# SUPPORTS AND FILTRATIONS IN ALGEBRAIC GEOMETRY AND MODULAR REPRESENTATION THEORY

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Abstract. Working with tensor triangulated categories, we prove two theorems relative to supports and then discuss their incarnations in algebraic geometry and in modular representation theory. First, we show that an indecomposable object has connected support. Then we consider the filtrations by dimension or codimension of the support and prove that the associated subquotients decompose as sums of local terms.

## Introduction

We proved in [1] that schemes $X$ in algebraic geometry and projective support varieties $\mathcal{V}_G(k)$ in modular representation theory both appear as the *spectrum* of prime ideals, $\mathrm{Spc}(\mathcal{K})$, for suitable tensor triangulated categories $\mathcal{K}$, namely and respectively for $\mathcal{K} = \mathrm{D}^{\mathrm{perf}}(X)$, the derived category of perfect complexes over $X$, and for $\mathcal{K} = \mathrm{stab}(kG)$, the stable category of finitely generated $kG$-modules modulo projectives. (This is recalled in Theorem 1.4 below.)

In this spectrum $\mathrm{Spc}(\mathcal{K})$, one can construct for every object $a \in \mathcal{K}$ a closed subset $\mathrm{supp}(a) \subset \mathrm{Spc}(\mathcal{K})$ called the *support* of $a$. This support provides a unified approach to the homological support $\mathrm{supph}(E) \subset X$ of a perfect complex $E \in \mathrm{D}^{\mathrm{perf}}(X)$ and to the projective support variety $\mathcal{V}_G(M) \subset \mathcal{V}_G(k)$ of a $kG$-module $M \in \mathrm{stab}(kG)$.

Since tensor triangulated categories also appear in topology, motivic theory, $KK$-theory, and other areas, general results about spectrum and support have a wide potential of application. This level of generality, that we dub *tensor triangular geometry*, is the one of our main results. The first one is Theorem 2.11, which says:

**Theorem.** *Let $\mathcal{K}$ be a strongly closed tensor triangulated category. Assume that $\mathcal{K}$ is idempotent complete. Then, if the support of an object $a \in \mathcal{K}$ can be decomposed as $\mathrm{supp}(a) = Y_1 \cup Y_2$ for disjoint closed subsets $Y_1$, $Y_2 \subset \mathrm{Spc}(\mathcal{K})$, with each open complement $\mathrm{Spc}(\mathcal{K}) \smallsetminus Y_i$ quasi-compact, then the object itself can be decomposed as a direct sum $a \simeq a_1 \oplus a_2$ with $\mathrm{supp}(a_i) = Y_i$ for $i = 1, 2$.*

We call a tensor triangulated category $\mathcal{K}$ *strongly closed* when the symmetric monoidal structure $\otimes : \mathcal{K} \times \mathcal{K} \longrightarrow \mathcal{K}$ is closed, *i.e.* admits an internal hom functor which is bi-exact, and when all objects are strongly dualizable (Def. 2.1). We explain in Remark 2.2 how such categories proliferate in Nature. An additive category $\mathcal{K}$ is *idempotent complete* if all its idempotents split (Rem. 2.10) and we show in Example 2.13 that this hypothesis is necessary. In any case, we can embed any triangulated category $\mathcal{K}$ into an idempotent complete one $\mathcal{K} \hookrightarrow \widetilde{\mathcal{K}}$, see [2].

The above result has the following avatars:

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**Corollary.** *Let $X$ be a quasi-compact and quasi-separated scheme (e.g. $X$ noetherian) and let $E \in \mathrm{D}^{\mathrm{perf}}(X)$ be a perfect complex on $X$. Let $Y_1$ and $Y_2$ be disjoint closed subsets of $X$ such that $X \smallsetminus Y_1$ and $X \smallsetminus Y_2$ are quasi-compact (automatic if $X$ is noetherian). Assume that the complex $E$ is acyclic outside $Y_1 \cup Y_2$. Then there exist perfect complexes $E_1$ and $E_2$ over $X$ such that $E \simeq E_1 \oplus E_2$ in $\mathrm{D}^{\mathrm{perf}}(X)$ and such that $E_i$ is acyclic outside $Y_i$ for $i = 1, 2$.*

**Corollary.** *Let $k$ be a field, $G$ a finite group scheme and $M$ an indecomposable $kG$-module. Then its projective support variety $\mathcal{V}_G(M)$ is connected.*

For $G$ a finite group, the latter is a celebrated theorem of J. F. Carlson. (Read at least the title of [6] for $k$ algebraically closed and see Benson [3, Part II, Thm. 5.12.1, p. 194] in general.) See also later developments in Rickard [18], Krause [15] or Chebolu [9].

In the next statement, for a closed subset $Z \subset \mathrm{Spc}(\mathcal{K})$ of the spectrum, $\dim(Z)$ refers to its Krull dimension. This is a special case of our second main result, Theorem 3.24, where we shall consider arbitrary dimension functions (Def. 3.1).

**Theorem.** *Let $\mathcal{K}$ be a strongly closed tensor triangulated category such that $\mathrm{Spc}(\mathcal{K})$ is noetherian. Let $0 = \mathcal{K}_{(-1)} \subset \mathcal{K}_{(0)} \subset \cdots \subset \mathcal{K}_{(p-1)} \subset \mathcal{K}_{(p)} \subset \cdots \subset \mathcal{K}$ be the filtration by dimension of the support, that is $\mathcal{K}_{(p)} := \{a \in \mathcal{K} \mid \dim(\mathrm{supp}(a)) \le p\}$. Then the associated quotients decompose as sums of local terms. More precisely, consider the functor $a \mapsto \big\{q_{\mathcal{P}}(a)\big\}_{\mathcal{P}}$ induced by the localizations $q_{\mathcal{P}} : \mathcal{K} \to \mathcal{K}/\mathcal{P}$*

$$\mathcal{K}_{(p)} \big/ \mathcal{K}_{(p-1)} \quad \longrightarrow \quad \coprod_{\substack{\mathcal{P} \in \mathrm{Spc}(\mathcal{K}) \\ \dim \overline{\{\mathcal{P}\}} \,= p}} (\mathcal{K}/\mathcal{P})_{(0)} \, .$$

*Then this functor is fully faithful and has cofinal image, i.e. every object in the right-hand category is a direct summand of an object coming from the left. Equivalently this functor induces an equivalence after idempotent completion on both sides.*

If one understands $\mathcal{K}/\mathcal{P}$ as the "local category at the point $\mathcal{P} \in \mathrm{Spc}(\mathcal{K})$" then the full subcategory $(\mathcal{K}/\mathcal{P})_{(0)} \subset \mathcal{K}/\mathcal{P}$ consists of those objects which are supported on the unique closed point of the spectrum of $\mathcal{K}/\mathcal{P}$. So, by analogy with commutative algebra, one can think of $(\mathcal{K}/\mathcal{P})_{(0)}$ as the "finite length" objects of tensor triangular geometry. (See Def. 3.7.) As before, we have immediate applications:

**Corollary.** *Let $X$ be a noetherian scheme and let $\mathrm{D}^{\mathrm{perf}}_{(p)}(X)$ be the full subcategory of $\mathrm{D}^{\mathrm{perf}}(X)$ of those perfect complexes $E$ such that $\dim_{\mathrm{Krull}}(\mathrm{supph}(E)) \le p$. Then localization at the points $x \in X_{(p)}$ (generic points of irreducible subschemes of $X$ of dimension $p$) induces the following equivalence of categories:*

$$\widetilde{\mathrm{D}^{\mathrm{perf}}_{(p)}(X) \big/ \mathrm{D}^{\mathrm{perf}}_{(p-1)}(X)} \quad \overset{\sim}{\longrightarrow} \quad \coprod_{x \in X_{(p)}} \mathrm{K}^{\mathrm{b}}_{\mathrm{fin.lg.}}(\mathcal{O}_{X,x}\text{ -free})$$

*between the idempotent completion of $\mathrm{D}^{\mathrm{perf}}_{(p)}(X) \,/\, \mathrm{D}^{\mathrm{perf}}_{(p-1)}(X)$ and the right-hand coproduct, where we denote by $\mathrm{K}^{\mathrm{b}}_{\mathrm{fin.lg.}}(\mathcal{O}_{X,x}\text{ -free})$ the homotopy category of bounded complexes of free $\mathcal{O}_{X,x}$-modules of finite rank with finite length homology.*

This result is well-known for regular schemes, and is the keystone of several local-global spectral sequences, like for instance the Brown-Gersten spectral sequences in algebraic $K$-theory and in Witt theory. Such a decomposition might be folklore for non-regular schemes as well but does not seem to exist in the literature.

