcal{S}_1, s_2 \in \mathcal{S}_2\}$. We want to express the weak cartesian nature of the following diagram :

$$\begin{tikzcd}
\mathcal{T} \arrow[r] \arrow[d] & \mathcal{T}/\mathcal{S}_1 \arrow[d] \\
\mathcal{T}/\mathcal{S}_2 \arrow[r] & \mathcal{T}/(\mathcal{S}_1 \oplus \mathcal{S}_2) .
\end{tikzcd}$$

2.11. **Theorem** (Gluing of objects [7, Thm. 4.3])**.** *Let $\mathcal{T}$, $\mathcal{S}_1$ and $\mathcal{S}_2$ be as above. Set $\mathcal{S}_{12} = \mathcal{S}_1 \oplus \mathcal{S}_2$. Assume that $\mathcal{T}/\mathcal{S}_i$ has small* Hom *sets, for $i \in \{1, 2, 12\}$ (e.g. $\mathcal{S}_i$ Bousfield). Let $t_1 \in \mathcal{T}/\mathcal{S}_1$ and $t_2 \in \mathcal{T}/\mathcal{S}_2$ be two objects and $\sigma : t_1 \overset{\sim}{\to} t_2$ an isomorphism in $\mathcal{T}/\mathcal{S}_{12}$. Then there exists an object $t \in \mathcal{T}$ and isomorphisms $t \simeq t_i$ in $\mathcal{T}/\mathcal{S}_i$ for $i = 1, 2$, compatible with σ in $\mathcal{T}/\mathcal{S}_{12}$. The object t so obtained is unique up to possibly non unique isomorphism and is called* a gluing of t_1 and t_2 along σ.

Alternative proof in the Bousfield case : For $i \in \{1, 2, 12\}$, we further suppose that $\mathcal{S}_i$ is Bousfield (Def. 2.8) with localization functors $L_i : \mathcal{T} \to \mathcal{T}$. By Thm. 2.6 (vi), we can identify $\mathcal{T}/\mathcal{S}_i$ with $\mathcal{S}_i^{\perp} = \mathrm{Im}(L_i)$ and view the isomorphism $\sigma : t_1 \to t_2$ in $\mathcal{T}/\mathcal{S}_{12}$ as an isomorphism $\sigma' : L_{12}(t_1) \overset{\sim}{\to} L_{12}(t_2)$ in $\mathcal{T}$. By Rem. 2.9, the inclusion $\mathcal{S}_i \subset \mathcal{S}_{12}$ induces morphisms of localizations $\varphi_{12,j} : L_j \to L_{12}$ for $j = 1, 2$. Now form a weak pull-back in $\mathcal{T}$ of $\varphi_{12,2}$ and $\sigma' \circ \varphi_{12,1}$ as follows

$$\begin{tikzcd}
t \arrow[r, dashed, "\alpha_1"] \arrow[d, dashed, "\alpha_2"'] \arrow[dr, phantom, "\lrcorner"] & L_1(t_1) \arrow[d] \arrow[r, "\varphi_{12,1}"] & L_{12}(t_1) \arrow[dl, "\simeq", "\sigma'"'] \\
L_2(t_2) \arrow[r, "\varphi_{12,2}"'] & L_{12}(t_2) &
\end{tikzcd}$$

From this construction, it is an exercise to see that t has the properties of a gluing. Compare [6, § 6]. The desired isomorphisms correspond to $L_1(\alpha_1)$ and $L_2(\alpha_2)$. □

Bousfield localization also yields an alternative proof of the following :

2.12. **Theorem** (Gluing of morphisms [7, Thm. 3.5])**.** *Under the same hypotheses as Thm. 2.11, one can also glue morphisms in $\mathcal{T}$: Given objects $s, t \in \mathcal{T}$ and morphisms $f_i : s \to t$ in $\mathcal{T}/\mathcal{S}_i$ for $i = 1, 2$, which agree in $\mathcal{T}/\mathcal{S}_{12}$, there is a (non necessarily unique) morphism $f : s \to t$ in $\mathcal{T}$ whose image in $\mathcal{T}/\mathcal{S}_i$ is equal to f_i, for $i = 1, 2$.*

So far, we have not used the tensor structure on $\mathcal{T}$. Here it comes.

2.13. **Theorem** (Smashing localization)**.** *Let $(\mathcal{T}, \otimes, \mathbb{1})$ be a $\otimes$-triangulated category and let $\mathcal{S}$ be a Bousfield subcategory (Def. 2.8). Let $\Delta_{\mathcal{S}} :\ \Gamma_{\mathcal{S}} \xrightarrow{\gamma} \mathrm{Id} \xrightarrow{\lambda} L_{\mathcal{S}} \to \Sigma\Gamma_{\mathcal{S}}$ be the localization triangle for $\mathcal{S}$, see (2.7). Now, suppose in addition that $\mathcal{S}$ is a $\otimes$-ideal (Def. 2.1). Then the following conditions are equivalent:*

(i) The subcategory $\mathcal{S}^{\perp}$ is also a $\otimes$-ideal.
(ii) There is an isomorphism of functors $L_{\mathcal{S}} \simeq L_{\mathcal{S}}(\mathbb{1}) \otimes -$.
(iii) There is an isomorphism of functors $\Delta_{\mathcal{S}} \simeq \Delta_{\mathcal{S}}(\mathbb{1}) \otimes -$.

Proof. This is also standard. We drop the subscripts $_{\mathcal{S}}$ from the notation. Assume that (i) holds. For every t in $\mathcal{T}$ consider the exact triangle $\Delta(\mathbb{1}) \otimes t$:

$$\Gamma(\mathbb{1}) \otimes t \longrightarrow t \longrightarrow L(\mathbb{1}) \otimes t \longrightarrow \Sigma\, \Gamma(\mathbb{1}) \otimes t.$$

Since $\mathcal{S}$ is $\otimes$-ideal, we have $\Gamma(\mathbb{1}) \otimes t \in \mathcal{S}$. Since we assume $\mathcal{S}^{\perp}$ $\otimes$-ideal, we also have $L(\mathbb{1}) \otimes t \in \mathcal{S}^{\perp}$. By uniqueness of the triangle $\Delta(t)$ we get (iii). Clearly (iii)$\Rightarrow$(ii). To show (ii)$\Rightarrow$(i), let $t \in \mathcal{S}^{\perp} = \mathrm{Im}(L)$ and $t' \in \mathcal{T}$ arbitrary. Then $t \otimes t' \simeq L(t) \otimes t' \simeq L(\mathbb{1}) \otimes t \otimes t' \simeq L(t \otimes t')$ hence $t \otimes t' \in \mathrm{Im}(L) = \mathcal{S}^{\perp}$. $\square$

2.14. *Remark.* The isomorphisms in Thm. 2.13 (ii)-(iii) are unique, if they are the identity morphism on the t in the second term of $\Delta_{\mathcal{S}}(t)$. We leave this to the reader.

2.15. **Definition.** A $\otimes$-ideal Bousfield subcategory $\mathcal{S}$ of $\mathcal{T}$ is called a *smashing ideal* if $\mathcal{S}$ satisfies the equivalent conditions of Theorem 2.13. We will denote by $\mathbb{S}(\mathcal{T})$ the class of all smashing ideals of $\mathcal{T}$ and order it by inclusion $\subset$.

2.16. *Remark.* Under Hypotheses 1.1, this $\mathbb{S}(\mathcal{T})$ is indeed a *set* by Krause [25, Thm. 4.9]. Also, the notion of smashing subcategory is sometimes formulated without $\otimes$-structure, by asking the localization functor $L_{\mathcal{S}}$ to preserve arbitrary coproducts, see [26, Prop. 5.5.1]. These two notions coincide here, see [22, Def. 3.3.2].

2.17. *Remark.* Let $\mathcal{T}$ be a $\otimes$-triangulated category and $\mathcal{S}$ be a smashing ideal of $\mathcal{T}$. Then there is an inclusion-preserving bijection between the smashing ideals of $\mathcal{T}$ which contain $\mathcal{S}$ and the smashing ideals of $\mathcal{T}/\mathcal{S}$, given by $\mathcal{S}' \mapsto \mathcal{S}'/\mathcal{S} \subset \mathcal{T}/\mathcal{S}$.

3. Tensor idempotents

Instead of smashing ideals (Def. 2.15) we can equivalently consider $\otimes$-idempotents.

3.1. **Proposition.** *Let $\mathcal{T}$ be a $\otimes$-triangulated category and let $\Delta :\ e \xrightarrow{\gamma} \mathbb{1} \xrightarrow{\lambda} f \to \Sigma e$ be an exact triangle. Then the following are equivalent:*

(i) $\gamma \otimes \mathrm{id}_e$ is an isomorphism $e \otimes e \xrightarrow{\sim} e$.
(ii) $e \otimes f = 0$.
(iii) $\lambda \otimes \mathrm{id}_f$ is an isomorphism $f \xrightarrow{\sim} f \otimes f$.

Moreover, in that case, $\mathrm{Ker}(- \otimes f) = \mathrm{Im}(- \otimes e)$ *and* $\mathrm{Ker}(- \otimes e) = \mathrm{Im}(- \otimes f)$.

Proof. Triangles $\Delta\otimes e$ and $\Delta\otimes f$ give (i)$\Leftrightarrow$(ii) and (ii)$\Leftrightarrow$(iii), respectively. The moreover part can be read on $\Delta\otimes t$ for arbitrary $t\in\mathcal{T}$. $\square$

3.2. **Definition.** An exact triangle $\Delta:\ e\overset{\gamma}{\to}\mathbb{1}\overset{\lambda}{\to}f\to\Sigma e$ satisfying the equivalent conditions of Prop. 3.1 will be called an *idempotent triangle*. Moreover, $\gamma: e\to\mathbb{1}$ will be called a *left idempotent* and $\lambda:\mathbb{1}\to f$ a *right idempotent*. A morphism of idempotent triangles $\Delta\to\Delta'$ is a morphism of triangles of the form $(\epsilon,\mathrm{id}_{\mathbb{1}},\varphi)$:

$$\begin{tikzcd} e \arrow[r, "\gamma"] \arrow[d, "\epsilon"'] & \mathbb{1} \arrow[r, "\lambda"] \arrow[d, equal] & f \arrow[r] \arrow[d, "\varphi"] & \Sigma e \arrow[d, "\Sigma\epsilon"] \\ e' \arrow[r, "\gamma'"] & \mathbb{1} \arrow[r, "\lambda'"] & f' \arrow[r] & \Sigma e'. \end{tikzcd} \tag{3.3}$$

Morphisms of left and right idempotents are defined in the obvious similar way, keeping only the square with ϵ and φ respectively. We denote by

$$\mathbb{D}(\mathcal{T}),\ \mathbb{E}(\mathcal{T}) \text{ and } \mathbb{F}(\mathcal{T})$$

the collection of isomorphism classes of idempotent triangles, of left idempotents and of right idempotents respectively. We denote by $[-]$ these isomorphism classes. All three $\mathbb{D}(\mathcal{T})$, $\mathbb{E}(\mathcal{T})$ and $\mathbb{F}(\mathcal{T})$ carry a partial order induced by morphisms. For instance, $[\Delta]\le[\Delta']$ if and only if there exist a morphism of idempotent triangles $(\epsilon,\mathrm{id},\varphi):\Delta\to\Delta'$. We will often simply write $\Delta\le\Delta'$, $e\le e'$ or $f\le f'$.

3.4. *Remark.* Following Proposition 3.1 and its proof, one sees that the (partially ordered) classes $\mathbb{D}(\mathcal{T})$, $\mathbb{E}(\mathcal{T})$ and $\mathbb{F}(\mathcal{T})$ are isomorphic via the forgetful maps $\mathbb{D}(\mathcal{T})\to\mathbb{E}(\mathcal{T}),\mathbb{F}(\mathcal{T})$, whose inverses come from completing morphisms into exact triangles. We now prove that they also agree with the class of smashing ideals $\mathbb{S}(\mathcal{T})$, see Def. 2.15.

3.5. **Theorem** (Compare [22, Lem. 3.1.6])**.** *Let $(\mathcal{T},\otimes,\mathbb{1})$ be a $\otimes$-triangulated category. All four partially ordered classes above, $\mathbb{S}(\mathcal{T})$, $\mathbb{D}(\mathcal{T})$, $\mathbb{E}(\mathcal{T})$ and $\mathbb{F}(\mathcal{T})$, are isomorphic. More precisely:*

- *(a) For every smashing ideal $\mathcal{S}\subset\mathcal{T}$, the triangle $\Delta_{\mathcal{S}}(\mathbb{1})$ defined in (2.7) is an idempotent triangle, which will be denoted*

$$\Delta(\mathcal{S}) := \Big(\ e(\mathcal{S})\xrightarrow{\gamma}\mathbb{1}\xrightarrow{\lambda}f(\mathcal{S})\longrightarrow\Sigma e(\mathcal{S})\ \Big). \tag{3.6}$$

  Up to unique isomorphism of idempotent triangles, $\Delta(\mathcal{S})$ is characterized by the following three properties: $\Delta(\mathcal{S})$ is exact, $e(\mathcal{S})\in\mathcal{S}$ and $f(\mathcal{S})\in\mathcal{S}^{\perp}$.
- *(b) For every idempotent triangle $\Delta:\ e\xrightarrow{\gamma}\mathbb{1}\xrightarrow{\lambda}f\longrightarrow\Sigma e$, the functor L_f defined by $-\otimes f:\mathcal{T}\to\mathcal{T}$ is a localization functor, whose kernel $\mathrm{Ker}(-\otimes f)$ is a smashing ideal, denoted $\mathcal{S}(\Delta)$. We have $\mathcal{S}(\Delta)=\mathrm{Ker}(-\otimes f)=\mathrm{Im}(-\otimes e)$ whereas $\mathcal{S}(\Delta)^{\perp}=\mathrm{Ker}(-\otimes e)=\mathrm{Im}(-\otimes f)$.*

The constructions $\mathcal{S}\mapsto\Delta(\mathcal{S})$ and $\Delta\mapsto\mathcal{S}(\Delta)$, described in (a) and (b), are order-preserving, mutually inverse bijections between $\mathbb{S}(\mathcal{T})$ and $\mathbb{D}(\mathcal{T})$.

Proof. For (a), since $\mathcal{S}$ and $\mathcal{S}^{\perp}$ are $\otimes$-ideals, $\Gamma_{\mathcal{S}}(\mathbb{1})\otimes L_{\mathcal{S}}(\mathbb{1})\in\mathcal{S}\cap\mathcal{S}^{\perp}=0$. We conclude by Proposition 3.1. For (b), the functor $L_f=-\otimes f:\mathcal{T}\to\mathcal{T}$ is a localization with $\lambda:\mathrm{Id}\to L_f$ induced by $\lambda:\mathbb{1}\to f$. It is clear that L_f satisfies Theorem 2.13 (ii). Hence $\mathcal{S}(\Delta):=\mathrm{Ker}(-\otimes f)$ is a smashing ideal. We claim that $f\in\mathcal{S}(\Delta)^{\perp}$. Indeed, if $\alpha : t \to f$ is a morphism with $t \in \mathcal{S}(\Delta)$, then tensoring α with $\lambda : \mathbb{1} \to f$ gives the commutative square

$$\begin{tikzcd} t \arrow[r] \arrow[d, "\alpha"'] & t \otimes f = 0 \arrow[d] \\ f \arrow[r, "\simeq"'] & f \otimes f \end{tikzcd}$$

which implies $\alpha = 0$. Thus $f \in \mathcal{S}(\Delta)^{\perp}$ and $e \in \mathcal{S}(\Delta)$. In short, $\Delta : \ e \to \mathbb{1} \to f \to \Sigma e$ must be the unique exact triangle with these properties, *i.e.* $\Delta \simeq \Delta_{\mathcal{S}(\Delta)}(\mathbb{1})$. It is now easy to see that $\Delta \mapsto \mathcal{S}(\Delta)$ and $\mathcal{S} \mapsto \Delta(\mathcal{S})$ are inverse bijections and that the latter is order-preserving, see Rem. 2.9. Finally, we show that $\Delta \mapsto \mathcal{S}(\Delta)$ is order-preserving. Let $(\epsilon, \mathrm{id}, \varphi) : \Delta \to \Delta'$ be a morphism of idempotent triangles as in (3.3) above. Applying $e \otimes -$ to (3.3) and using that $e \otimes f = 0$ shows that $\mathrm{id}_e \otimes \lambda' : e \to e \otimes f' = L_{f'}(e)$ is zero. This implies $L_{f'}(e) = 0$ by universality of $e \to L_{f'}(e)$ (or just by applying $- \otimes f'$), hence $e \in \mathrm{Ker}(- \otimes f')$. We get as wanted $\mathcal{S}(\Delta) = \mathrm{Im}(- \otimes e) \subset \mathrm{Ker}(- \otimes f') = \mathcal{S}(\Delta')$. □

3.7. **Corollary.** *Let $(\mathcal{T}, \otimes, \mathbb{1})$ be a $\otimes$-triangulated category.*

(a) *Let $\Delta = (e \to \mathbb{1} \to f \to \Sigma e)$ be an idempotent triangle. Then we have* $\mathrm{Im}(- \otimes e) = \mathrm{Ker}(- \otimes f) \ \perp \ \mathrm{Im}(- \otimes f) = \mathrm{Ker}(- \otimes e)$.

(b) *If $[\Delta] \le [\Delta']$ then there exists a* unique *morphism of idempotent triangles $\Delta \to \Delta'$. Similarly for left idempotents and for right idempotents.*

(c) *If two right idempotents f and f' are isomorphic as objects of $\mathcal{T}$, via an isomorphism $\alpha : f \overset{\sim}{\to} f'$, then there is a unique isomorphism of right idempotents $\varphi : f \overset{\sim}{\to} f'$ (possibly different from α). Same with left idempotents and idempotent triangles, mutatis mutandis.*

Proof. We use Theorem 3.5 throughout: (a) is clear from $\mathrm{Ker}(- \otimes f) = \mathcal{S}(\Delta) \perp \mathcal{S}(\Delta)^{\perp} = \mathrm{Im}(- \otimes f)$. For (b), if $\Delta \le \Delta'$, we have $\mathcal{S}(\Delta) \subset \mathcal{S}(\Delta')$ and uniqueness of $\Delta \to \Delta'$ is Remark 2.9. For (c), use that $\mathrm{Ker}(- \otimes f) = \mathrm{Ker}(- \otimes f')$ in $\mathbb{S}(\mathcal{T})$. □

3.8. *Remark.* For $\mathcal{T}$ compactly generated, $\mathbb{D}(\mathcal{T})$, $\mathbb{E}(\mathcal{T})$ and $\mathbb{F}(\mathcal{T})$ are sets (Rem. 2.16).

3.9. *Remark.* Any $\otimes$-triangulated functor $q : \mathcal{T} \to \mathcal{T}'$ induces three obvious compatible maps $\mathbb{D}(\mathcal{T}) \to \mathbb{D}(\mathcal{T}')$, $\mathbb{E}(\mathcal{T}) \to \mathbb{E}(\mathcal{T}')$ and $\mathbb{F}(\mathcal{T}) \to \mathbb{F}(\mathcal{T}')$ and consequently a fourth one $\mathbb{S}(\mathcal{T}) \to \mathbb{S}(\mathcal{T}')$ by the bijection of Thm. 3.5. If moreover $\mathcal{T}'$ is of the form $\mathcal{T}/\mathcal{S}$ for some smashing subcategory $\mathcal{S}$ of $\mathcal{T}$ then $\mathbb{S}(\mathcal{T}) \to \mathbb{S}(\mathcal{T}/\mathcal{S})$ is nothing but the map of Remark 2.17. We leave this verification to the reader.

3.10. **Notation.** When $e \to \mathbb{1} \to f \to \Sigma e$ is an idempotent triangle, we use ${}^{\perp}f$ to mean e and similarly $e^{\perp}$ to mean f. For any right idempotent f, we thus have $({}^{\perp}f)^{\perp} = f$ and $f \otimes {}^{\perp}f = 0$ as well as $\mathrm{Im}(- \otimes {}^{\perp}f) = \mathrm{Ker}(- \otimes f)$.

3.11. **Proposition.** *Let $\mathcal{T}$ be a $\otimes$-triangulated category. Then the four isomorphic partial orders $(\mathbb{S}(\mathcal{T}), \subset)$, $(\mathbb{D}(\mathcal{T}), \le)$, $(\mathbb{E}(\mathcal{T}), \le)$ and $(\mathbb{F}(\mathcal{T}), \le)$ are* lattices, *i.e. any finite subset has a supremum and an infimum. The tensor product provides the meet $\wedge$ (pairwise infimum) on $\mathbb{E}(\mathcal{T})$ and the join $\vee$ (pairwise supremum) on $\mathbb{F}(\mathcal{T})$. Explicitly, for any pair of left idempotents, we have $e \wedge e' = e \otimes e'$ and for any pair of right idempotents we have $f \vee f' = f \otimes f'$. On $\mathbb{S}(\mathcal{T})$, this becomes: If $\mathcal{S}$ and $\mathcal{S}'$ are smashing ideals, then $\mathcal{S} \wedge \mathcal{S}' = \mathcal{S} \cap \mathcal{S}'$ whereas $\mathcal{S} \vee \mathcal{S}' = \mathrm{Ker}(- \otimes f(\mathcal{S}) \otimes f(\mathcal{S}'))$.*

Proof. Tensor clearly preserves left and right idempotents. If $\alpha : f \to f''$ and $\alpha' : f' \to f''$ are morphisms of right idempotents then $\alpha \otimes \alpha' : f \otimes f' \to f'' \otimes f'' \simeq f''$ is the unique (Cor. 3.7) morphism of right idempotents making the following diagram commute

$$\begin{tikzcd}
\mathbb{1} \arrow[r, "\lambda"] \arrow[d, "\lambda'"'] & f \arrow[d, "1\otimes\lambda'"'] \arrow[ddr, bend left, "\alpha"] & \\
f' \arrow[r, "\lambda\otimes 1"] \arrow[drr, bend right, "\alpha'"'] & f\otimes f' \arrow[dr, "\alpha\otimes\alpha'"'] & \\
& & f''.
\end{tikzcd}$$

This shows that $-\otimes-$ gives coproducts of right idempotents, hence joins in $\mathbb{F}(\mathcal{T})$. Dually, $-\otimes-$ gives products of left idempotents, hence meets in $\mathbb{E}(\mathcal{T})$. The bijections $\mathbb{D}(\mathcal{T}) \simeq \mathbb{E}(\mathcal{T}) \simeq \mathbb{F}(\mathcal{T}) \simeq \mathbb{S}(\mathcal{T})$ of Thm. 3.5 yield joins and meets everywhere. One checks that $\mathrm{Im}(-\otimes e)\wedge\mathrm{Im}(-\otimes e') = \mathrm{Im}(-\otimes e\otimes e') = \mathrm{Im}(-\otimes e)\cap\mathrm{Im}(-\otimes e')$. $\square$

3.12. **Corollary** (Compare [33, Lem. 7.2])**.** *If $e \le e'$ in $\mathbb{E}(\mathcal{T})$, then $e\otimes(e')^{\perp} = 0$ and there is a unique isomorphism of left idempotents $e\otimes e' \cong e$. Similarly, if $f \le f'$ in $\mathbb{F}(\mathcal{T})$, then $({}^{\perp}f)\otimes f' = 0$ and $f\otimes f' \cong f'$. (See Notation 3.10.)*

Proof. $e \le e'$ implies $e\wedge e' = e$ and $\wedge$ is given by $\otimes$ on left idempotents by Prop. 3.11. Hence the unique isomorphism of left idempotents $e\otimes e' \cong e$ by Cor. 3.7. It follows that $e\otimes(e')^{\perp} \cong e\otimes e'\otimes(e')^{\perp} = 0$. The dual is dual. $\square$

We can now state and prove an abstract version of Rickard's Mayer-Vietoris triangles, see [33, Thm. 8.1]. The idea of the proof is the same as in *loc. cit.*

3.13. **Theorem** (Abstract Mayer-Vietoris triangles; see Rouquier [34, Prop. 5.10])**.** *Let $e, e' \in \mathbb{E}(\mathcal{T})$ be left idempotents and $f, f' \in \mathbb{F}(\mathcal{T})$ be right idempotents in any $\otimes$-triangulated category $\mathcal{T}$. Then there exist exact triangles in $\mathcal{T}$*

(a) $e\otimes e' \to e\oplus e' \to e\vee e' \to \Sigma(e\otimes e')$,

(b) $f\wedge f' \to f\oplus f' \to f\otimes f' \to \Sigma(f\wedge f')$.

Proof. Let $\lambda : \mathbb{1} \to f$ and $\lambda' : \mathbb{1} \to f'$ be in $\mathbb{F}(\mathcal{T})$. By Prop. 3.11, there is a commutative square of right idempotents as in the outer square of the following diagram:

$$\begin{tikzcd}
f\wedge f' \arrow[rr, "i"] \arrow[dr, dashed, "v"] \arrow[dd, "i'"'] & & f \arrow[dd, "p"] \\
& d \arrow[ur, "q"'] \arrow[dl, "q'"] \arrow[dr, phantom, "\lrcorner"] & \\
f' \arrow[rr, "p'"'] & & f\otimes f',
\end{tikzcd}$$

which we complete by introducing the weak pull-back d of $p = \mathrm{id}_f\otimes\lambda'$ and $p' = \lambda\otimes\mathrm{id}_{f'}$, as well as a corner morphism $v : f\wedge f' \to d$ making the diagram commute. Let us prove that $c := \mathrm{cone}(v)$ is zero. First note that all the objects $f, f', f\wedge f'$ and $f\otimes f'$ become zero when we tensor them by ${}^{\perp}(f\wedge f') = {}^{\perp}f\otimes{}^{\perp}f'$. Thus $d\otimes{}^{\perp}(f\wedge f') = 0$ and $c\otimes{}^{\perp}(f\wedge f') = 0$ as well. It now suffices to prove that $c\otimes{}^{\perp}(f\wedge f') \simeq c$, that is, $c \in \mathrm{Im}(-\otimes{}^{\perp}(f\wedge f')) = \mathrm{Ker}(-\otimes(f\wedge f')) = \mathrm{Ker}(-\otimes f)\cap\mathrm{Ker}(-\otimes f')$, using Proposition 3.11 again.

Observe that $\mathrm{id}_f\otimes p' = \mathrm{id}_f\otimes\lambda\otimes\mathrm{id}_{f'}$ is an isomorphism since f is idempotent. Applying $f\otimes-$ to the above weak pull-back gives another weak pull-back, from which we deduce that $\mathrm{id}_f \otimes q$ is an isomorphism as well. On the other hand, the same is true for i by Corollary 3.12 applied to $f \wedge f' \leq f$, that is, $\mathrm{id}_f \otimes i$ is an isomorphism. By 2-out-of-3, we see that v has the same property : it becomes an isomorphism under $f \otimes -$. This implies that $f \otimes c \simeq 0$, that is, $c \in \mathrm{Ker}(- \otimes f)$. Symmetrically, $c \in \mathrm{Ker}(- \otimes f')$ hence $c \in \mathrm{Ker}(- \otimes f) \cap \mathrm{Ker}(- \otimes f')$ as wanted.

So, $c \simeq 0$ and v is an isomorphism. This gives the exact triangle in (b). The proof for the triangle in (a) is dual. □

3.14. *Remark.* The proof gives more precisely two weakly (bi) cartesian squares

$$\begin{tikzcd} e \otimes e' = e \wedge e' \arrow[r] \arrow[d] & e \arrow[d] \\ e' \arrow[r] & e \vee e' \end{tikzcd} \quad \text{and} \quad \begin{tikzcd} f \wedge f' \arrow[r] \arrow[d] & f \arrow[d] \\ f' \arrow[r] & f \vee f' = f \otimes f' \end{tikzcd}$$

in $\mathcal{T}$, where all the maps are morphisms of idempotents, as given by Proposition 3.11. This characterizes those maps uniquely by Corollary 3.7.