For the case of $\mathcal{K} = \mathrm{stab}(kG)$, for $G$ a finite group, the analogous filtration has already been described in Carlson-Donovan-Wheeler [8], at least when the field $k$ is algebraically closed. Here, we obtain:

**Corollary.** *Let $k$ be a field, $G$ a finite group scheme and $\mathrm{stab}(kG)$ the category of $kG$-modules modulo projectives. Consider the projective support variety $\mathcal{V}_G(k) = \mathrm{Proj}(H^\bullet(G,k))$. For $p \geq 0$, consider $\mathrm{stab}_{(p)}(kG)$ the full subcategory of $\mathrm{stab}(kG)$ of those $kG$-modules $M$ such that $\dim_{\mathrm{Krull}}(\mathcal{V}_G(M)) \leq p$. (This $\mathrm{stab}_{(p)}(kG)$ is $\mathcal{M}_{p+1}$ in the special case of [8].) Then we have a fully faithful functor*

$$\frac{\mathrm{stab}_{(p)}(kG)}{\mathrm{stab}_{(p-1)}(kG)} \hookrightarrow \coprod_{x \in \mathcal{V}_G(k)_{(p)}} \frac{\mathrm{stab}(kG)}{\{\, M \in \mathrm{stab}(kG) \,\big|\, x \notin \mathcal{V}_G(M) \,\}}$$

*whose image lies inside the coproduct of "finite length" objects in each localization, and it is cofinal therein. Saying that a $kG$-module $M \in \mathrm{stab}(kG)$ is of "finite length" in the localization at a point $x \in \mathcal{V}_G(k)$ is equivalent to: either $M$ is zero at $x$, that is $x \notin \mathcal{V}_G(M)$, or $\overline{\{x\}}$ is an irreducible component of the support $\mathcal{V}_G(M)$.*

See more in Section 4, where we explain the Corollaries presented in this Introduction. The organization of the paper should now be clear from the following table of contents. Let us moreover mention that our second main result also applies to the filtration by Krull codimension instead of dimension, see Theorem 3.26.

## Contents



## 1. Basic elements of tensor triangular geometry

We survey the main concepts and results of [1]. Standard notions about triangulated categories can be found in Verdier [21] or Neeman [17].

**Definition 1.1.** A *tensor triangulated* category $(\mathcal{K}, \otimes, 1)$ is a triangulated category $\mathcal{K}$ with a symmetric monoidal structure $\otimes : \mathcal{K} \times \mathcal{K} \longrightarrow \mathcal{K}$, $(a,b) \mapsto a \otimes b$ such that the functors $a \otimes -$ and $- \otimes b$ are exact for every $a, b \in \mathcal{K}$. We have in particular $a \otimes b \cong b \otimes a$ and $1 \otimes a \cong a$ for the *unit* $1 \in \mathcal{K}$.

A *prime ideal* $\mathcal{P} \subsetneq \mathcal{K}$ is a proper subcategory such that (1)-(4) below hold true:

(1) $\mathcal{P}$ is a full triangulated subcategory, *i.e.* $0 \in \mathcal{P}$, $a \in \mathcal{P} \Leftrightarrow T(a) \in \mathcal{P}$ and if $a \to b \to c \to T(a)$ is distinguished in $\mathcal{K}$ and $a, b \in \mathcal{P}$ then $c \in \mathcal{P}$;
(2) $\mathcal{P}$ is thick, *i.e.* if $a \oplus b \in \mathcal{P}$ then $a, b \in \mathcal{P}$;
(3) $\mathcal{P}$ is a $\otimes$-ideal, *i.e.* if $a \in \mathcal{P}$ then $a \otimes b \in \mathcal{P}$ for all $b \in \mathcal{K}$;
(4) $\mathcal{P}$ is *prime*, *i.e.* if $a \otimes b \in \mathcal{P}$ then $a \in \mathcal{P}$ or $b \in \mathcal{P}$.

Only (4) was introduced in [1]. Properties (1)-(3) are standard and we shall actually say that a subcategory $\mathcal{J} \subset \mathcal{K}$ satisfying (1), (2) and (3) is a *thick $\otimes$-ideal.*

We tacitly assume that $\mathcal{K}$ is essentially small, *i.e.* it has a set of isomorphism classes of objects. The *spectrum* $\mathrm{Spc}(\mathcal{K})$ is the set of primes $\mathcal{P} \subset \mathcal{K}$. The *support* of an object $a \in \mathcal{K}$ is defined as the subset $\mathrm{supp}(a) = \{\mathcal{P} \in \mathrm{Spc}(\mathcal{K}) \mid a \notin \mathcal{P}\} \subset \mathrm{Spc}(\mathcal{K})$. The complements $\mathrm{U}(a) = \{\mathcal{P} \in \mathrm{Spc}(\mathcal{K}) \mid a \in \mathcal{P}\}$ of these supports form a basis $\{\mathrm{U}(a)\}_{a\in\mathcal{K}}$ of the so-called *Zariski topology* on the spectrum.

*Remark* 1.2. Since a prime $\mathcal{P} \subset \mathcal{K}$ is thick, we can construct the Verdier localization

$$q_{\mathcal{P}} : \mathcal{K} \longrightarrow \mathcal{K}/\mathcal{P}$$

*i.e.* the universal functor from $\mathcal{K}$ to a triangulated category which maps all objects of $\mathcal{P}$ to zero. Equivalently, one can describe $\mathcal{K}/\mathcal{P}$ as having the same objects as $\mathcal{K}$ but morphisms being equivalence classes of fractions $\overset{s}{\leftarrow} \rightarrow$ with respect to those morphisms $s$ whose cone belongs to $\mathcal{P}$. With this in mind, it is maybe more intuitive to think of $\mathcal{K}/\mathcal{P}$ as the "local category at a point of the spectrum" and to think of $\mathcal{P}$ as those objects which vanish at that point. Since the support of an object is the locus where the object does *not* vanish, this justifies the above definition of the support, which also reads $\mathrm{supp}(a) = \{\mathcal{P} \in \mathrm{Spc}(\mathcal{K}) \mid q_{\mathcal{P}}(a) \neq 0\}$.