Always walking in Rickard's path (see [33, Cor. 8.2] and its proof) we deduce a generalized version of a celebrated theorem of Jon Carlson (see [18]).

3.15. **Corollary** (Generalised Carlson Theorem)**.** *Let $\mathcal{S}, \mathcal{S}'$ be smashing ideals in $\mathcal{T}$.*

(a) If $\mathcal{S} \cap \mathcal{S}' = 0$, then we have $e(\mathcal{S} \vee \mathcal{S}') \simeq e(\mathcal{S}) \oplus e(\mathcal{S}')$ and $\mathcal{S} \vee \mathcal{S}' = \mathcal{S} \oplus \mathcal{S}'$, that is, any $t \in \mathcal{S} \vee \mathcal{S}'$ is decomposable as $t \simeq s \oplus s'$ where $s \in \mathcal{S}$ and $s' \in \mathcal{S}'$.

(b) If $\mathcal{S} \vee \mathcal{S}' = \mathcal{T}$, then $f(\mathcal{S} \cap \mathcal{S}') \simeq f(\mathcal{S}) \oplus f(\mathcal{S}')$ and $(\mathcal{S} \cap \mathcal{S}')^{\perp} = \mathcal{S}^{\perp} \oplus (\mathcal{S}')^{\perp}$.

Proof. When $\mathcal{S} \cap \mathcal{S}' = 0$, we have $e(\mathcal{S} \wedge \mathcal{S}') = e(0) = 0$ and $e(\mathcal{S} \vee \mathcal{S}') \simeq e(\mathcal{S}) \oplus e(\mathcal{S}')$ by the Mayer-Vietoris triangle in Thm. 3.13 (a). For $t \in \mathcal{S} \vee \mathcal{S}'$, we have $t \simeq t \otimes e(\mathcal{S} \vee \mathcal{S}')$, hence $t \simeq (t \otimes e(\mathcal{S})) \oplus (t \otimes e(\mathcal{S}')) \in \mathcal{S} \oplus \mathcal{S}'$, which gives (a). Part (b) is dual. □

4. Inflating and Telescope

So far, we have used the triangular structure of $\mathcal{T}$ (for localization) and the $\otimes$-structure (for smashing and idempotents). We now throw in the assumption that $\mathcal{T}$ is also compactly generated. The main device throughout the following chapters is what we informally call *inflation*, that is the assignment of the smashing ideal $\langle \mathcal{C} \rangle \subset \mathcal{T}$ (Notation 2.2) to every thick ideal $\mathcal{C} \subset \mathcal{K}$ of compact objects.

4.1. **Theorem** (Miller, Neeman)**.** *Under Hypotheses 1.1, let $\mathcal{C}$ be a thick $\otimes$-ideal of $\mathcal{K} = \mathcal{T}^c$. Then, we have :*

$$\begin{tikzcd} \langle \mathcal{C} \rangle \arrow[r, tail] & \langle \mathcal{K} \rangle = \mathcal{T} \arrow[r, two heads] & \mathcal{T}/\langle \mathcal{C} \rangle \\ \mathcal{C} \arrow[r, tail] \arrow[u, hook] & \mathcal{K} \arrow[r, two heads] \arrow[u, hook] & \mathcal{K}/\mathcal{C} \arrow[u, hook] \end{tikzcd}$$

(a) $\langle \mathcal{C} \rangle$ is a smashing $\otimes$-ideal of $\mathcal{T}$ and $\langle \mathcal{C} \rangle^c = \langle \mathcal{C} \rangle \cap \mathcal{K} = \mathcal{C}$.

(b) $\mathcal{T}/\langle \mathcal{C} \rangle$ has small hom sets and is a compactly generated $\otimes$-triangulated category in the sense of Hypotheses 1.1.

(c) $\mathcal{K}/\mathcal{C}$ fully faithfully embeds into the compact objects of $\mathcal{T}/\langle \mathcal{C} \rangle$ and the thick closure of $\mathcal{K}/\mathcal{C}$ is exactly $(\mathcal{T}/\langle \mathcal{C} \rangle)^c$.

Proof. (a) is Miller's theorem, see [29] or [22, Thm. 3.3.3]. Parts (b) and (c) are Neeman's results [31, Cor. 4.4.3, Thm. 4.4.9]. This is also nicely done in [26, Thm. 5.6.1]. In order to fulfil our Hypotheses 1.1, $\bar{\mathcal{T}} := \mathcal{T}/\langle\mathcal{C}\rangle$ must also carry a closed $\otimes$-structure. The tensor on $\bar{\mathcal{T}}$ is induced by that of $\mathcal{T}$ since $\langle\mathcal{C}\rangle$ is a $\otimes$-ideal. Since $\bar{\mathcal{T}}$ is compactly generated it follows that this $\otimes$-structure is closed as well. Explicitly, let $r : \bar{\mathcal{T}} \longrightarrow \langle\mathcal{C}\rangle^{\perp} \subset \mathcal{T}$ be a right-adjoint right-inverse of the localization functor $q : \mathcal{T} \longrightarrow \bar{\mathcal{T}}$. Then the tensor on $\bar{\mathcal{T}}$ is given by $x \otimes y = q(r(x) \otimes r(y))$ and the internal hom can be given by $(x, z) \mapsto q(\underline{\mathrm{hom}}(r(x), r(z)))$. Indeed, we have

$$\begin{aligned}\bar{\mathcal{T}}(x \otimes y, z) = \bar{\mathcal{T}}\big(q(r(x) \otimes r(y)), z\big) &\cong \mathcal{T}\big(r(x) \otimes r(y), r(z)\big) \cong \mathcal{T}\big(r(y), \underline{\mathrm{hom}}(r(x), r(z))\big)\\ &\cong \mathcal{T}\Big(r(y), rq\big(\underline{\mathrm{hom}}(r(x), r(z))\big)\Big) \cong \bar{\mathcal{T}}\Big(y, q\big(\underline{\mathrm{hom}}(r(x), r(z))\big)\Big)\end{aligned}$$

where the $\cong$ at the beginning of the second row uses the fact that $\underline{\mathrm{hom}}(\mathcal{T}, \langle\mathcal{C}\rangle^{\perp}) \subset \langle\mathcal{C}\rangle^{\perp}$, which holds since $\langle\mathcal{C}\rangle$ is $\otimes$-ideal. Since q is a $\otimes$-functor, it preserves rigid objects by [27, Prop. 1.9]. Therefore, by (c), all compact objects of $\bar{\mathcal{T}}$ are rigid. By (c) again, the unit $\mathbb{1}_{\bar{\mathcal{T}}} = q(\mathbb{1}_{\mathcal{T}})$ remains compact in $\bar{\mathcal{T}}$. ☐

4.2. **Definition.** Under Hypotheses 1.1, we say that the *Telescope Conjecture holds for* $\mathcal{T}$ if every smashing $\otimes$-ideal $\mathcal{S}$ of $\mathcal{T}$ is inflated, *i.e.* $\mathcal{S} = \langle\mathcal{C}\rangle$ where $\mathcal{C}$ is a thick $\otimes$-ideal of $\mathcal{K}$ (necessarily equal to $\mathcal{S} \cap \mathcal{K}$). We often just write *"TC holds for $\mathcal{T}$"*.

4.3. *Remark.* The Telescope Conjecture, also known as the Smashing Conjecture, was first formulated in topology for $\mathcal{T} = \mathrm{SH}$, see Bousfield [15, 3.4] and Ravenel [32, 1.33], where it remains open. See a modern approach in Krause [25]. The conjecture fails for general compactly generated triangulated categories, see Keller [24].

4.4. **Proposition.** *Under Hypotheses 1.1, let $\mathcal{S} \subset \mathcal{T}$ be a smashing ideal. If TC holds for $\mathcal{T}$ then TC holds for $\mathcal{T}/\mathcal{S}$.*

Proof. Since TC holds for $\mathcal{T}$, we have $\mathcal{S} = \langle\mathcal{C}\rangle$ for $\mathcal{C} := \mathcal{S} \cap \mathcal{K} \subset \mathcal{K}$ and we know that $\mathcal{T}/\langle\mathcal{C}\rangle$ is compactly generated by Thm. 4.1. Let now $\bar{\mathcal{S}}$ be a smashing $\otimes$-ideal of $\mathcal{T}/\mathcal{S}$. It is easy to see that there is a smashing $\otimes$-ideal $\mathcal{S}'$ of $\mathcal{T}$, that contains $\mathcal{S}$, whose image under $q : \mathcal{T} \to \mathcal{T}/\mathcal{S}$ is $\bar{\mathcal{S}}$ (Rem. 2.17). Since TC holds for $\mathcal{T}$, we have that $\mathcal{S}' = \langle\mathcal{C}'\rangle$ for $\mathcal{C}' = \mathcal{S}' \cap \mathcal{K}$. Then $\bar{\mathcal{S}} = q(\langle\mathcal{C}'\rangle) = \langle q(\mathcal{C}')\rangle$ is easy to check. ☐

Recall from Prop. 3.11 that the join $\mathcal{S} \vee \mathcal{S}'$ is defined to be $\mathrm{Ker}(- \otimes f(\mathcal{S}) \otimes f(\mathcal{S}'))$ and is the smallest smashing ideal containing both smashing ideals $\mathcal{S}$ and $\mathcal{S}'$.

4.5. **Lemma.** *Under Hypotheses 1.1, let $\mathcal{C}, \mathcal{C}'$ be thick $\otimes$-ideals of $\mathcal{K} = \mathcal{T}^c$. Then*

(a) $\langle\mathcal{C}\rangle \cap \langle\mathcal{C}'\rangle = \langle\mathcal{C} \cap \mathcal{C}'\rangle$.
(b) If $\mathcal{C}'' \subset \mathcal{K}$ is the thick $\otimes$-ideal generated by $\mathcal{C}$ and $\mathcal{C}'$, then $\langle\mathcal{C}''\rangle = \langle\mathcal{C}\rangle \vee \langle\mathcal{C}'\rangle$.
(c) If $\mathcal{C} \perp \mathcal{C}'$ (Def. 2.5) then $\langle\mathcal{C}\rangle \perp \langle\mathcal{C}'\rangle$.
(d) If $\mathcal{C} \perp \mathcal{C}'$ and $\mathcal{C}' \perp \mathcal{C}$ then $\langle\mathcal{C} \oplus \mathcal{C}'\rangle = \langle\mathcal{C}\rangle \oplus \langle\mathcal{C}'\rangle$

Proof. The non-trivial inclusion $\langle\mathcal{C}\rangle \cap \langle\mathcal{C}'\rangle \subset \langle\mathcal{C} \cap \mathcal{C}'\rangle$ in (a) uses the equality

$$\langle\mathcal{C}\rangle = \left\{ t \in \mathcal{T} \;\middle|\; \begin{array}{c} \text{for every } f : x \to t \text{ with } x \in \mathcal{K}, \text{ there exists } c \in \mathcal{C} \\ \text{and a factorization } \begin{tikzcd}[column sep=small] x \arrow[rr, "f"] \arrow[dr] & & t \\ & c \arrow[ur] & \end{tikzcd} \end{array} \right\} \tag{4.6}$$

(see [31, Thm. 4.3.3]) and the rigid $\otimes$-structure on $\mathcal{K}$. Indeed, using (4.6) twice, we reduce to show that any morphism $g : c \to c'$ in $\mathcal{K}$ with $c \in \mathcal{C}$ and $c' \in \mathcal{C}'$ factors by an object of $\mathcal{C} \cap \mathcal{C}'$. If $h : \mathbb{1} \to \underline{\mathrm{hom}}(c, \mathbb{1}) \otimes c'$ is the adjoint of g, then g factors as

$$\begin{tikzcd} c \arrow[rr, "g"] \arrow[dr, "\mathbb{1}\otimes h"'] & & c' \\ & c \otimes \underline{\mathrm{hom}}(c, \mathbb{1}) \otimes c' \arrow[ur, "\epsilon\otimes\mathbb{1}"'] & \end{tikzcd}$$

where ϵ is the counit of the adjunction. Of course $c \otimes \underline{\mathrm{hom}}(c, \mathbb{1}) \otimes c' \in \mathcal{C} \cap \mathcal{C}'$. This gives (a). Both inclusions in (b) are very easy. For (c), note that (4.6) implies $\mathcal{C} \subset {}^{\perp}\langle \mathcal{C}' \rangle$ and then use that ${}^{\perp}\langle \mathcal{C}' \rangle$ is localizing. Finally, the non-trivial inclusion $\langle \mathcal{C} \oplus \mathcal{C}' \rangle \subset \langle \mathcal{C} \rangle \oplus \langle \mathcal{C}' \rangle$ in (d) comes from the fact that $\langle \mathcal{C} \rangle \oplus \langle \mathcal{C}' \rangle$ is localizing (triangulated is the issue). In fact, $\langle \mathcal{C} \rangle \oplus \langle \mathcal{C}' \rangle = \langle \mathcal{C} \rangle \vee \langle \mathcal{C}' \rangle$ by Corollary 3.15 (a). ☐

5. Tensor idempotents and geometry

We quickly recall elements of tensor triangular geometry from [2], [3] and [7].

5.1. **Definition** (See [2])**.** Let $(\mathcal{K}, \otimes, \mathbb{1})$ be an essentially small $\otimes$-triangulated category. A *prime ideal* $\mathcal{P} \subsetneq \mathcal{K}$ is a proper thick $\otimes$-ideal such that $a \otimes b \in \mathcal{P}$ forces $a \in \mathcal{P}$ or $b \in \mathcal{P}$. The *spectrum* $\mathrm{Spc}(\mathcal{K})$ is the set of prime ideals $\mathcal{P} \subset \mathcal{K}$. The *support* of an object $a \in \mathcal{K}$ is defined as the subset $\mathrm{supp}(a) = \{\mathcal{P} \in \mathrm{Spc}(\mathcal{K}) \mid a \notin \mathcal{P}\}$. The complements $U(a) = \{\mathcal{P} \in \mathrm{Spc}(\mathcal{K}) \mid a \in \mathcal{P}\}$ of these supports form a basis $\{U(a)\}_{a \in \mathcal{K}}$ of the so-called *Zariski topology* on the spectrum.

5.2. **Examples.** Here are the spectra $\mathrm{Spc}(\mathcal{K})$ for the standard $\mathcal{K} = \mathcal{T}^c$ of Exas 1.2.

(1) For $\mathcal{K} = \mathrm{SH}^{\mathrm{fin}}$, one has $\mathrm{Spc}(\mathcal{K}) = \{\mathrm{SH}^{\mathrm{fin}}_{\mathrm{tor}}\} \cup \{\, \mathcal{P}_{p,n} \,|\, p \text{ prime and } 1 \le n \le \infty \,\}$, where $\mathcal{P}_{p,n}$ is the kernel of the n^{th} Morava K-theory at p for $n < \infty$, where $\mathcal{P}_{p,\infty} = \cap_{n \ge 1} \mathcal{P}_{p,n} = \mathrm{Ker}(\mathrm{SH}^{\mathrm{fin}} \to \mathrm{SH}^{\mathrm{fin}}_{(p)})$ and where $\mathrm{SH}^{\mathrm{fin}}_{\mathrm{tor}}$ is the category of torsion spectra (which is also $\mathcal{P}_{p,0}$ for all p). The main reference is Hopkins-Smith [21], with explanations in [5, § 9].
(2) For $\mathcal{K} = \mathrm{D}^{\mathrm{perf}}(X)$, one recovers $\mathrm{Spc}(\mathcal{K}) \cong X$ itself. See [2, Thm. 6.3].
(3) For $\mathcal{K} = \mathrm{stab}(kG)$, one has $\mathrm{Spc}(\mathcal{K}) \cong \mathrm{Proj}(\mathrm{H}^{\bullet}(G, k))$, which is the projective support variety $\mathcal{V}_G$ of G over k. See [2, Thm. 6.3] as well.

5.3. *Remark.* The pair $(\mathrm{Spc}(\mathcal{K}), \mathrm{supp})$ is a *support datum* in the sense of [2, Def. 3.1], *i.e.* $\mathrm{supp}(0) = \varnothing$, $\mathrm{supp}(\mathbb{1}) = \mathrm{Spc}(\mathcal{K})$, $\mathrm{supp}(a \oplus b) = \mathrm{supp}(a) \cup \mathrm{supp}(b)$, $\mathrm{supp}(\Sigma a) = \mathrm{supp}(a)$, $\mathrm{supp}(c) \subset \mathrm{supp}(a) \cup \mathrm{supp}(b)$ for every exact triangle $a \to b \to c \to \Sigma a$ and finally $\mathrm{supp}(a \otimes b) = \mathrm{supp}(a) \cap \mathrm{supp}(b)$. All this for $a, b, c \in \mathcal{K}$.

5.4. *Remark.* We have $\overline{\{\mathcal{P}\}} = \{\, \mathcal{Q} \in \mathrm{Spc}(\mathcal{K}) \,|\, \mathcal{Q} \subset \mathcal{P} \,\}$ by [2, Prop. 2.9].

5.5. **Definition.** Let $Y \subset \mathrm{Spc}(\mathcal{K})$. We denote by $\mathcal{K}_Y$ the thick $\otimes$-ideal of $\mathcal{K}$

$$\mathcal{K}_Y := \{a \in \mathcal{K} \mid \mathrm{supp}(a) \subset Y\} \tag{5.6}$$

of objects *supported on* Y. It is *radical*, that is, it contains $a^{\otimes n}$ only if it contains a.

5.7. **Definition.** For a topological space X, a subset Y of the form $Y = \cup_{i \in I} Y_i$ where all Y_i are closed and have quasi-compact complement, is called a *Thomason subset of* X (or *dual open*). We write $\mathrm{Th}(X)$ for the set of Thomason subsets of X.

5.8. **Examples.** In Examples 1.2, when $\mathcal{K} = \mathrm{D}^{\mathrm{perf}}(X)$ for X a *noetherian* scheme, or when $\mathcal{K} = \mathrm{stab}(kG)$ for G a finite group, then a Thomason subset $Y \subset \mathrm{Spc}(\mathcal{K})$ is simply a specialization-closed one, meaning $y \in Y \Rightarrow \overline{\{y\}} \subset Y$ for all $y \in Y$, or equivalently, Y is a union of closed subsets. Compare Proposition 7.13 below. For the other running example, $\mathcal{K} = \mathrm{SH}^{\mathrm{fin}}$, a Thomason subset is a union of closed subsets of the form $\overline{\{\mathcal{P}_{p,n}\}}$ for n finite. One excludes $\overline{\{\mathcal{P}_{p,\infty}\}} = \{\mathcal{P}_{p,\infty}\}$ because its open complement is not quasi-compact. See more in Section 7.

5.9. **Theorem** (Classification Theorem, [2, Thm. 4.10])**.** *Let $\mathcal{K}$ be a $\otimes$-triangulated category as above. Let $\mathbb{R}(\mathcal{K})$ be the set of all thick radical $\otimes$-ideals of $\mathcal{K}$. Then there is a bijection* $\mathrm{Th}(\mathrm{Spc}(\mathcal{K})) \to \mathbb{R}(\mathcal{K})$ *mapping every Thomason subset $Y \subset \mathrm{Spc}(\mathcal{K})$ to $\mathcal{K}_Y$, see* (5.6)*. Its inverse is given by $\mathcal{J} \mapsto \mathrm{supp}(\mathcal{J}) := \cup_{a \in \mathcal{J}} \mathrm{supp}(a)$.*

5.10. *Remark.* When $\mathcal{K}$ is rigid (as it is for our $\mathcal{K} = \mathcal{T}^c$), every thick $\otimes$-ideal is automatically radical by [3, Prop. 4.2]. So "radical" is irrelevant for us here.

5.11. *Remark.* The space $\mathrm{Spc}(\mathcal{K})$ is always *spectral* in the sense of Hochster [20], *i.e.* it has a basis of quasi-compact open subsets and every irreducible subset of X has a unique generic point (in particular X itself is assumed quasi-compact and it is T_0). Note that every quasi-compact open subset U of a spectral space X is again spectral. To each spectral space X one can associate its *Hochster dual* X^* which is again a spectral space. The underlying space of X^* is the set X itself, while the dual-open subsets are given by the collection of what we called Thomason subsets of X (Def. 5.7). Hochster proves that $(X^*)^* = X$ in [20, Prop. 8].

5.12. **Definition.** Under Hypotheses 1.1, let U be a quasi-compact open subset of $\mathrm{Spc}(\mathcal{K})$ and Z its closed complement. We set

$$\mathcal{T}(U) := \mathcal{T}/\langle \mathcal{K}_Z \rangle \qquad \text{and} \qquad \mathcal{K}(U) = \mathcal{T}(U)^c.$$

We call $\mathcal{T}(U)$ the *category* $\mathcal{T}$ *on* U. We will denote by $\mathrm{res}_U : \mathcal{T} \to \mathcal{T}(U)$ the corresponding localization functor. The assignment $U \mapsto \mathcal{T}(U)$ is a "presheaf" of triangulated categories on $\mathrm{Spc}(\mathcal{K})$. By Theorem 4.1, $\mathcal{T}(U)$ is a compactly generated $\otimes$-triangulated category whose subcategory of compact objects $\mathcal{K}(U)$ is the idempotent completion of $\mathcal{K}/\mathcal{K}_Z$. We have $\mathrm{Spc}(\mathcal{K}(U)) \cong U$ by [7, Prop. 1.11].

5.13. *Remark.* If $\mathcal{T} = \mathrm{D}(X)$ for a quasi-compact quasi-separated scheme X, see Ex. 1.2 (2), and if U is a quasi-compact open of $X \simeq \mathrm{Spc}(\mathrm{D}^{\mathrm{perf}}(X))$, then the above category $\mathcal{T}(U)$ is nothing but $\mathrm{D}(U)$. Indeed, $\mathrm{D}(U) \cong \mathrm{D}(X)/\,\mathrm{D}_Z(X)$, where $\mathrm{D}_Z(X)$ is the subcategory of $\mathrm{D}(X)$ consisting of complexes supported on $Z = X \smallsetminus U$; see Thomason-Trobaugh [36] or Jørgensen [23, Thm. 1]. The key point is now that $\langle \mathrm{D}^{\mathrm{perf}}_Z(X) \rangle = \mathrm{D}_Z(X)$, as proven in Rouquier [34, Thm. 6.8] for instance.

5.14. **Theorem.** *Under Hypotheses 1.1, let $Y, Z \subset \mathrm{Spc}(\mathcal{K})$ be disjoint Thomason subsets. Then $\langle \mathcal{K}_Y \rangle$ and $\langle \mathcal{K}_Z \rangle$ are in a formal Mayer-Vietoris situation (Def. 2.10). Hence we have $\langle \mathcal{K}_Y \rangle \oplus \langle \mathcal{K}_Z \rangle = \langle \mathcal{K}_Y \oplus \mathcal{K}_Z \rangle = \langle \mathcal{K}_{Y \cup Z} \rangle$.*

Proof. Since $\mathcal{K}$ is rigid, [3, Cor. 2.8 and Thm. 2.11] give $\mathcal{K}_Y \perp \mathcal{K}_Z \perp \mathcal{K}_Y$ and $\mathcal{K}_{Y \cup Z} = \mathcal{K}_Y \oplus \mathcal{K}_Z$. The result follows from Lemma 4.5 (c) and (d). $\square$

5.15. **Corollary.** *If* $\mathrm{Spc}(\mathcal{K}) = U_1 \cup U_2$ *with* U_i *quasi-compact open,* $i = 1, 2$*, then*

$$\begin{tikzcd} \mathcal{T} \arrow[r] \arrow[d] & \mathcal{T}(U_1) \arrow[d] \\ \mathcal{T}(U_2) \arrow[r] & \mathcal{T}(U_1 \cap U_2)\,. \end{tikzcd}$$

satisfies gluing of objects and morphisms as in Theorems 2.11 and 2.12. □

5.16. **Definition.** Under Hypotheses 1.1, let $Y \subset \mathrm{Spc}\,\mathcal{K}$ be a Thomason subset. Set

$$\mathcal{T}_Y := \langle \mathcal{K}_Y \rangle \overset{(5.6)}{=} \langle \{\, a \in \mathcal{K} \,|\, \mathrm{supp}(a) \subset Y \,\} \rangle \subset \mathcal{T}\,.$$

By Thm. 4.1, $\mathcal{T}_Y$ is a smashing ideal of $\mathcal{T}$. We define $e(Y) := e(\mathcal{T}_Y)$ and $f(Y) := f(\mathcal{T}_Y)$. We call these the *(left and right) tensor idempotents* associated to the subset $Y \subset \mathrm{Spc}(\mathcal{K})$. As usual, they are characterized by the exact triangle

$$\Delta(Y) := \Big(\, e(Y) \longrightarrow \mathbb{1} \longrightarrow f(Y) \longrightarrow \Sigma e(Y) \,\Big) \tag{5.17}$$

such that $e(Y) \in \mathcal{T}_Y$ and $f(Y) \in (\mathcal{T}_Y)^{\perp}$. We have $e(Y) \otimes f(Y) = 0$ and $\mathcal{T}_Y = \mathrm{Im}(-\otimes e(Y)) = \mathrm{Ker}(-\otimes f(Y))$; see Thm. 3.5.

The following result generalizes Prop. 6.2, Thm. 7.5 and Thm. 8.1 in [33].

5.18. **Theorem** (Geometric Mayer-Vietoris triangles)**.** *Under Hypotheses 1.1, let* $Y_1, Y_2 \subset \mathrm{Spc}(\mathcal{K})$ *be Thomason subsets. There are unique isomorphisms of left idempotents*

$$e(Y_1 \cap Y_2) \cong e(Y_1) \otimes e(Y_2) \qquad \text{and} \qquad e(Y_1 \cup Y_2) \cong e(Y_1) \vee e(Y_2)$$

in $\mathcal{T}$*. Similarly, there are unique isomorphisms of right idempotents*

$$f(Y_1 \cap Y_2) \cong f(Y_1) \wedge f(Y_2) \qquad \text{and} \qquad f(Y_1 \cup Y_2) \cong f(Y_1) \otimes f(Y_2)\,.$$

Finally, there exist exact triangles in $\mathcal{T}$*:*

$$e(Y_1 \cap Y_2) \to e(Y_1) \oplus e(Y_2) \to e(Y_1 \cup Y_2) \to \Sigma(e(Y_1 \cap Y_2)) \quad \text{and}$$
$$f(Y_1 \cap Y_2) \to f(Y_1) \oplus f(Y_2) \to f(Y_1 \cup Y_2) \to \Sigma(f(Y_1 \cap Y_2)).$$

Proof. First note that by definition, $\mathcal{K}_{Y_1 \cap Y_2} = \mathcal{K}_{Y_1} \cap \mathcal{K}_{Y_2}$. By Lemma 4.5 (a), it follows that $\langle \mathcal{K}_{Y_1 \cap Y_2} \rangle = \langle \mathcal{K}_{Y_1} \rangle \cap \langle \mathcal{K}_{Y_2} \rangle = \langle \mathcal{K}_{Y_1} \rangle \wedge \langle \mathcal{K}_{Y_2} \rangle$. On the other hand, by the Classification Theorem 5.9, the thick $\otimes$-ideal of $\mathcal{K}$ generated by $\mathcal{K}_{Y_1}$ and $\mathcal{K}_{Y_2}$ must be $\mathcal{K}_{Y_1 \cup Y_2}$, hence $\langle \mathcal{K}_{Y_1 \cup Y_2} \rangle = \langle \mathcal{K}_{Y_1} \rangle \vee \langle \mathcal{K}_{Y_2} \rangle$ by Lemma 4.5 (b). We have proved:

$$\mathcal{T}_{Y_1 \cap Y_2} = \mathcal{T}_{Y_1} \wedge \mathcal{T}_{Y_2} \qquad \text{and} \qquad \mathcal{T}_{Y_1 \cup Y_2} = \mathcal{T}_{Y_1} \vee \mathcal{T}_{Y_2}\,. \tag{5.19}$$

Proposition 3.11 translates this into the four stated isomorphisms of idempotents, which can then be used in the abstract Mayer-Vietoris triangles of Thm. 3.13. □

5.20. *Remark.* Following-up on Remark 3.14, we have more precisely constructed two weakly (bi) cartesian squares in $\mathcal{T}$:

$$\begin{tikzcd} e(Y_1 \cap Y_2) \arrow[r] \arrow[d] & e(Y_1) \arrow[d] \\ e(Y_2) \arrow[r] & e(Y_1 \cup Y_2) \end{tikzcd} \qquad \text{and} \qquad \begin{tikzcd} f(Y_1 \cap Y_2) \arrow[r] \arrow[d] & f(Y_1) \arrow[d] \\ f(Y_2) \arrow[r] & f(Y_1 \cup Y_2) \end{tikzcd}$$

in which all the morphisms are the unique morphisms of idempotents induced by the four inclusions: $Y_1 \cap Y_2 \subset Y_i$ and $Y_i \subset Y_1 \cup Y_2$ for $i = 1, 2$.