**Proposition 1.3.** *Let $\mathcal{K}$ be a tensor triangulated category. Then we have*

(i) $\mathrm{supp}(0) = \varnothing$ *and* $\mathrm{supp}(1) = \mathrm{Spc}(\mathcal{K})$.
(ii) $\mathrm{supp}(a \oplus b) = \mathrm{supp}(a) \cup \mathrm{supp}(b)$.
(iii) $\mathrm{supp}(Ta) = \mathrm{supp}(a)$ *where $T : \mathcal{K} \to \mathcal{K}$ is the suspension.*
(iv) $\mathrm{supp}(a) \subset \mathrm{supp}(b) \cup \mathrm{supp}(c)$ *for any distinguished $a \to b \to c \to T(a)$.*
(v) $\mathrm{supp}(a \otimes b) = \mathrm{supp}(a) \cap \mathrm{supp}(b)$.

*(In fact,* $(\mathrm{Spc}(\mathcal{K}), \mathrm{supp})$ *is universal for these properties, see* [1, Thm. 3.2]*.)*

Tensor triangular geometry contains for example classical algebraic geometry and the theory of support varieties in modular representation theory. Indeed :

**Theorem 1.4.** *We have homeomorphisms of underlying topological spaces :*

(i) $X \xrightarrow{\sim} \mathrm{Spc}(\mathrm{D}^{\mathrm{perf}}(X))$ *for every quasi-compact and quasi-separated scheme $X$.*
(ii) $\mathcal{V}_G(k) \xrightarrow{\sim} \mathrm{Spc}(\mathrm{stab}(kG))$ *for every field $k$ of characteristic $p > 0$ and every finite group scheme $G$. Recall that $\mathcal{V}_G(k) \overset{def.}{=} \mathrm{Proj}(H^\bullet(G,k))$ where the commutative graded ring $H^\bullet(G,k)$ is defined to be $\oplus_{i\in\mathbb{N}} H^i(G,k)$ when $p = 2$ and $\oplus_{i\in\mathbb{N}} H^{2i}(G,k)$ when $p > 2$.*

*We can upgrade these homeomorphisms to scheme isomorphisms if we equip* $\mathrm{Spc}(\mathcal{K})$ *with the sheaf of rings $\mathcal{O}_{\mathcal{K}}$ obtained locally from the endomorphisms of the unit.*

As explained in [1], this result follows from suitable classification of thick $\otimes$-ideals of $\mathrm{D}^{\mathrm{perf}}(X)$ and of $\mathrm{stab}(kG)$. In fact, point (i) is only proved in [1] for $X$ topologically noetherian and the reader can restrict herself to such schemes if he prefers. Buan, Krause and Solberg recently gave the above generalization in [5].

We established in [1], that $\mathrm{Spc}(\mathcal{K})$ allows a classification of thick $\otimes$-ideals for general tensor triangulated categories $\mathcal{K}$. This is the key ingredient in our proof of Theorem 2.11. We need the following convenient notation.

**Notation 1.5.** Let $Y \subset \mathrm{Spc}(\mathcal{K})$. We denote by $\mathcal{K}_Y$ the full subcategory $\mathcal{K}_Y := \{a \in \mathcal{K} \mid \mathrm{supp}(a) \subset Y\}$ of those objects *supported on $Y$*.

*Remark* 1.6. By Proposition 1.3 (v), such a subcategory $\mathcal{K}_Y$ is always *radical*, that is, $a^{\otimes n} \in \mathcal{K}_Y \Rightarrow a \in \mathcal{K}_Y$. For $\mathcal{K}$ "strongly closed", all thick $\otimes$-ideals are radical (Prop. 2.4), so one can as well ignore this assumption in the following result.

**Theorem 1.7** (Classification, Thm. 4.10 of [1])**.** *There is a bijection*

$$\left\{ \begin{array}{r} Y \subset \mathrm{Spc}(\mathcal{K}) \textit{ s.t. } Y = \cup Y_\alpha \textit{ for } Y_\alpha \textit{ closed} \\ \textit{with } \mathrm{Spc}(\mathcal{K}) \smallsetminus Y_\alpha \textit{ quasi-compact} \end{array} \right\} \xrightarrow{\sim} \left\{ \mathcal{J} \subset \mathcal{K} \textit{ radical thick } \otimes\textit{-ideal} \right\}$$

*given by* $Y \mapsto \mathcal{K}_Y$ *(see Notation 1.5), with inverse* $\mathcal{J} \mapsto \mathrm{supp}(\mathcal{J}) := \bigcup_{a \in \mathcal{J}} \mathrm{supp}(a)$.

*Remark* 1.8. The above condition for $Y \subset \mathrm{Spc}(\mathcal{K})$ to be a union of closed subsets $Y_\alpha$ with quasi-compact complements was used by Thomason in [19] in the case of schemes. Therefore, we shall refer to such subsets $Y$ of $\mathrm{Spc}(\mathcal{K})$ as *Thomason subsets*. They are specialization closed ($y \in Y \Rightarrow \overline{\{y\}} \in Y$). If moreover $\mathrm{Spc}(\mathcal{K})$ is noetherian then conversely every specialization closed subset is a Thomason subset.

Results from [1] which we use only once in this paper will be quoted when needed. On the other hand, we shall use several times, even tacitly, the following facts:

**Proposition 1.9** ([1, Prop. 2.9])**.** *The closure of a point* $\mathcal{P} \in \mathrm{Spc}(\mathcal{K})$ *is* $\overline{\{\mathcal{P}\}} = \{\mathcal{Q} \in \mathrm{Spc}(\mathcal{K}) \mid \mathcal{Q} \subset \mathcal{P}\}$.

**Proposition 1.10** ([1, Prop. 2.18])**.** *Let* $Z \neq \varnothing$ *be a non-empty irreducible closed subset of* $\mathrm{Spc}(\mathcal{K})$. *Then there exists a unique point* $\mathcal{P} \in \mathrm{Spc}(\mathcal{K})$ *such that* $\overline{\{\mathcal{P}\}} = Z$.

**Proposition 1.11** ([1, Prop. 3.11])**.** *Let* $\mathcal{J} \subset \mathcal{K}$ *be a thick* $\otimes$*-ideal and consider* $q : \mathcal{K} \longrightarrow \mathcal{K}/\mathcal{J}$ *the localization. Then the natural map* $\mathrm{Spc}(q) : \mathrm{Spc}(\mathcal{K}/\mathcal{J}) \longrightarrow \mathrm{Spc}(\mathcal{K})$ *is injective and gives a homeomorphism* $\mathrm{Spc}(\mathcal{K}/\mathcal{J}) \xrightarrow{\sim} \{\mathcal{P} \in \mathrm{Spc}(\mathcal{K}) \mid \mathcal{P} \supset \mathcal{J}\}$ *with inverse* $\mathcal{P} \mapsto \mathcal{P}/\mathcal{J}$. *Via this identification, we have* $\mathrm{supp}(q(a)) = \mathrm{supp}(a) \cap \{\mathcal{P} \supset \mathcal{J}\}$ *for every object* $a \in \mathcal{K}$. *Moreover, we have* $(\mathcal{K}/\mathcal{J}) \,/\, (\mathcal{P}/\mathcal{J}) = \mathcal{K}/\mathcal{P}$.

Indeed the result about $\mathrm{supp}(q(a))$ is a special case of the functoriality of $\mathrm{Spc}(-)$ as explained in [1, Prop. 3.6]. This functoriality is also used in the next Proposition. The fact that $(\mathcal{K}/\mathcal{J}) \,/\, (\mathcal{P}/\mathcal{J}) = \mathcal{K}/\mathcal{P}$ is an easy exercise about localization.

**Proposition 1.12** ([1, Prop. 3.13])**.** *Let* $\iota : \mathcal{K} \hookrightarrow \widetilde{\mathcal{K}}$ *be the idempotent completion (Rem. 2.10). Then* $\mathrm{Spc}(\iota) : \mathrm{Spc}(\widetilde{\mathcal{K}}) \xrightarrow{\sim} \mathrm{Spc}(\mathcal{K})$ *is a homeomorphism with inverse* $\mathcal{P} \mapsto \widetilde{\mathcal{P}} \subset \widetilde{\mathcal{K}}$. *Via this identification, we have* $\mathrm{supp}(a) = \mathrm{supp}(\iota(a))$ *for all* $a \in \mathcal{K}$.