5.21. *Remark.* Let $\rho : \mathrm{Spc}(\mathcal{K}) \to X$ be a map to a spectral topological space X, see Rem. 5.11. Assume that ρ is spectral (*i.e.* is continuous for the given topologies and for their dual). Then this yields left and right idempotents in $\mathcal{T}$ by sending any Thomason subset $W \subset X$ to $e(\rho^{-1}(W))$ and $f(\rho^{-1}(W))$. Mayer-Vietoris triangles in this setting are immediate consequences of Theorem 5.18 and the trivial fact that $W \mapsto \rho^{-1}(W)$ commutes with union and intersection.

5.22. **Example.** Following [33, § 6], we can describe the idempotents $e(Y)$ and $f(Y)$ when $Y \subset \mathrm{Spc}(\mathcal{K})$ is *principal* in the following sense. Let $\zeta : \mathbb{1} \to \Sigma^d(\mathbb{1})$ be a homogeneous element of degree $d \in \mathbb{Z}$ in the graded central ring

$$\mathrm{R}^\bullet_{\mathcal{K}} := \mathrm{End}^\bullet(\mathbb{1}) = \mathrm{Hom}_{\mathcal{K}}(\mathbb{1}, \Sigma^\bullet \mathbb{1})$$

using notation of [5]. Then the closed subset $V(\zeta) = \{\, \mathfrak{p} \,|\, \zeta \in \mathfrak{p} \,\}$ in the homogeneous spectrum $\mathrm{Spec}^{\mathrm{h}}(\mathrm{R}^\bullet_{\mathcal{K}})$ defines a closed subset $(\rho^\bullet)^{-1}(V(\zeta)) \subset \mathrm{Spc}(\mathcal{K})$, where $\rho^\bullet : \mathrm{Spc}(\mathcal{K}) \to \mathrm{Spec}^{\mathrm{h}}(\mathrm{R}^\bullet_{\mathcal{K}})$ is the spectral map defined in [5, Def. 5.1]. By [5, Thm. 5.3], this closed subset is simply $\mathrm{supp}(\mathrm{cone}(\zeta)) \subset \mathrm{Spc}(\mathcal{K})$, the support of the cone of the chosen ζ. Let $\Delta(\zeta) := \Delta(\mathrm{supp}(\mathrm{cone}(\zeta)))$ be the corresponding idempotent triangle

$$\Delta(\zeta) = \Big(e(\zeta) \longrightarrow \mathbb{1} \longrightarrow f(\zeta) \longrightarrow \Sigma e(\zeta) \Big) \tag{5.23}$$

as in (5.17) for $Y = \mathrm{supp}(\mathrm{cone}(\zeta))$. Here is an alternative description of $f(\zeta)$.

5.24. **Theorem.** *Under Hypotheses 1.1, let $\zeta : \mathbb{1} \to \Sigma^d(\mathbb{1})$, $d \in \mathbb{Z}$. Consider the homotopy colimit* $\mathrm{hocolim}_i\, \Sigma^{id}(\mathbb{1})$ *(see [31, Def. 1.6.4]) of the following sequence:*

$$\mathbb{1} \xrightarrow{\zeta} \Sigma^d(\mathbb{1}) \xrightarrow{\zeta} \Sigma^{2d}(\mathbb{1}) \longrightarrow \cdots \longrightarrow \Sigma^{id}(\mathbb{1}) \xrightarrow{\zeta} \Sigma^{(i+1)d}(\mathbb{1}) \longrightarrow \cdots$$

Then $\mathbb{1} \longrightarrow \mathrm{hocolim}_i\, \Sigma^{id}(\mathbb{1})$ *is a right idempotent isomorphic to* $\mathbb{1} \to f(\zeta)$.

Proof. Let $Y := \mathrm{supp}(\mathrm{cone}(\zeta))$. Let $\zeta^\infty : \mathbb{1} \to \mathrm{hocolim}_i\, \Sigma^{id}(\mathbb{1}) =: h$ be the map of the statement and choose an exact triangle $g \longrightarrow \mathbb{1} \xrightarrow{\zeta^\infty} h \longrightarrow \Sigma g$. It is easy to check that $\zeta \otimes \mathrm{id}_h : h \xrightarrow{\sim} \Sigma^d(h)$ is an isomorphism, using [31, Lem. 1.7.1] for instance. Hence $\mathrm{cone}(\zeta) \otimes h = 0$. Then the localizing subcategory $\mathrm{Ker}(- \otimes h) \subset \mathcal{T}$ contains $\mathrm{cone}(\zeta)$, hence also contains $\langle \mathrm{cone}(\zeta) \rangle = \langle \mathcal{K}_Y \rangle = \mathcal{T}_Y \ni e(Y)$. So, $e(Y) \otimes h = 0$, that is, $h \in \mathrm{Ker}(- \otimes e(Y)) = (\mathcal{T}_Y)^\perp$. From the commutative diagram

$$\begin{tikzcd}
\mathbb{1} \arrow[r, equal] \arrow[d, "\zeta"] & \mathbb{1} \arrow[r, dotted, no head] \arrow[d, "\zeta^2"] & \mathbb{1} \arrow[r, equal] \arrow[d, "\zeta^{i-1}"] & \mathbb{1} \arrow[r, equal] \arrow[d, "\zeta^i"] & \mathbb{1} \arrow[r, equal] \arrow[d, "\zeta^{i+1}"] & \cdots \arrow[r, equal] & \mathbb{1} \arrow[d, "\zeta^\infty"] \\
\Sigma^d(\mathbb{1}) \arrow[r, "\zeta"] & \Sigma^{2d}(\mathbb{1}) \arrow[r, dotted] & \Sigma^{(i-1)d}(\mathbb{1}) \arrow[r, "\zeta"] & \Sigma^{id}(\mathbb{1}) \arrow[r, "\zeta"] & \Sigma^{(i+1)d}(\mathbb{1}) \arrow[r] & \cdots \arrow[r] & h
\end{tikzcd}$$

one proves easily that $\mathrm{cone}(\zeta^\infty)$ belongs to $\langle \{\, \mathrm{cone}(\zeta^i) \,|\, i \in \mathbb{N} \,\} \rangle = \langle \mathrm{cone}(\zeta) \rangle = \mathcal{T}_Y$. Hence $g \in \mathcal{T}_Y$. In short, we have an exact triangle $g \to \mathbb{1} \to h \to \Sigma g$ with $g \in \mathcal{T}_Y$ and $h \in (\mathcal{T}_Y)^\perp$. This characterizes the triangle $\Delta(\mathcal{T}_Y) = \Delta(Y) = \Delta(\zeta)$, see (3.6), (5.17), (5.23). In particular, $h \cong f(\zeta)$ as claimed. ☐

5.25. *Remark.* When (the spectrum of) the ring $\mathrm{R}^\bullet_{\mathcal{K}} = \mathrm{End}^\bullet(\mathbb{1})$ is noetherian, one can extend the above definition of $e(\zeta)$ and $f(\zeta)$ to any closed $W \subset \mathrm{Spec}^{\mathrm{h}}(\mathrm{R}^\bullet_{\mathcal{K}})$ by choosing $\zeta_1, \ldots, \zeta_r$ such that $W = \cap_{i=1}^r V(\zeta_i)$ and setting $e(W) = e(\zeta_1) \otimes \cdots \otimes e(\zeta_r)$ and $f(W) = f(\zeta_1) \wedge \cdots \wedge f(\zeta_r)$ or equivalently by setting directly $e(W) = e(Y)$ and $f(W) = f(Y)$ where $Y = (\rho^\bullet)^{-1}(W)$ is the preimage of W in $\mathrm{Spc}(\mathcal{K})$ under the continuous map $\rho^\bullet : \mathrm{Spc}(\mathcal{K}) \to \mathrm{Spec}^{\mathrm{h}}(\mathrm{R}^\bullet_{\mathcal{K}})$ of [5]. These coincide; see Rem. 5.21.

5.26. *Remark.* There are other generalizations of Rickard's idempotents on the market, mostly using the spectrum of some ring acting on $\mathcal{T}$, instead of our $\mathrm{Spc}(\mathcal{T}^c)$. For instance, Hovey *et al.* [22, Def. 6.1.6] consider supports in the spectrum of the endomorphism ring of the unit of $\mathcal{T}$. See also Benson-Iyengar-Krause [11] for a graded version. Actually, any support datum on $\mathcal{K} = \mathcal{T}^c$ (see Rem. 5.3), say $\{\sigma(a) \subset X\}_{a\in\mathcal{K}}$ in some space X, allows an *ad hoc* definition of the subcategory $\mathcal{K}_W$ as $\{\, a \in \mathcal{K} \,|\, \sigma(a) \subset W \,\}$ for $W \subset X$. One could then apply the machinery to $\mathcal{T}_W := \langle \mathcal{K}_W \rangle$. Yet, it is a theorem that the best support datum on $\mathcal{K}$ is the one provided by $\mathrm{Spc}(\mathcal{K})$; see [2, Thm. 3.2]. Any variation is at most as good as this one.

Specifically, Theorem 5.24 shows how the tensor idempotents $e(\zeta)$ and $f(\zeta)$ coming from the homogenous spectrum of the central graded ring $\mathrm{R}^\bullet_{\mathcal{K}} = \mathrm{End}^\bullet_{\mathcal{K}}(\mathbb{1})$ are just special cases of our general construction. Note that $\mathrm{Spec}^{\mathrm{h}}(\mathrm{R}^\bullet_{\mathcal{K}})$ fails to recover the spectrum $\mathrm{Spc}(\mathcal{K})$ for general $\mathcal{T}$, as illustrated in topology; see [5, § 9]. So, $\mathrm{Spc}(\mathcal{K})$ is *strictly* better than $\mathrm{Spec}^{\mathrm{h}}(\mathrm{R}^\bullet_{\mathcal{K}})$.

Another variation on the theme would be to consider support data $\{\sigma(t) \subset X\}_{t\in\mathcal{T}}$ on the whole of $\mathcal{T}$ and to define $\mathcal{T}_W$ boldly as $\{\, t \in \mathcal{T} \,|\, \sigma(t) \subset W \,\}$. This is simply useless unless we can decide when $\mathcal{T}_W$ is smashing.

In conclusion, the balanced path seems to follow Rickard's original idea, rooting back to Miller's theorem in topology, *i.e.* to create smashing subcategories of $\mathcal{T}$ by inflating thick subcategories of $\mathcal{T}^c$ and parametrizing the latter with some subsets of some topological space, the best choice for such a space being $\mathrm{Spc}(\mathcal{T}^c)$.

6. Rickard map and Telescope Conjecture

6.1. **Definition.** Under Hypotheses 1.1, recall that $\mathrm{Th}(\mathrm{Spc}\,\mathcal{K})$ is the set of Thomason subsets of the spectrum of the subcategory of compact objects $\mathcal{K} = \mathcal{T}^c$ (Def. 5.7) and that $\mathbb{D}(\mathcal{T})$ is the set of idempotent triangles in $\mathcal{T}$ (Def. 3.2). We define *the Rickard map of* $\mathcal{T}$ as the assignment of Definition 5.16

$$\begin{array}{rccl} \Delta: & \mathrm{Th}(\mathrm{Spc}\,\mathcal{K}) & \longrightarrow & \mathbb{D}(\mathcal{T}) \\ & Y & \longmapsto & \Delta(Y) = \big(\, e(Y) \longrightarrow \mathbb{1} \longrightarrow f(Y) \longrightarrow \Sigma e(Y) \,\big) \end{array}$$

of an idempotent triangle $\Delta(Y)$ to every Thomason subset $Y \subset \mathrm{Spc}(\mathcal{K})$. Recall that $\Delta(Y)$ is the localization triangle of the smashing ideal $\mathcal{T}_Y := \langle \mathcal{K}_Y \rangle$ inflated from the thick ideal $\mathcal{K}_Y = \{\, a \in \mathcal{K} \,|\, \mathrm{supp}(a) \subset Y \,\}$ of compact objects supported on Y. So, $\Delta(Y)$ is an exact triangle, with left idempotent $e(Y)$ in $\mathcal{T}_Y$ and right idempotent $f(Y)$ such that $-\otimes f(Y) : \mathcal{T} \to \mathcal{T}$ gives localization modulo $\mathcal{T}_Y$.

6.2. **Proposition.** *Under Hypotheses 1.1, the Rickard map* $\mathrm{Th}(\mathrm{Spc}\,\mathcal{K}) \xrightarrow{\Delta} \mathbb{D}(\mathcal{T})$ *is always injective. It is surjective if and only if the Telescope Conjecture holds for* $\mathcal{T}$.