We cannot express the local category $\widetilde{\mathcal{K}}/\widetilde{\mathcal{P}}$ in terms of $\mathcal{K}/\mathcal{P}$ although the following result describes a general relation between idempotent completion and localization.

**Proposition 1.13.** *Let* $\mathcal{J} \subset \mathcal{K}$ *be a thick subcategory of a triangulated category* $\mathcal{K}$. *Consider the Verdier localization functor* $q : \mathcal{K} \to \mathcal{K}/\mathcal{J}$ *and the idempotent completion functor* $\iota : \mathcal{K} \to \widetilde{\mathcal{K}}$. *Then we have a commutative diagram:*

$$\begin{tikzcd}
\mathcal{J} \arrow[r, tail] \arrow[d, "\iota"'] & \mathcal{K} \arrow[r, two heads, "q"] \arrow[d, "\iota"'] & \mathcal{K}/\mathcal{J} \arrow[d, "F"] \\
\widetilde{\mathcal{J}} \arrow[r, tail] & \widetilde{\mathcal{K}} \arrow[r, two heads, "q'"'] & \widetilde{\mathcal{K}}/\widetilde{\mathcal{J}}
\end{tikzcd}$$

*for a unique fully faithful functor* $F : \mathcal{K}/\mathcal{J} \to \widetilde{\mathcal{K}}/\widetilde{\mathcal{J}}$ *which identifies* $\mathcal{K}/\mathcal{J}$ *with a cofinal subcategory of* $\widetilde{\mathcal{K}}/\widetilde{\mathcal{J}}$.

*Proof.* The functor $F$ is induced by the universal property of the localization since obviously $q' \circ \iota$ vanishes on $\mathcal{J}$. Cofinality of $F$ is obvious. Let us see that $F$ is faithful. It suffices to see that a morphism $f : a \to b$ in $\mathcal{K}$ such that $q'(\iota(f)) = 0$ satisfies $q(f) = 0$. The former means that $\iota(f)$ factors via some object of $\widetilde{\mathcal{J}}$ in $\widetilde{\mathcal{K}}$ and this implies, $\mathcal{J}$ being cofinal in $\widetilde{\mathcal{J}}$, that $\iota(f)$ factors via some object of $\mathcal{J}$ in $\widetilde{\mathcal{K}}$, and hence in $\mathcal{K}$ as well, since $\iota$ is fully faithful. Let us see that $F$ is full. Let

$$g : \quad \iota(a) \xleftarrow{\ s\ } z \xrightarrow{\ f\ } \iota(b) \tag{1.1}$$

be a morphism $g : q'(\iota(a)) \to q'(\iota(b))$ between two objects $a, b \in \mathcal{K}$, given by a fraction in $\widetilde{\mathcal{K}}$ such that $\mathrm{cone}(s) \in \widetilde{\mathcal{J}}$. The morphism $g$ is also given by the fraction

$$\iota(a) \xleftarrow{\ (s \quad 0)\ } z \oplus \mathrm{cone}(s) \xrightarrow{\ (f \quad 0)\ } \iota(b)\,. \tag{1.2}$$

There exists a distinguished triangle in $\widetilde{\mathcal{K}}$ on the "denominator" $\begin{pmatrix} s & 0 \end{pmatrix}$ as follows:

$$z \oplus \mathrm{cone}(s) \xrightarrow{\ (s \quad 0)\ } \iota(a) \xrightarrow{\ \begin{pmatrix} s_1 \\ 0 \end{pmatrix}\ } \mathrm{cone}(s) \oplus T(\mathrm{cone}(s)) \xrightarrow{\ \begin{pmatrix} s_2 & 0 \\ 0 & 1 \end{pmatrix}\ } T(z) \oplus T(\mathrm{cone}(s))\,.$$

Note that the second and third objects belong to $\mathcal{K}$. Hence $z \oplus \mathrm{cone}(s) \in \mathcal{K}$ too and since $\mathrm{cone}(s) \oplus T(\mathrm{cone}(s)) \in \mathcal{J}$, the fraction pictured in (1.2) defines a morphism between $q(a)$ and $q(b)$ in $\mathcal{K}/\mathcal{J}$, whose image by $F$ is the morphism $g$ of (1.1). $\square$

*Remark* 1.14. In the above Proposition, we do not claim that $F$ realizes the idempotent completion of $\mathcal{K}/\mathcal{J}$ since there is no reason for $\widetilde{\mathcal{K}}/\widetilde{\mathcal{J}}$ to be idempotent complete. Take for instance $\mathcal{K} = \widetilde{\mathcal{K}} = \mathrm{D}^{\mathrm{perf}}(X)$ and $\mathcal{J} = \widetilde{\mathcal{J}} = \mathrm{D}^{\mathrm{perf}}_Z(X)$ for some closed subset $Z \subset X$ of a scheme $X$ for which $K_0(X) \to K_0(X \smallsetminus Z)$ is not surjective and use $\widetilde{\mathcal{K}/\mathcal{J}} \cong \mathrm{D}^{\mathrm{perf}}(X \smallsetminus Z)$, see Thomason [20, § 5].

## 2. Supports in strongly closed tensor triangulated categories

**Definition 2.1.** We call a tensor triangulated category $(\mathcal{K}, \otimes, 1)$ *strongly closed* if there exists a bi-exact functor $\mathrm{hom} : \mathcal{K}^{\mathrm{op}} \times \mathcal{K} \longrightarrow \mathcal{K}$ with natural isomorphisms

$$\mathrm{Hom}_{\mathcal{K}}(a \otimes b, c) \cong \mathrm{Hom}_{\mathcal{K}}(a, \mathrm{hom}(b, c)) \tag{2.1}$$

and such that all objects are *strongly dualizable*, *i.e.* the natural morphism

$$D(a) \otimes b \xrightarrow{\sim} \mathrm{hom}(a, b) \tag{2.2}$$

is an isomorphism for all $a, b \in \mathcal{K}$, where we denote by $D(a)$ the dual $D(a) := \mathrm{hom}(a, 1)$ of an object $a \in \mathcal{K}$. More details can be found in [14, App. A], for instance. It follows from (2.2) that

$$D(D(a)) \cong a \tag{2.3}$$

for all $a \in \mathcal{K}$; see for instance [14, Thm. A.2.5 (b)].

*Remark* 2.2. Many examples of triangulated categories appear inside some huge tensor triangulated categories $\mathcal{C}$ with infinite coproducts. This is the approach of Hovey-Palmieri-Strickland [14], where $\mathcal{C}$ is also assumed to be generated (as a triangulated category with coproducts) by a set $\mathcal{G}$ of strongly dualizable objects and to satisfy a Brown representability property. If all objects $g \in \mathcal{G}$ are compact ("small" in *loc. cit.*), that is, $\mathrm{Hom}_{\mathcal{K}}(g,-)$ commutes with coproducts, then $\mathcal{C}$ is called algebraic. If moreover the unit $1 \in \mathcal{C}$ is compact as well, then $\mathcal{C}$ is called a *unital algebraic stable homotopy category.* Several examples of such categories are given in [14, Ex. 1.2.3 and § 9] ; see in particular Examples 1.2.3 (c) and (e) *loc. cit.*

To such a unital algebraic stable homotopy category $\mathcal{C}$, we can associate the strongly closed tensor triangulated category $\mathcal{K}$ of its compact objects (see [14, Thm. 2.1.3 (d)]). Starting with $\mathcal{C} = \mathrm{D}(\mathrm{Qcoh}_X)$ the derived category of quasi-coherent $\mathcal{O}_X$-modules over a noetherian separated scheme $X$, we obtain $\mathcal{K} = \mathrm{D}^{\mathrm{perf}}(X)$. See more in § 4.1.