Proof. The Rickard map Δ is the composition of the following three maps:

$$\begin{array}{ccc} \mathrm{Th}(\mathrm{Spc}\,\mathcal{K}) & \xrightarrow{\sim} & \mathbb{R}(\mathcal{K}) \\ Y & \longmapsto & \mathcal{K}_Y \end{array}\ ,\qquad \begin{array}{ccc} \mathbb{R}(\mathcal{K}) & \hookrightarrow & \mathbb{S}(\mathcal{T}) \\ \mathcal{C} & \longmapsto & \langle \mathcal{C} \rangle \end{array} \qquad \text{and} \qquad \begin{array}{ccc} \mathbb{S}(\mathcal{T}) & \xrightarrow{\sim} & \mathbb{D}(\mathcal{T}) \\ \mathcal{S} & \longmapsto & \Delta(\mathcal{S})\,. \end{array}$$

Here, $\mathbb{R}(\mathcal{K})$ is the set of (radical) thick $\otimes$-ideals of $\mathcal{K}$, as in the Classification Theorem 5.9, which also gives the bijectivity of the first map; see also Rem. 5.10. The second map is the "inflation" of Theorem 4.1 (a), which also provides injectivity: actually a retraction $\mathbb{S}(\mathcal{T}) \to \mathbb{R}(\mathcal{T})$ is given by $\mathcal{S} \mapsto \mathcal{S} \cap \mathcal{K}$. The surjectivity of the second map is exactly the Telescope property of Definition 4.2. Finally, the last map is the bijection of Theorem 3.5. $\square$

The above simple result is the key to the understanding of the Telescope Conjecture in a local way. But first, let us express the naturality of our Rickard map.

6.3. **Theorem.** *Let $\mathcal{T}$ and $\mathcal{T}'$ be compactly generated tensor triangulated as in Hypotheses 1.1 and let $\mathcal{K} = \mathcal{T}^c$ and $\mathcal{K}' = (\mathcal{T}')^c$. Let $F : \mathcal{T} \to \mathcal{T}'$ be a $\otimes$-triangulated functor such that $F(\mathcal{K}) \subset \mathcal{K}'$ and such that F preserves small coproducts. (Equivalently, F has a right adjoint which itself preserves small coproducts, see* [26, Prop. 5.3.1 and Lem. 5.4.1]*.) Then the induced map $\Phi := \mathrm{Spc}(F) : \mathrm{Spc}(\mathcal{K}') \to \mathrm{Spc}(\mathcal{K})$ induces another map $\mathrm{Th}(\mathrm{Spc}\,\mathcal{K}) \to \mathrm{Th}(\mathrm{Spc}\,\mathcal{K}')$, defined by $Y \mapsto \Phi^{-1}(Y)$, that we simply denote F. Moreover the following diagram commutes:*

$$\begin{tikzcd}
\mathrm{Th}(\mathrm{Spc}\,\mathcal{K}) \arrow[r, "\Delta"] \arrow[d, "F"'] & \mathbb{D}(\mathcal{T}) \arrow[d, "F"] \\
\mathrm{Th}(\mathrm{Spc}\,\mathcal{K}') \arrow[r, "\Delta"] & \mathbb{D}(\mathcal{T}') .
\end{tikzcd}$$

Proof. The continuous map $\Phi = \mathrm{Spc}(F) : \mathrm{Spc}(\mathcal{K}') \to \mathrm{Spc}(\mathcal{K})$ is spectral (*cf.* Rem. 5.21) which amounts to say that the preimage under Φ of a quasi-compact open is quasi-compact; to check this, recall from [2, Prop. 2.14] that the quasi-compact open subsets of $\mathrm{Spc}(\mathcal{K})$ are the $U(a) = \mathrm{Spc}(\mathcal{K}) \smallsetminus \mathrm{supp}(a)$, for $a \in \mathcal{K}$, and from [2, Prop. 3.6] the first equality below (the second equality is just the notation):

$$\mathrm{supp}(F(a)) = \Phi^{-1}(\mathrm{supp}(a)) = F(\mathrm{supp}(a))\,. \tag{6.4}$$

It follows that the preimage under Φ of every Thomason subset of $\mathrm{Spc}(\mathcal{K})$ is still Thomason; see Def. 5.7. Hence the map denoted $F : \mathrm{Th}(\mathrm{Spc}\,\mathcal{K}) \to \mathrm{Th}(\mathrm{Spc}\,\mathcal{K}')$ and given by $Y \mapsto \Phi^{-1}(Y)$ is well-defined. Note that $F : \mathrm{Th}(\mathrm{Spc}\,\mathcal{K}) \to \mathrm{Th}(\mathrm{Spc}\,\mathcal{K}')$ preserves inclusion. The obvious map $F : \mathbb{D}(\mathcal{T}) \to \mathbb{D}(\mathcal{T}')$ sends an idempotent triangle $\Delta = (e \to \mathbb{1} \to f \to \Sigma e)$ to $F(\Delta) := (F(e) \to \mathbb{1} \to F(f) \to \Sigma F(e))$, which is still idempotent since $F(e) \otimes F(f) \simeq F(e \otimes f) = F(0) = 0$; see Def. 3.2.

Let us prove the last statement. Let $Y \subset \mathrm{Spc}(\mathcal{K})$ be a Thomason subset and

$$\Delta(Y) = \Big(\, e(Y) \longrightarrow \mathbb{1} \longrightarrow f(Y) \longrightarrow \Sigma e(Y) \,\Big)$$

the associated idempotent triangle (5.17). Let $\mathcal{J}' \subset \mathcal{K}'$ the thick $\otimes$-ideal of $\mathcal{K}'$ generated by $F(\mathcal{K}_Y)$. By (6.4), $\mathcal{J}' \subset \mathcal{K}'_{F(Y)}$. Therefore $\mathrm{supp}(\mathcal{J}') \subset F(Y)$. Conversely, for any $a \in \mathcal{K}_Y$, we have $F(a) \in \mathcal{J}'$ hence $\mathrm{supp}(\mathcal{J}') \supset \mathrm{supp}(F(a)) = F(\mathrm{supp}(a))$ by (6.4) again. Hence $F(Y) = F(\mathrm{supp}(\mathcal{K}_Y)) \subset \cup_{a \in \mathcal{K}_Y} F(\mathrm{supp}(a)) \subset \mathrm{supp}(\mathcal{J}')$. In conclusion, $\mathrm{supp}(\mathcal{J}') = F(Y)$, which proves that $\mathcal{J}' = \mathcal{K}'_{F(Y)}$, by the Classification Theorem 5.9. Therefore, in $\mathcal{T}'$, we have $\langle F(\mathcal{K}_Y)\rangle = \langle \mathcal{K}'_{F(Y)}\rangle \overset{\text{def.}}{=} \mathcal{T}'_{F(Y)}$. Hence $F(\mathcal{T}_Y) = F(\langle \mathcal{K}_Y\rangle) \subset \langle F(\mathcal{K}_Y)\rangle = \mathcal{T}'_{F(Y)}$. In particular, $F(e(Y)) \in \mathcal{T}'_{F(Y)}$. On the other hand, $\mathcal{T}'_{F(Y)} = \langle \mathcal{K}'_{F(Y)}\rangle = \langle F(\mathcal{K}_Y)\rangle \subset \langle F(\langle \mathcal{K}_Y\rangle)\rangle = \langle F(\mathcal{T}_Y)\rangle = \langle F(\mathrm{Im}(- \otimes e(Y)))\rangle \subset \langle \mathrm{Im}(- \otimes F(e(Y)))\rangle = \mathrm{Im}(- \otimes F(e(Y)))$ since the latter is a localizing ideal. Putting everything together, we have $\mathcal{T}'_{F(Y)} = \mathrm{Im}(- \otimes F(e(Y)))$. So, $F(e(Y))$ is the left idempotent generating the smashing ideal $\mathcal{T}'_{F(Y)}$, *i.e.* it is $e(\mathcal{T}'_{F(Y)})$, which is denoted $e(F(Y))$ in $\mathcal{T}'$. Consequently, $F(\Delta(Y)) = \Delta(F(Y))$. $\square$

6.5. **Corollary.** *Under Hypotheses 1.1, let $U \subset \mathrm{Spc}(\mathcal{K})$ be quasi-compact open. Recall the localization $\mathrm{res}_U : \mathcal{T} \longrightarrow \mathcal{T}(U)$ of Definition 5.12. Recall that $(\mathcal{T}(U))^c = \mathcal{K}(U)$ and that $\mathrm{Spc}(\mathcal{K}(U)) \cong U$. Then the following diagram commutes*

$$\begin{tikzcd}
Y \arrow[d, mapsto] & \in & \mathrm{Th}(\mathrm{Spc}\,\mathcal{K}) \arrow[r, "\Delta"] \arrow[d] & \mathbb{D}(\mathcal{T}) \arrow[d, "\mathrm{res}_U"] \\
Y \cap U & \in & \mathrm{Th}(U) \cong \mathrm{Th}(\mathrm{Spc}\,\mathcal{K}(U)) \arrow[r, "\Delta"] & \mathbb{D}(\mathcal{T}(U))
\end{tikzcd}$$

Proof. This is a special case of Theorem 6.3. Indeed, the homeomorphism $U \cong \mathrm{Spc}(\mathcal{K}(U))$ is such that $\mathrm{supp}(\mathrm{res}_U(a)) = \mathrm{supp}(a) \cap U$, see [7, Prop. 1.11]. So the map $\mathrm{res}_U : \mathrm{Th}(\mathrm{Spc}\,\mathcal{K}) \to \mathrm{Th}(\mathrm{Spc}\,\mathcal{K}(U))$ of Theorem 6.3 is indeed $Y \mapsto Y \cap U$. $\square$

We now state the main result of this section which says that in order to prove the Telescope Conjecture it is enough to prove it locally.

6.6. **Theorem.** *Under Hypotheses 1.1, suppose that the spectrum $\mathrm{Spc}(\mathcal{K}) = \bigcup_{i\in I} U_i$ is covered by quasi-compact open U_i. Then TC holds for $\mathcal{T}(U_i)$ for all $i \in I$ if and only if TC holds for $\mathcal{T}$.*

Proof. We have already seen $\Leftarrow$ in Proposition 4.4. For the opposite implication, quasi-compactness of $\mathrm{Spc}(\mathcal{K})$ and a simple induction argument reduce the proof to the case $\mathrm{Spc}\,\mathcal{K} = U_1 \cup U_2$. By Corollary 6.5, we have a commutative cube of sets:

$$\begin{tikzcd}[row sep=small, column sep=small]
\mathrm{Th}(\mathrm{Spc}\,\mathcal{K}) \arrow[rr] \arrow[dd] \arrow[dr, "\Delta"'] & & \mathrm{Th}(U_1) \arrow[dd] \arrow[dr, "\Delta"] & \\
& \mathbb{D}(\mathcal{T}) \arrow[rr] \arrow[dd] & & \mathbb{D}(\mathcal{T}(U_1)) \arrow[dd] \\
\mathrm{Th}(U_2) \arrow[rr] \arrow[dr, "\Delta"'] & & \mathrm{Th}(U_{12}) \arrow[dr, "\Delta"] & \\
& \mathbb{D}(\mathcal{T}(U_2)) \arrow[rr] & & \mathbb{D}(\mathcal{T}(U_{12}))
\end{tikzcd} \tag{6.7}$$

where $U_{12} := U_1 \cap U_2$, where the maps $\Delta : \mathrm{Th} \to \mathbb{D}$ are the Rickard maps, where the maps in the back are of the form $Y \mapsto Y \cap U_i$ for $i \in \{1, 2, 12\}$ and where the maps in the front are the obvious localizations.

We will show that the two faces in the front and in the back are pullbacks.

For the square involving Thomason subsets, it suffices to show that in a spectral space X, covered by two quasi-compact open $X = U_1 \cup U_2$, a subset $Y \subset X$ is Thomason if $Y \cap U_i$ is Thomason for $i = 1, 2$. Since all spaces involved are spectral we can consider the dual topologies (Rem. 5.11). Then, both subsets $Y \cap U_i$ are open in the closed subspaces U_i^* of $X^* = U_1^* \cup U_2^*$ hence Y is open in X^* as wanted.

The square involving idempotents is a pullback by a gluing argument in triangulated categories. We leave the gluing of idempotents as an exercise for we do not really need it below but we check uniqueness. That is, we check that if two idempotent triangles $\Delta = (e \to \mathbb{1} \to f \to \Sigma e)$ and $\Delta' = (e' \to \mathbb{1} \to f' \to \Sigma e')$ in $\mathbb{D}(\mathcal{T})$ are isomorphic over U_1 and U_2, that is, in $\mathcal{T}(U_i)$ for $i = 1, 2$, then they are already isomorphic in $\mathcal{T}$. By uniqueness of the isomorphism (Cor. 3.7), the two isomorphisms $\mathrm{res}_{U_i}(\Delta) \overset{\sim}{\to} \mathrm{res}_{U_i}(\Delta')$ in $\mathcal{T}(U_i)$, $i = 1, 2$, must be equal in $\mathcal{T}(U_{12})$. Therefore, by Theorem 2.12, there exists an isomorphism $\varphi : f \overset{\sim}{\to} f'$, as objects of $\mathcal{T}$. But Corollary 3.7 implies that $\Delta \simeq \Delta'$ as idempotent triangles.

So, the commutative cube (6.7) has cartesian front and back faces. We can now conclude the proof. Suppose that TC holds for both $\mathcal{T}(U_1)$ and $\mathcal{T}(U_2)$. Then the Rickard map $\Delta : \mathrm{Th}(\mathrm{Spc}\,\mathcal{K}) \to \mathbb{D}(\mathcal{T})$ is surjective by an easy diagram chase in the cube (6.7), using the bijectivity of $\Delta : \mathrm{Th}(U_i) \to \mathbb{D}(\mathcal{T}(U_i))$ for $i = 1, 2$ and the injectivity only on U_{12}; see Proposition 6.2. ☐

As an application of Theorem 6.6, we deduce the Telescope Conjecture for noetherian schemes, from the affine case proven by Neeman [30]. Let then X be a noetherian scheme and let $\mathcal{T} = \mathrm{D}(X) = \mathrm{D}_{\mathrm{Qcoh}}(\mathcal{O}_X)$, as in Ex. 1.2 (2). (Recall that $\mathrm{D}(X) \cong \mathrm{D}(\mathrm{Qcoh}(X))$ when X is moreover separated.) We have seen that $\mathcal{T}$ is compactly generated and that $\mathcal{K} = \mathcal{T}^c = \mathrm{D}^{\mathrm{perf}}(X)$. Moreover we know that $\mathrm{Spc}(\mathrm{D}^{\mathrm{perf}}(X)) \simeq X$, see [2, Cor. 5.6] for X (topologically) noetherian, or [16, Thm. 8.5] in maximal generality, *i.e.* for X quasi-compact and quasi-separated.