This method gives an industrial way of producing strongly closed tensor triangulated categories. Note however that very nice and important examples can also be manufactured directly by hand, like for instance the stable category $\mathcal{K} = \mathrm{stab}(kG)$, with tensor product $- \underset{k}{\otimes} -$ over $k$, obvious unit and internal hom. See more in § 4.2.

**Lemma 2.3.** *Let $\mathcal{K}$ be a strongly closed tensor triangulated category and let $a \in \mathcal{K}$. Then the object $a$ is a direct summand of $a \otimes a \otimes D(a)$.*

*Proof.* Use the unit-counit relation for the adjunction $- \otimes a : \mathcal{K} \rightleftarrows \mathcal{K} : D(a) \otimes -$ which itself follows from (2.1) and (2.2). See alternatively [14, Lem. A.2.6]. □

**Proposition 2.4.** *Let $\mathcal{K}$ be a strongly closed tensor triangulated category and let $\mathcal{J} \subset \mathcal{K}$ be a thick $\otimes$-ideal. Then $\mathcal{J}$ is radical, that is, $a^{\otimes n} \in \mathcal{J} \Rightarrow a \in \mathcal{J}$.*

*Proof.* It suffices to prove $a^{\otimes 2} \in \mathcal{J} \Rightarrow a \in \mathcal{J}$. From $a \otimes a \in \mathcal{J}$ we have $a \otimes a \otimes D(a) \in \mathcal{J}$ (since $\mathcal{J}$ is a $\otimes$-ideal) and by Lemma 2.3 we have $a \in \mathcal{J}$ (since $\mathcal{J}$ is thick). □

**Corollary 2.5.** *Let $\mathcal{K}$ be a strongly closed tensor triangulated category and let $a \in \mathcal{K}$ be an object. Then* $\mathrm{supp}(a) = \varnothing$ *if and only if* $a = 0$.

*Proof.* We know from [1, Cor. 2.4] that $\mathrm{supp}(a) = \varnothing$ is equivalent to $a^{\otimes n} = 0$ for some $n \in \mathbb{N}$ and therefore to $a = 0$ by the above Proposition applied to $\mathcal{J} = 0$. □

**Proposition 2.6.** *Let $\mathcal{K}$ be a strongly closed tensor triangulated category and let $\mathcal{J} \subset \mathcal{K}$ be a thick $\otimes$-ideal. Then $D(\mathcal{J}) = \mathcal{J}$.*

*Proof.* Let $b \in \mathcal{J}$. From $\mathcal{J}$ being a $\otimes$-ideal it contains $D(b) \otimes D(b) \otimes b$ and then also $D(b) \otimes D(b) \otimes D^2(b)$ since $D^2(b) \cong b$. Applying Lemma 2.3 to $a = D(b)$, we know that $D(b)$ is a direct summand of $D(b) \otimes D(b) \otimes D^2(b) \in \mathcal{J}$ and hence $D(b) \in \mathcal{J}$ since $\mathcal{J}$ is thick. So, we have the inclusion $D(\mathcal{J}) \subset \mathcal{J}$. We get the other inclusion by applying $D(-)$ to this one, using again that $D^2 \simeq \mathrm{Id}_{\mathcal{K}}$, see (2.3). □

**Proposition 2.7.** *Let $a, b \in \mathcal{K}$ be objects in a strongly closed tensor triangulated category $\mathcal{K}$. We have :*

(i) $\mathrm{supp}(D(a)) = \mathrm{supp}(a)$.

(ii) $\mathrm{supp}(\mathrm{hom}(a,b)) = \mathrm{supp}(a) \cap \mathrm{supp}(b)$.

*Proof.* Let $\mathcal{P} \in \mathrm{Spc}(\mathcal{K})$ be a prime. By Proposition 2.6, we have $a \in \mathcal{P} \Leftrightarrow D(a) \in \mathcal{P}$. By definition of the support (see Def. 2.1) we then have $\mathrm{supp}(D(a)) = \{\mathcal{P} \in \mathrm{Spc}(\mathcal{K}) \mid D(a) \notin \mathcal{P}\} = \{\mathcal{P} \in \mathrm{Spc}(\mathcal{K}) \mid a \notin \mathcal{P}\} = \mathrm{supp}(a)$, hence (i). Part (ii) follows from (i), from strong dualizability (2.2) and from Proposition 1.3 (v). □

**Corollary 2.8.** *Let $\mathcal{K}$ be a strongly closed tensor triangulated category. Suppose that the supports of two objects do not meet :* $\mathrm{supp}(a) \cap \mathrm{supp}(b) = \varnothing$. *Then there is no non-trivial morphism between them :* $\mathrm{Hom}_{\mathcal{K}}(a,b) = 0$.

*Proof.* By Proposition 2.7 (ii) and Corollary 2.5, we have $\mathrm{hom}(a,b)=0$. But then $\mathrm{Hom}_{\mathcal{K}}(a,b)=\mathrm{Hom}_{\mathcal{K}}(1\otimes a,b)\overset{(2.1)}{=}\mathrm{Hom}_{\mathcal{K}}(1,\mathrm{hom}(a,b))=0$. □

*Remark* 2.9. For $f:a\to b$ with $\mathrm{supp}(a)\cap\mathrm{supp}(b)=\varnothing$, we have $q_{\mathcal{P}}(f)=0$ in $\mathcal{K}/\mathcal{P}$ for every $\mathcal{P}\in\mathrm{Spc}(\mathcal{K})$. However, from the local vanishing of a morphism $f$, we cannot deduce $f=0$ but only $f^{\otimes n}=0$ for some $n\geq 1$ (see [1, Prop. 2.21]). For instance, let $E'\rightarrowtail E\twoheadrightarrow E''$ be an exact sequence of vector bundles over a scheme $X$. In $\mathrm{D^b}(\mathrm{VB}_X)$, we have an associated distinguished triangle whose third map $E''\to T(E')$ is zero if and only if the original short exact sequence splits. So, in general, this map is locally zero but is not zero in $\mathrm{D^b}(\mathrm{VB}_X)$. (Here, $\mathrm{supp}(E'')$ and $\mathrm{supp}(T(E'))$ can intersect non-trivially, so Corollary 2.8 does not apply.)

*Remark* 2.10. Recall that an additive category $\mathcal{K}$ is *idempotent complete* (or pseudo-abelian or karoubian) if all idempotents of all objects split, that is, if $e\in\mathrm{Hom}_{\mathcal{K}}(a,a)$ with $e^2=e$ then the object $a$ decomposes as a direct sum $a\simeq a'\oplus a''$ on which $e$ becomes $\left(\begin{smallmatrix}1&0\\0&0\end{smallmatrix}\right)$, that is, $a\simeq\mathrm{Im}(e)\oplus\mathrm{Ker}(e)$. One can always "idempotent complete" an additive category $\mathcal{K}\hookrightarrow\widetilde{\mathcal{K}}$ and $\widetilde{\mathcal{K}}$ remains triangulated, if $\mathcal{K}$ was, see [2].

**Theorem 2.11.** *Let $\mathcal{K}$ be an idempotent complete, strongly closed tensor triangulated category. Let $Y_1$, $Y_2\subset\mathrm{Spc}(\mathcal{K})$ be disjoint Thomason subsets of $\mathrm{Spc}(\mathcal{K})$ (see Rem. 1.8). Then the subcategory $\mathcal{K}_{Y_1\cup Y_2}$ supported on $Y_1\cup Y_2$ (see Not. 1.5) coincides with $\mathcal{K}_{Y_1}\oplus\mathcal{K}_{Y_2}:=\{a\in\mathcal{K}\mid a\simeq a_1\oplus a_2$ with $a_i\in\mathcal{K}_{Y_i}\}$.*

*Proof.* In fact, the core of the proof consists in proving that $\mathcal{K}_{Y_1}\oplus\mathcal{K}_{Y_2}$ *is indeed* a thick $\otimes$-ideal of $\mathcal{K}$ (see Def. 1.1). It is clearly $\otimes$-ideal.

Let us see that $\mathcal{K}_{Y_1}\oplus\mathcal{K}_{Y_2}$ is a triangulated subcategory. Let

$$a_1\oplus a_2\xrightarrow{u}b_1\oplus b_2\xrightarrow{v}c\xrightarrow{w}T(a_1\oplus a_2)$$

be a distinguished triangle in $\mathcal{K}$ with $\mathrm{supp}(a_i)\subset Y_i$ and $\mathrm{supp}(b_i)\subset Y_i$ for $i=1,2$. Since $\mathrm{supp}(a_1)\cap\mathrm{supp}(b_2)\subset Y_1\cap Y_2=\varnothing$, we know by Corollary 2.8 that $\mathrm{Hom}_{\mathcal{K}}(a_1,b_2)=0$. Similarly $\mathrm{Hom}_{\mathcal{K}}(a_2,b_1)=0$. Therefore $u=\left(\begin{smallmatrix}u_1&0\\0&u_2\end{smallmatrix}\right)$ where $u_i:a_i\to b_i$ for $i=1,2$. Note that $\mathrm{cone}(u_i)\in\mathcal{K}_{Y_i}$. By uniqueness of the cone, we have $c\simeq\mathrm{cone}(u_1)\oplus\mathrm{cone}(u_2)$ and so $c\in\mathcal{K}_{Y_1}\oplus\mathcal{K}_{Y_2}$.

Let us then prove that $\mathcal{K}_{Y_1}\oplus\mathcal{K}_{Y_2}$ is thick. Indeed, let $e=e^2$ be an idempotent on an object of the form $a_1\oplus a_2$ with $\mathrm{supp}(a_i)\subset Y_i$ for $i=1,2$. As above, we have $\mathrm{Hom}_{\mathcal{K}}(a_1,a_2)=\mathrm{Hom}_{\mathcal{K}}(a_2,a_1)=0$ so the idempotent has the form $e=\left(\begin{smallmatrix}e_1&0\\0&e_2\end{smallmatrix}\right)$ for idempotents $e_i=(e_i)^2:a_i\to a_i$. Now, since the ambient category $\mathcal{K}$ is idempotent complete, each $a_i$ decomposes as $a_i=\mathrm{Im}(e_i)\oplus\mathrm{Ker}(e_i)$. Note that $\mathrm{supp}(\mathrm{Im}(e_i))\subset\mathrm{supp}(a_i)\subset Y_i$. So, since we have $\mathrm{Im}(e)\simeq\mathrm{Im}(e_1)\oplus\mathrm{Im}(e_2)$, this means that the direct summand $\mathrm{Im}(e)$ of $a_1\oplus a_2$ also belongs to our subcategory $\mathcal{K}_{Y_1}\oplus\mathcal{K}_{Y_2}$. Every direct summand is the image of an idempotent thus we have proved that the subcategory $\mathcal{K}_{Y_1}\oplus\mathcal{K}_{Y_2}$ is closed under taking direct summands.

Therefore, the subcategory $\mathcal{K}_{Y_1}\oplus\mathcal{K}_{Y_2}$ is thick, $\otimes$-ideal and triangulated. It is also radical like every thick $\otimes$-ideal by Proposition 2.4. Then, because of the classification of such subcategories given in Theorem 1.7, there is a Thomason subset $Y\subset\mathrm{Spc}(\mathcal{K})$ such that $\mathcal{K}_{Y_1}\oplus\mathcal{K}_{Y_2}=\mathcal{K}_Y$. On the other hand $\mathcal{K}_{Y_1}\oplus\mathcal{K}_{Y_2}$ clearly is the smallest thick $\otimes$-ideal containing $\mathcal{K}_{Y_1}$ and $\mathcal{K}_{Y_2}$. By the classification again, the subset $Y$ must similarly be the smallest Thomason subset of $\mathrm{Spc}(\mathcal{K})$ containing $Y_1$ and $Y_2$, that is, $Y=Y_1\cup Y_2$. □

*Remark* 2.12. If $\mathrm{supp}(a) = Y_1 \cup Y_2$ with $Y_1$ and $Y_2$ disjoint, closed, and with quasi-compact complement, then Theorem 2.11 gives a decomposition $a \simeq a_1 \oplus a_2$ with $\mathrm{supp}(a_i) \subset Y_i$. Now, since $Y_1 \cup Y_2 = \mathrm{supp}(a) = \mathrm{supp}(a_1) \cup \mathrm{supp}(a_2)$ and since $Y_1 \cap Y_2 = \varnothing$, we necessarily have equality $\mathrm{supp}(a_i) = Y_i$. This was the formulation given in the Introduction.

**Example 2.13.** Let $\mathcal{L}$ be a strongly closed tensor triangulated category such that $K_0(\mathcal{L}) \neq 0$. Define a category $\mathcal{K}$ as the following full subcategory of $\mathcal{L} \times \mathcal{L}$:

$$\mathcal{K} := \left\{ (x_1 , x_2) \in \mathcal{L} \times \mathcal{L} \,\middle|\, [x_1] = [x_2] \text{ in } K_0(\mathcal{L}) \right\}.$$

It is easy to check that $\mathcal{K}$ inherits from $\mathcal{L} \times \mathcal{L}$ the structure of a tensor triangulated category with $1_{\mathcal{K}} = (1_{\mathcal{L}} , 1_{\mathcal{L}})$. Moreover, $\mathcal{K}$ is cofinal in $\mathcal{L} \times \mathcal{L}$, that is, every object of $\mathcal{L} \times \mathcal{L}$ is a direct summand of an object of $\mathcal{K}$. (Indeed, Thomason proved in [19] that cofinal subcategories are exactly characterized by subgroups of $K_0$.) By Proposition 1.12, $\mathrm{Spc}(\mathcal{K}) = \mathrm{Spc}(\mathcal{L} \times \mathcal{L}) = \mathrm{Spc}(\mathcal{L}) \sqcup \mathrm{Spc}(\mathcal{L})$. Let us denote by $Y_1$ and $Y_2$ these two disjoint copies of $\mathrm{Spc}(\mathcal{L})$ in $\mathrm{Spc}(\mathcal{K})$. For an object $x = (x_1 , x_2) \in \mathcal{L} \times \mathcal{L}$, we have $\mathrm{supp}(x) = \mathrm{supp}(x_1) \sqcup \mathrm{supp}(x_2)$ with $\mathrm{supp}(x_i) \subset Y_i$ and therefore the condition $\mathrm{supp}(x) \subset Y_1$ is equivalent to $x_2 = 0$. The element $1_{\mathcal{K}}$ has support $Y_1 \sqcup Y_2$ but cannot be decomposed as $a_1 \oplus a_2$ with $a_i \in \mathcal{K}$ and $\mathrm{supp}(a_i) \subset Y_i$, for this would imply that $a_1 = (1_{\mathcal{L}}, 0) \in \mathcal{K}$ and therefore $[1_{\mathcal{L}}] = 0$, which contradicts $K_0(\mathcal{L}) \neq 0$. Of course, such a decomposition $1_{\mathcal{K}} = (1_{\mathcal{L}}, 0) \oplus (0, 1_{\mathcal{L}})$ exists in the bigger category $\mathcal{L} \times \mathcal{L}$.