6.8. **Corollary** (See Alonso *et al.* [1, Thm. 5.8])**.** *Let X be a noetherian scheme. Then the Telescope Conjecture holds for $\mathrm{D}(X) = \mathrm{D}_{\mathrm{Qcoh}}(\mathcal{O}_X)$. Consequently every smashing subcategory of $\mathrm{D}(X)$ is of the following form, for a unique specialization closed subset $Y \subset X$:*

$$\mathrm{D}_{\mathrm{Qcoh},Y}(\mathcal{O}_X) := \left\{ E \in \mathrm{D}_{\mathrm{Qcoh}}(\mathcal{O}_X) \,\middle|\, E_x \simeq 0 \text{ in } \mathrm{D}(\mathcal{O}_{X,x}) \text{ for all } x \in X \smallsetminus Y \right\}.$$

Proof. One can cover $\mathrm{Spc}(\mathcal{K}) \simeq X = \cup_{i\in I} U_i$ by (quasi-compact) affine open subschemes $U_i = \mathrm{Spec}(R_i)$, with R_i noetherian. By Rem. 5.13 we have that $\mathcal{T}(U_i)$ is equivalent to $\mathrm{D}(U_i) \cong \mathrm{D}(R_i)$. Now by a result of Neeman [30, Cor. 3.4] we know that TC holds for each $\mathrm{D}(R_i)$. We conclude by Theorem 6.6 that TC holds for $\mathcal{T} = \mathrm{D}(X)$. The classification of thick $\otimes$-ideals in $\mathcal{T}^c = \mathrm{D}^{\mathrm{perf}}(X)$ is due to Thomason [35, Thm. 3.15], as in Thm. 5.9 above. Hence the second part. ☐

7. Residue objects and supports for big objects

We keep $\mathcal{T}$ compactly generated $\otimes$-triangulated as in Hypotheses 1.1 and $\mathrm{Spc}(\mathcal{K})$ the spectrum of the subcategory of compact objects $\mathcal{K} = \mathcal{T}^c$. We now define residue objects and supports for not necessarily compact objects, generalizing [10].

7.1. *Remark.* In Section 5, we constructed a left and a right idempotent $e(Y)$ and $f(Y)$ fitting in an exact triangle $\Delta(Y) :\; e(Y) \to \mathbb{1} \to f(Y) \to \Sigma e(Y)$, for every Thomason subset $Y \subset \mathrm{Spc}(\mathcal{K})$; see (5.17). We can combine left and right to produce

$$g(Y, Z) := e(Y) \otimes f(Z) \tag{7.2}$$

in $\mathcal{T}$, for any pair of Thomason subsets $Y, Z \subset \mathrm{Spc}(\mathcal{K})$. This is neither a right, nor a left idempotent but it is a $\otimes$-idempotent: $g(Y, Z) \otimes g(Y, Z) \simeq g(Y, Z)$.

7.3. **Notation.** Let us write $(-)^c$ for the complement of subsets in $\mathrm{Spc}(\mathcal{K})$.

7.4. **Lemma.** *Under Hypotheses 1.1, let $Y, Y', Z, Z' \in \mathrm{Th}(\mathrm{Spc}\,\mathcal{K})$.*

(a) Suppose that $Y \cap Z^c = Y' \cap Z^c$, then $g(Y, Z) \cong g(Y', Z)$.
(b) Suppose that $Y \cap Z^c = Y \cap (Z')^c$, then $g(Y, Z) \cong g(Y, Z')$.

Proof. Replacing Y' by $Y \cup Y'$, we easily reduce (a) to the case $Y \subset Y'$. Hence $\langle \mathcal{K}_Y \rangle \subset \langle \mathcal{K}_{Y'} \rangle$ and we have a (unique) morphism of idempotent triangles

$$\begin{tikzcd} e(Y) \arrow[r, "\gamma"] \arrow[d, "\epsilon"'] & \mathbb{1} \arrow[r, "\lambda"] \arrow[d, equal] & f(Y) \arrow[r] \arrow[d, "\varphi"] & \Sigma e(Y) \arrow[d, "\Sigma\epsilon"] \\ e(Y') \arrow[r, "\gamma'"] & \mathbb{1} \arrow[r, "\lambda'"] & f(Y') \arrow[r] & \Sigma e(Y')\,. \end{tikzcd}$$

Since $Y \cup Z = (Y \cap Z^c) \cup Z = (Y' \cap Z^c) \cup Z = Y' \cup Z$, Theorem 5.18 gives an isomorphism $f(Y) \otimes f(Z) \cong f(Y \cup Z) = f(Y' \cup Z) \cong f(Y') \otimes f(Z)$ which must be $\varphi \otimes \mathrm{id}_{f(Z)}$ (Cor. 3.7). Applying $- \otimes f(Z)$ to the above diagram, it follows that $\epsilon \otimes \mathrm{id}_{f(Z)}$ is also an isomorphism $e(Y) \otimes f(Z) \overset{\sim}{\to} e(Y') \otimes f(Z)$ as wanted in (a). The proof of (b) is similar. □

7.5. **Corollary.** *Up to isomorphism in $\mathcal{T}$, the object $g(Y, Z) = e(Y) \otimes f(Z)$ only depends on the subset $Y \cap Z^c$ in* $\mathrm{Spc}(\mathcal{K})$. □

7.6. *Remark.* Let X be a (spectral) topological space. We could call $A \subset X$ *trop-beau* if it is the intersection of a Thomason subset and the complement of a Thomason subset : $A = Y \cap Z^c$, with $Y, Z \in \mathrm{Th}(X)$. Every Thomason is trop-beau. By Corollary 7.5, for A trop-beau, we can define a $\otimes$-idempotent $g(A)$ in $\mathcal{T}$ as

$$g(A) := e(Y) \otimes f(Z) \tag{7.7}$$

for any choice of $Y, Z \in \mathrm{Th}(\mathrm{Spc}\,\mathcal{K})$ such that $A = Y \cap Z^c$. It is easy to check that if $A, A' \subset \mathrm{Spc}(\mathcal{K})$ are trop-beau subsets then so is $A \cap A'$ and $g(A \cap A') \cong g(A) \otimes g(A')$ in $\mathcal{T}$. Indeed, say $A = Y \cap Z^c$ and $A' = Y' \cap (Z')^c$ with $Y, Y', Z, Z' \in \mathrm{Th}(\mathrm{Spc}\,\mathcal{K})$. Then $A \cap A' = (Y \cap Y') \cap (Z \cup Z')^c$ and Theorem 5.18 gives us that $g(A \cap A') = e(Y \cap Y') \otimes f(Z \cup Z') \cong e(Y) \otimes e(Y') \otimes f(Z) \otimes f(Z') \cong g(A) \otimes g(A')$.

7.8. **Lemma** (See [30, Lem. 1.4]). *For $\mathcal{P} \in \mathrm{Spc}(\mathcal{K})$, the following are equivalent :*

(i) The open $\mathrm{Spc}(\mathcal{K}) \smallsetminus \overline{\{\mathcal{P}\}}$ *is quasi-compact.*
(ii) There exists $s \in \mathcal{K}$ such that $\overline{\{\mathcal{P}\}} = \mathrm{supp}(s)$.
(iii) $\overline{\{\mathcal{P}\}}$ is a Thomason subset of $\mathrm{Spc}(\mathcal{K})$.

If this holds, then

(a) The singleton $\{\mathcal{P}\}$ is trop-beau (Rem. 7.6). More precisely $\{\mathcal{P}\} = \overline{\{\mathcal{P}\}} \cap (Y_{\mathcal{P}})^c$ where $Y_{\mathcal{P}} := \{\, \mathcal{Q} \in \mathrm{Spc}(\mathcal{K}) \,|\, \mathcal{P} \not\subset \mathcal{Q} \,\}$ is the Thomason subset such that $\mathcal{K}_{Y_{\mathcal{P}}} = \mathcal{P}$, that is, $Y_{\mathcal{P}} = \mathrm{supp}(\mathcal{P})$; see the Classification Theorem 5.9.
(b) For any $s \in \mathcal{K}$ such that $\mathrm{supp}(s) = \overline{\{\mathcal{P}\}}$, we have $s \otimes e(\overline{\{\mathcal{P}\}}) \simeq s$.
(c) The idempotent $g(\{\mathcal{P}\}) = e(\overline{\{\mathcal{P}\}}) \otimes f(Y_{\mathcal{P}})$ (see (7.7)) is non-zero.

Proof. The equivalence (i)⇔(ii) is [2, Prop. 2.14], (i)⇒(iii) is trivial and (iii)⇒(i) is easy, as in the proof of Prop. 7.13 (iv)⇒(v) below. Now, suppose (i)–(iii) hold. The equality $\{\mathcal{P}\} = \overline{\{\mathcal{P}\}} \cap (Y_{\mathcal{P}})^c$ follows from $\overline{\{\mathcal{P}\}} = \{\, \mathcal{Q} \,|\, \mathcal{Q} \subset \mathcal{P} \,\}$ (Rem. 5.4) and from $(Y_{\mathcal{P}})^c = \{\, \mathcal{Q} \,|\, \mathcal{P} \subset \mathcal{Q} \,\}$. Moreover, it is easy to check that $Y_{\mathcal{P}} = \cup_{a \in \mathcal{P}} \mathrm{supp}(a) = \mathrm{supp}(\mathcal{P})$, as in Theorem 5.9. So $Y_{\mathcal{P}}$ is Thomason and we get (a). For (b), since $s \in \mathcal{K}_{\mathrm{supp}(s)} = \mathcal{K}_{\overline{\{\mathcal{P}\}}}$, we have that $s \in \langle \mathcal{K}_{\overline{\{\mathcal{P}\}}} \rangle = \mathrm{Im}(- \otimes e(\overline{\{\mathcal{P}\}}))$. Now, for (c), suppose *ab absurdo* that $g(\{\mathcal{P}\}) = 0$. Then $0 = s \otimes g(\{\mathcal{P}\}) = s \otimes e(\overline{\{\mathcal{P}\}}) \otimes f(Y_{\mathcal{P}}) \simeq s \otimes f(Y_{\mathcal{P}})$ by (b). This implies $s \in \mathrm{Ker}(- \otimes f(Y_{\mathcal{P}})) = \langle \mathcal{K}_{Y_{\mathcal{P}}} \rangle = \langle \mathcal{P} \rangle$. So, $s \in \mathcal{K} \cap \langle \mathcal{P} \rangle = \mathcal{P}$ (Thm. 4.1 (a)). The latter reads $\mathcal{P} \notin \mathrm{supp}(s)$ which contradicts $\mathrm{supp}(s) = \overline{\{\mathcal{P}\}}$. □

7.9. **Definition.** Under Hypotheses 1.1, a prime $\mathcal{P} \in \operatorname{Spc}(\mathcal{K})$ is called *visible* if it satisfies the equivalent conditions (i)–(iii) of Lemma 7.8. We define the *residue object at* $\mathcal{P}$ as the $\otimes$-idempotent $g(\{\mathcal{P}\})$ of (7.7), that is:

$$\kappa(\mathcal{P}) := g(\overline{\{\mathcal{P}\}}, Y_{\mathcal{P}}) \overset{\text{def.}}{=} e(\overline{\{\mathcal{P}\}}) \otimes f(Y_{\mathcal{P}})\,. \tag{7.10}$$

7.11. **Example.** Let p be a prime number and $\mathcal{T} = \mathrm{SH}_{(p)}$ the p-local stable homotopy category. As in Exa. 5.2 (1), the primes of $\mathcal{K} = \mathrm{SH}^{\mathrm{fin}}_{(p)}$ are exactly

$$\mathcal{C}_\infty = 0 \subset \cdots \subset \mathcal{C}_n \subset \mathcal{C}_{n-1} \subset \cdots \subset \mathcal{C}_2 \subset \mathcal{C}_1 \subset \mathcal{C}_0$$

where $\mathcal{C}_n = (\mathcal{P}_{p,n})_{(p)} \subset \mathrm{SH}^{\mathrm{fin}}_{(p)}$ is the kernel of the n^{th} Morava K-theory at p, for $n \geq 0$, see [5, Thm. 9.1]. There is a unique closed point $\{\mathcal{C}_\infty\}$ but there is no $s \in \mathrm{SH}^{\mathrm{fin}}_{(p)}$ such that $\operatorname{supp}(s) = \{\mathcal{C}_\infty\}$ for this would mean $s \in \cap_{n\geq 1}\mathcal{C}_n$ and the latter intersection is zero, contradicting $\mathcal{C}_\infty \in \operatorname{supp}(s)$ which reads $s \notin \mathcal{C}_\infty = 0$. In other words, the prime $\mathcal{P} = \mathcal{C}_\infty$ is not visible in the sense of Definition 7.9.

7.12. **Example.** Let R be a local ring and $\mathcal{T} = \mathrm{D}(R)$ as in Ex. 1.2 (2). Under $\operatorname{Spec}(R) \simeq \operatorname{Spc}(\mathcal{K})$, the maximal ideal $\mathfrak{m} \subset R$ corresponds to the triangular prime $\mathcal{P} = 0$. So, $e(Y_{\mathcal{P}}) \in \langle \mathcal{P} \rangle = 0$ and $f(Y_{\mathcal{P}}) = \mathbb{1}$. Hence, $\kappa(\mathcal{P}) = e(\overline{\{\mathcal{P}\}})$ is a *left* idempotent. It is therefore not to be confused with the residue field $R \to R/\mathfrak{m}$.

7.13. **Proposition.** *Let X be a spectral topological space (Rem. 5.11). The following are equivalent:*

(i) X is noetherian (any non-empty family of closed subsets has a minimal one).
(ii) Every open subset of X is quasi-compact.
(iii) Thomason subsets of X (Def. 5.7) coincide with specialization closed ones.
(iv) For every $x \in X$, the closed subset $\overline{\{x\}}$ is Thomason.
(v) For every $x \in X$, the open subset $X \smallsetminus \overline{\{x\}}$ is quasi-compact.