Therefore our Theorem 2.11 fails if we do not assume $\mathcal{K}$ idempotent complete.

*Remark* 2.14. So far, in addition to being idempotent complete (which can be "arranged"), we only needed the category $\mathcal{K}$ to satisfy the following two properties:

(1) Every thick $\otimes$-ideal of $\mathcal{K}$ is radical. (Prop. 2.4)

(2) If $\mathrm{supp}(a) \cap \mathrm{supp}(b) = \varnothing$ then $\mathrm{Hom}_{\mathcal{K}}(a, b) = 0$. (Cor. 2.8)

Note however that we do not see why these conditions pass to localizations $\mathcal{K}/\mathcal{J}$, a fact we shall need in the sequel. Hence our *a priori* more restrictive condition that the tensor triangulated category $\mathcal{K}$ is strongly closed behaves better than just asking for $\mathcal{K}$ to satisfy (1) and (2) above. This is what we briefly sketch now.

**Proposition 2.15.** *Let $\mathcal{K}$ be a strongly closed tensor triangulated category.*

(i) *Consider the idempotent completion $\iota : \mathcal{K} \hookrightarrow \widetilde{\mathcal{K}}$. Then there exists a well-defined structure of strongly closed tensor triangulated category on $\widetilde{\mathcal{K}}$ such that $\iota(a) \otimes \iota(b) \cong \iota(a \otimes b)$ and $\mathrm{hom}(\iota(a), \iota(b)) \cong \iota(\mathrm{hom}(a, b))$.*

(ii) *Let $\mathcal{J} \subset \mathcal{K}$ be a thick $\otimes$-ideal and consider the corresponding Verdier localization $q : \mathcal{K} \to \mathcal{K}/\mathcal{J}$. Then there exists a well-defined structure of strongly closed tensor triangulated category on $\mathcal{K}/\mathcal{J}$ such that $q(a) \otimes q(b) \cong q(a \otimes b)$ and $\mathrm{hom}(q(a), q(b)) \cong q(\mathrm{hom}(a, b))$.*

*Proof.* To see (i), recall that the objects of $\widetilde{\mathcal{K}}$ are pairs $(a, e)$ with $a \in \mathcal{K}$ and $e = e^2 \in \mathrm{Hom}_{\mathcal{K}}(a, a)$. One can then define the product on $\widetilde{\mathcal{K}}$ by $(a, e) \otimes (b, f) := (a \otimes b\, ,\, e \otimes f)$ and the internal hom by $\mathrm{hom}\big((a, e)\, ,\, (b, f)\big) = \big(\mathrm{hom}(a, b)\, ,\, \mathrm{hom}(e, f)\big)$.

To see (ii), it suffices to observe that the functors $a \otimes -$, $- \otimes b$, $\mathrm{hom}(a, -)$ and $\mathrm{hom}(-, b)$ are exact and that they send $\mathcal{J}$ to itself. For $\mathrm{hom}(-, -)$, it suffices to see that $D(\mathcal{J}) \subset \mathcal{J}$ which has been established in Proposition 2.6.

In both cases, the relevant adjunctions are easily constructed via their unit and counit, by exporting those existing on $\mathcal{K}$. Details are left to the reader. □

## 3. Dimension functions and quotients of the associated filtrations

Extend the usual order from $\mathbb{Z}$ to $\mathbb{Z}\cup\{\pm\infty\}$ by $-\infty < d < +\infty$ for all $d \in \mathbb{Z}$.

**Definition 3.1.** Let $\mathcal{K}$ be a tensor triangulated category. A *dimension function on* $\mathcal{K}$ is a function $\dim : \operatorname{Spc}(\mathcal{K}) \longrightarrow \mathbb{Z}\cup\{\pm\infty\}$ satisfying two conditions:

(1) If $\mathcal{Q} \subset \mathcal{P}$ then $\dim(\mathcal{Q}) \le \dim(\mathcal{P})$;

(2) If $\mathcal{Q} \subset \mathcal{P}$ and $\dim(\mathcal{Q}) = \dim(\mathcal{P})$ is finite then $\mathcal{Q} = \mathcal{P}$.

**Example 3.2.** The main example to keep in mind is $\dim(\mathcal{P}) = \dim_{\mathrm{Krull}}\big(\overline{\{\mathcal{P}\}}\big)$ the Krull dimension of the irreducible closed subset of $\operatorname{Spc}(\mathcal{K})$ generated by the point $\mathcal{P}$. By Propositions 1.9 and 1.10, this number $\dim(\mathcal{P}) \in \mathbb{Z}_{\ge 0}\cup\{+\infty\}$ is equal to the supremum of those $m \in \mathbb{Z}$ for which there exists a sequence of $m$ strict inclusions of primes ending at $\mathcal{P}$:

$$\mathcal{Q}_m \subsetneq \mathcal{Q}_{m-1} \subsetneq \cdots \subsetneq \mathcal{Q}_1 \subsetneq \mathcal{Q}_0 = \mathcal{P}\,.$$

**Example 3.3.** One can also consider $\dim(\mathcal{P}) = -\operatorname{codim}_{\mathrm{Krull}}\big(\overline{\{\mathcal{P}\}}\big)$ the opposite of the Krull codimension of $\overline{\{\mathcal{P}\}}$. This number $\dim(\mathcal{P}) \in \mathbb{Z}_{\le 0}\cup\{-\infty\}$ is the infimum of those $n \in \mathbb{Z}$ for which there exist $-n$ strict inclusions of primes starting at $\mathcal{P}$:

$$\mathcal{P} = \mathcal{Q}_0 \subsetneq \mathcal{Q}_{-1} \subsetneq \cdots \subsetneq \mathcal{Q}_{n+1} \subsetneq \mathcal{Q}_n\,.$$

**Notation 3.4.** Let $\dim : \operatorname{Spc}(\mathcal{K}) \longrightarrow \mathbb{Z}\cup\{\pm\infty\}$ be a dimension function on a tensor triangulated category $\mathcal{K}$. For every $p \in \mathbb{Z}\cup\{\pm\infty\}$, we define

$$\operatorname{Spc}(\mathcal{K})_{(p)} := \big\{\, \mathcal{P} \in \operatorname{Spc}(\mathcal{K}) \,\big|\, \dim(\mathcal{P}) = p \,\big\}\,.$$

**Definition 3.5.** Let $\dim : \operatorname{Spc}(\mathcal{K}) \to \mathbb{Z}\cup\{\pm\infty\}$ be a dimension function and let $Z \subset \operatorname{Spc}(\mathcal{K})$ be a closed subset. Then we define

$$\dim(Z) := \sup_{\mathcal{P}\in Z} \dim(\mathcal{P}) \tag{3.1}$$

for $Z$ non-empty and we set $\dim(\varnothing) = -\infty$.