Proof. (i)$\Leftrightarrow$(ii) is an easy exercise. To see (ii)$\Rightarrow$(iii), note that a Thomason subset is always specialization closed. Conversely, if $Y = \cup_{y\in Y}\overline{\{y\}}$, it is Thomason since every $X \smallsetminus \overline{\{y\}}$ is quasi-compact by (ii). The implication (iii)$\Rightarrow$(iv) is trivial. For (iv)$\Rightarrow$(v), if $\overline{\{x\}} = \cup_{i\in I} Y_i$ with $X \smallsetminus Y_i$ quasi-compact, then x belongs to some Y_i hence $\overline{\{x\}} = Y_i$ and $X \smallsetminus \overline{\{x\}} = X \smallsetminus Y_i$ is quasi-compact. Let us prove the non-trivial part: (v)$\Rightarrow$(ii). Let $\mathcal{F} = \big\{\, Z \subset X \text{ closed} \,\big|\, X \smallsetminus Z \text{ is not quasi-compact} \,\big\}$. Suppose, *ab absurdo*, that $\mathcal{F} \neq \varnothing$. Let $\mathcal{C} \subset \mathcal{F}$ be totally ordered with respect to inclusion and let $Z_0 = \cap_{Z\in\mathcal{C}} Z$. Then the open cover $X \smallsetminus Z_0 = \cup_{Z\in\mathcal{C}}(X \smallsetminus Z)$ *either* has no finite subcover and therefore $X \smallsetminus Z_0$ is not quasi-compact, *or* has a finite subcover in which case $X \smallsetminus Z_0 = X \smallsetminus Z$ for some $Z \in \mathcal{C}$ (since $\mathcal{C}$ is totally ordered) and $X \smallsetminus Z_0$ is therefore not quasi-compact again. In both cases, $Z_0 \in \mathcal{F}$. So Z_0 is a lower bound for $\mathcal{C}$ in $\mathcal{F}$. By Zorn, it follows that $\mathcal{F}$ has a minimal element for inclusion. Since X is spectral, this minimal element must be non-empty (X is quasi-compact) and irreducible (the intersection of two quasi-compact opens remains quasi-compact). Hence this minimal element is of the form $\overline{\{x\}}$. But $\overline{\{x\}} \in \mathcal{F}$ contradicts (v). $\square$

7.14. **Corollary.** *Every prime $\mathcal{P}$ in $\operatorname{Spc}(\mathcal{K})$ is visible (Def. 7.9) if and only if the spectrum $\operatorname{Spc}(\mathcal{K})$ is a noetherian topological space.* $\square$

7.15. **Example.** Inspecting the examples of 1.2, we get:

(1) $\operatorname{Spc}(\mathrm{SH}^{\mathrm{fin}})$ is not noetherian as already observable p-locally, see Ex. 7.11.
(2) $\operatorname{Spc}(\mathrm{D}^{\mathrm{perf}}(X)) \simeq X$ is noetherian if the scheme X is (topologically) noetherian.

(3) $\mathrm{Spc}(\mathrm{stab}(kG)) \simeq \mathcal{V}_G$ is always noetherian, see [8, Vol. II, Thm. 4.2.1].

From now on, we assume $\mathrm{Spc}(\mathcal{K})$ noetherian, *i.e.* every $\mathcal{P}$ is visible. So we have residue objects $\kappa(\mathcal{P}) = e(\overline{\{\mathcal{P}\}}) \otimes f(Y_\mathcal{P})$ in $\mathcal{T}$ for all $\mathcal{P} \in \mathrm{Spc}(\mathcal{K})$.

7.16. **Definition.** Under Hypotheses 1.1, suppose that $\mathrm{Spc}(\mathcal{K})$ is noetherian. For any object $t \in \mathcal{T}$ we define its *(big) support* as follows

$$\mathrm{Supp}(t) = \{\mathcal{P} \in \mathrm{Spc}(\mathcal{K}) \mid t \otimes \kappa(\mathcal{P}) \neq 0\}\,,$$

which we temporarily distinguish from the already defined $\mathrm{supp}(t)$ when $t \in \mathcal{T}^c$.

7.17. **Proposition.** *Under Hypotheses 1.1, assume that* $\mathrm{Spc}(\mathcal{K})$ *is noetherian. Then the assignment* $t \mapsto \mathrm{Supp}(t)$ *satisfies the following properties:*

(a) For every compact object $x \in \mathcal{K}$ *one has* $\mathrm{Supp}(x) = \mathrm{supp}(x)$*.*

(b) $\mathrm{Supp}(0) = \varnothing$, $\mathrm{Supp}(\mathbb{1}) = \mathrm{Spc}(\mathcal{K})$*.*

(c) $\mathrm{Supp}(\sqcup_{i\in I} t_i) = \cup_{i\in I} \mathrm{Supp}(t_i)$ *for any set* $\{t_i\}_{i\in I}$ *of objects of* $\mathcal{T}$*.*

(d) $\mathrm{Supp}(\Sigma t) = \mathrm{Supp}(t)$*.*

(e) $\mathrm{Supp}(t) \subset \mathrm{Supp}(t') \cup \mathrm{Supp}(t'')$ *for any exact triangle* $t \to t' \to t'' \to \Sigma t$*.*

(f) $\mathrm{Supp}(t \otimes t') \ \subset\ \mathrm{Supp}(t) \cap \mathrm{Supp}(t')$*.*

Proof. For point (a) we have to prove that for a prime $\mathcal{P}$ and for a compact object x one has $x \in \mathcal{P} \iff x \otimes \kappa(\mathcal{P}) = 0$. If $x \in \mathcal{P}$ then $x \in \langle\mathcal{P}\rangle = \mathcal{T}_{Y_\mathcal{P}}$ and thus $x \otimes f(Y_\mathcal{P}) = 0$ already. Whence $x \otimes \kappa(\mathcal{P}) = x \otimes e(\overline{\{\mathcal{P}\}}) \otimes f(Y_\mathcal{P}) = 0$. Conversely, suppose that $x \otimes \kappa(\mathcal{P}) = 0$. Since $\mathcal{P}$ is visible, there exists $s \in \mathcal{K}$ such that $\mathrm{supp}(s) = \overline{\{\mathcal{P}\}}$, see Lemma 7.8, whose part (b) yields

$$0 = x \otimes s \otimes \kappa(\mathcal{P}) = x \otimes s \otimes e(\overline{\{\mathcal{P}\}}) \otimes f(Y_\mathcal{P}) = x \otimes s \otimes f(Y_\mathcal{P}).$$

This implies that $x \otimes s \in \mathcal{T}_{Y_\mathcal{P}} \cap \mathcal{K} = \langle \mathcal{K}_{Y_\mathcal{P}}\rangle \cap \mathcal{K} = \mathcal{K}_{Y_\mathcal{P}} = \mathcal{P}$ (using Thm. 4.1 (a) again and $Y_\mathcal{P} = \mathrm{supp}(\mathcal{P})$). But $s \notin \mathcal{P}$ since $\mathcal{P} \in \mathrm{supp}(s)$, so $x \in \mathcal{P}$ by definition of a prime ideal. This finishes (a). The rest is straightforward. □

7.18. **Proposition.** *Under Hypotheses 1.1, suppose* $\mathrm{Spc}(\mathcal{K})$ *noetherian. Let* $Y \subset \mathrm{Spc}(\mathcal{K})$ *be a Thomason subset (i.e. specialization closed). Then we have*

$$\mathrm{Supp}(t \otimes e(Y)) = \mathrm{Supp}(t) \cap Y \quad \textit{and} \quad \mathrm{Supp}(t \otimes f(Y)) = \mathrm{Supp}(t) \cap Y^c$$

for every t *in* $\mathcal{T}$*. In particular we have* $\mathrm{Supp}(e(Y)) = Y$ *and* $\mathrm{Supp}(f(Y)) = Y^c$*. More generally if* $A \subset \mathrm{Spc}(\mathcal{K})$ *is trop-beau (Rem. 7.6) then* $\mathrm{Supp}(g(A)) = A$*.*

Proof. Notice first that if $\mathcal{P} \in Y$ then $\overline{\{\mathcal{P}\}} \subset Y$, since Y is specialization closed. Thus, by Thm. 5.18 we have $e(\overline{\{\mathcal{P}\}}) \otimes e(Y) \simeq e(\overline{\{\mathcal{P}\}})$. This implies that $t \otimes e(Y) \otimes \kappa(\mathcal{P}) \simeq t \otimes \kappa(\mathcal{P})$, for any t in $\mathcal{T}$. Thus $\mathrm{Supp}(t \otimes e(Y)) \cap Y = \mathrm{Supp}(t) \cap Y$. It remains to show $\mathrm{Supp}(t \otimes e(Y)) \subset Y$. Since $\mathrm{Supp}(t \otimes e(Y)) \subset \mathrm{Supp}(t) \cap \mathrm{Supp}(e(Y))$, it is enough to show that $\mathrm{Supp}(e(Y)) \subset Y$. Let us prove $Y^c \subset (\mathrm{Supp}(e(Y))^c$. Since Y is specialization closed, if $\mathcal{P} \notin Y$ then $Y \subset \left\{ \mathcal{Q} \,\middle|\, \mathcal{P} \notin \overline{\{\mathcal{Q}\}} \right\} = \left\{ \mathcal{Q} \,\middle|\, \mathcal{P} \not\subset \mathcal{Q} \right\}$, see Rem. 5.4. This reads $Y \subset Y_\mathcal{P}$, see Lemma 7.8. Hence $e(Y) \otimes e(Y_\mathcal{P}) \simeq e(Y)$ and, since $e(Y_\mathcal{P}) \otimes f(Y_\mathcal{P}) = 0$, we get $e(Y) \otimes \kappa(\mathcal{P}) \simeq e(Y) \otimes e(Y_\mathcal{P}) \otimes e(\overline{\{\mathcal{P}\}}) \otimes f(Y_\mathcal{P}) = 0$. The latter means $\mathcal{P} \notin \mathrm{Supp}(e(Y))$ as wanted. This gives the first desired equality. The other equality is obtained in a similar manner, interchanging e's and f's. □

7.19. **Corollary.** *Under Hypotheses 1.1, suppose* $\mathrm{Spc}(\mathcal{K})$ *noetherian. Let* $t \in \mathcal{T}$ *and* $\mathcal{P} \in \mathrm{Spc}(\mathcal{K})$*. Let* $t_\mathcal{P} = t \otimes f(Y_\mathcal{P}) = L_{\langle\mathcal{P}\rangle}(t)$ *the localization of* t *at* $\langle\mathcal{P}\rangle$*. Then* $\mathrm{Supp}(t_\mathcal{P}) = \mathrm{Supp}(t) \cap \{\mathcal{Q} \in \mathrm{Spc}(\mathcal{K}) \mid \mathcal{P} \subset \mathcal{Q}\}$ *in* $\mathrm{Spc}(\mathcal{K}/\mathcal{P}) \cong \left\{ \mathcal{Q} \,\middle|\, \mathcal{P} \subset \mathcal{Q} \right\}$*.* □

7.20. *Remark.* Proposition 7.18 is the first indication of a so-called $\otimes$-theorem for supports, see [10], that is, the hope to have $\operatorname{Supp}(t \otimes t') = \operatorname{Supp}(t) \cap \operatorname{Supp}(t')$ for all t, t' in $\mathcal{T}$. By Proposition 7.18, the $\otimes$-theorem holds when one of the two objects is $e(Y)$ or $f(Y)$. We want to prove another case of this $\otimes$-theorem.

7.21. **Lemma.** *Let s, s' in $\mathcal{T}$ generate the same localizing $\otimes$-ideal of $\mathcal{T}$, that is, $\langle s\rangle^{\otimes} = \langle s'\rangle^{\otimes}$. Then, $\operatorname{Supp}(t \otimes s) = \operatorname{Supp}(t \otimes s')$ for all $t \in \mathcal{T}$. In particular, $\operatorname{Supp}(s) = \operatorname{Supp}(s')$.*

Proof. We claim that for every $v \in \mathcal{T}$ we have $v \otimes s = 0$ if and only if $v \otimes s' = 0$. Indeed, $\operatorname{Ker}(v \otimes -)$ is a localizing $\otimes$-ideal and therefore contains s if and only if it contains $\langle s\rangle^{\otimes}$. The claim follows from $\langle s\rangle^{\otimes} = \langle s'\rangle^{\otimes}$. Now, for $\mathcal{P} \in \operatorname{Spc}(\mathcal{K})$, apply this claim to $v = t \otimes \kappa(\mathcal{P})$. This gives $\operatorname{Supp}(t \otimes s) = \operatorname{Supp}(t \otimes s')$ as wanted. $\square$

7.22. **Theorem** (Half $\otimes$-Theorem)**.** *Under Hypotheses 1.1 with $\operatorname{Spc}(\mathcal{K})$ noetherian, let t be in $\mathcal{T}$ and $x \in \mathcal{K}$ be a compact object. Then*

$$\operatorname{Supp}(t \otimes x) = \operatorname{Supp}(t) \cap \operatorname{supp}(x).$$

Proof. Since $\langle x\rangle^{\otimes} = \langle \mathcal{K}_{\operatorname{supp} x}\rangle = \langle e(\operatorname{supp} x)\rangle^{\otimes}$, the above lemma yields $\operatorname{Supp}(t \otimes x) = \operatorname{Supp}(t \otimes e(\operatorname{supp} x)) = \operatorname{Supp}(t) \cap \operatorname{supp}(x)$ by Proposition 7.18. $\square$

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Paul Balmer, UCLA Mathematics Department, Los Angeles, CA 90095-1555, USA
E-mail address: `balmer@math.ucla.edu`
URL: `http://www.math.ucla.edu/~balmer`

Giordano Favi, Department Mathematics, Universität Basel, 4051 Basel, Switzerland; also : Mathematics Section, EGG, EPFL, 1015 Lausanne, Switzerland
E-mail address: `giordano.favi@gmail.com`
URL: `http://www.math.unibas.ch/~giordano`