**Notation 3.6.** Let $\dim : \operatorname{Spc}(\mathcal{K}) \longrightarrow \mathbb{Z}\cup\{\pm\infty\}$ be a dimension function. For every $p \in \mathbb{Z}\cup\{\pm\infty\}$, we define a subcategory $\mathcal{K}_{(p)}$ of $\mathcal{K}$ as follows:

$$\mathcal{K}_{(p)} = \big\{\, a \in \mathcal{K} \,\big|\, \dim(\operatorname{supp}(a)) \le p \,\big\}\,.$$

We obviously have

$$\sqrt{0} \subset \mathcal{K}_{(-\infty)} \subset \cdots \subset \mathcal{K}_{(p-1)} \subset \mathcal{K}_{(p)} \subset \ldots \subset \mathcal{K}_{(+\infty)} = \mathcal{K}\,.$$

Note that $\mathcal{K}_{(-\infty)}$ can differ from $\{a \in \mathcal{K} \mid \operatorname{supp}(a) = \varnothing\} = \sqrt{0}$ as illustrated by the example of the constant dimension function $\dim(\mathcal{P}) = -\infty$, $\forall\, \mathcal{P} \in \operatorname{Spc}(\mathcal{K})$.

**Definition 3.7.** Let $\mathcal{L}$ be a *local* tensor triangulated category, in the sense that its spectrum has a unique closed point $x_0 \in \operatorname{Spc}(\mathcal{L})$. (Then $x_0$ necessarily is $x_0 = \sqrt{0}$ and when $\mathcal{L}$ is strongly closed, this closed point is $x_0 = 0$.) We define

$$\mathrm{FL}(\mathcal{L}) := \big\{\, a \in \mathcal{L} \,\big|\, \operatorname{supp}(a) \subset \{x_0\} \,\big\}$$

the category of *"finite length"* objects. This terminology is not justified intrisically but by analogy with commutative algebra (where a finitely generated module over a local ring is of finite length if and only if its support is contained in the unique closed point). However, more generally, the classification of Theorem 1.7 implies that $\mathrm{FL}(\mathcal{L})$ is the smallest non-zero thick $\otimes$-ideal of $\mathcal{L}$, at least when $\operatorname{Spc}(\mathcal{L})$ is noetherian, for $\{x_0\}$ clearly is the smallest non-empty Thomason subset of $\operatorname{Spc}(\mathcal{L})$.

**Proposition 3.8.** *Let $\mathcal{L}$ be a local (Def. 3.7) tensor triangulated category and let* $\dim_{\mathcal{L}} : \mathrm{Spc}(\mathcal{L}) \to \mathbb{Z}_{\geq 0} \cup \{+\infty\}$ *be the Krull dimension (see Ex. 3.2). Then*

$$\mathrm{FL}(\mathcal{L}) = \mathcal{L}_{(0)}\,.$$

*Proof.* We have $\mathrm{Spc}(\mathcal{L})_{(0)} = \{x_0\}$, where $x_0$ is the only closed point of $\mathrm{Spc}(\mathcal{L})$. Hence, for $Z \subset \mathrm{Spc}(\mathcal{L})$ closed, we have $\dim(Z) = 0$ if and only if $Z = \{x_0\}$. $\square$

*Remark* 3.9. Let $\dim : \mathrm{Spc}(\mathcal{K}) \longrightarrow \mathbb{Z} \cup \{\pm\infty\}$ be a dimension function and let $\mathcal{J} \subset \mathcal{K}$ be a thick $\otimes$-ideal. We can restrict the dimension function on the subspace $\mathrm{Spc}(\mathcal{K}/\mathcal{J}) \subset \mathrm{Spc}(\mathcal{K})$, see Prop. 1.11. This will be used intensively below. With this in mind, observe the difference between the previous proposition and the next one.

**Proposition 3.10.** *Let $\mathcal{K}$ be a tensor triangulated category and let* $\dim : \mathrm{Spc}(\mathcal{K}) \to \mathbb{Z} \cup \{\pm\infty\}$ *be a dimension function on $\mathcal{K}$. Let $p \in \mathbb{Z}$ and $\mathcal{P} \in \mathrm{Spc}(\mathcal{K})_{(p)}$. Consider the local category $\mathcal{L} = \mathcal{K}/\mathcal{P}$ and equip* $\mathrm{Spc}(\mathcal{L})$ *with the restriction of the given dimension function on the subspace* $\mathrm{Spc}(\mathcal{L}) \subset \mathrm{Spc}(\mathcal{K})$*. Then we have*

$$\mathrm{FL}(\mathcal{L}) = \mathcal{L}_{(p)}\,.$$

*Proof.* We have $\mathrm{Spc}(\mathcal{L})_{(p)} = \{\mathcal{P}\}$ (see Prop. 1.11 and Def. 3.1). Hence, for $Z \subset \mathrm{Spc}(\mathcal{L})$ closed, we have $\dim(Z) = p$ if and only if $Z = \{\mathcal{P}\}$. $\square$

**Proposition 3.11.** *Let* $\dim : \mathrm{Spc}(\mathcal{K}) \longrightarrow \mathbb{Z} \cup \{\pm\infty\}$ *be a dimension function. Let $p \in \mathbb{Z} \cup \{\pm\infty\}$. Let $\mathcal{J} \subset \mathcal{K}$ be a thick $\otimes$-ideal. (Recall Remark 3.9.) Then the localization functor $q : \mathcal{K} \to \mathcal{K}/\mathcal{J}$ maps $\mathcal{K}_{(p)}$ into $(\mathcal{K}/\mathcal{J})_{(p)}$.*

*Proof.* Suppose that $a \in \mathcal{K}_{(p)}$. Then, since $\mathrm{supp}(q(a)) = \mathrm{supp}(a) \cap \mathrm{Spc}(\mathcal{K}/\mathcal{J}) \subset \mathrm{supp}(a)$, we have $\dim(\mathrm{supp}(q(a))) \leq \dim(\mathrm{supp}(a)) \leq p$. $\square$

*Remark* 3.12. For every dimension function and every $p \in \mathbb{Z} \cup \{\pm\infty\}$, we have $\mathcal{K}_{(p)} \subset \widetilde{\mathcal{K}}_{(p)}$ and $\widetilde{\mathcal{K}}_{(p)}$ is the idempotent completion of $\mathcal{K}_{(p)}$. Indeed, $\widetilde{\mathcal{K}}$ is idempotent complete, hence so is every thick subcategory, like $\widetilde{\mathcal{K}}_{(p)}$ for instance, and $\mathcal{K}_{(p)} \subset \widetilde{\mathcal{K}}_{(p)}$ is cofinal since $\mathrm{supp}(a \oplus T(a)) = \mathrm{supp}(a)$ for every $a \in \widetilde{\mathcal{K}}$.

**Definition 3.13.** A tensor triangulated category $\mathcal{K}$ will be called *topologically noetherian* if the topological space $\mathrm{Spc}(\mathcal{K})$ is noetherian, *i.e.* if every non-empty family of its closed subsets contains a minimal element for inclusion. Recall that this forces every closed subset to be a finite union of irreducibles. Recall as well that every subspace of a noetherian topological space is again noetherian.

*Remark* 3.14. It sometimes simplifies things to assume $\mathcal{K}$ topologically noetherian. For instance, an immediate corollary of Theorem 2.11 is: *In a topologically noetherian, idempotent complete, strongly closed tensor triangulated category, the support of an indecomposable object is connected.*

*Remark* 3.15. Given a set $I$ and a collection $\{\mathcal{K}_i\}_{i\in I}$ of triangulated categories, the coproduct $\coprod_{i\in I} \mathcal{K}_i$ is the triangulated category whose objects are collections $\{a_i\}_{i\in I}$ of objects $a_i \in \mathcal{K}_i$ with $a_i = 0$ for all but finitely many indices $i \in I$ and with morphisms defined componentwise: $\mathrm{Hom}_{\coprod \mathcal{K}_i}(\{a_i\}\,,\,\{b_i\}) = \oplus_{i\in I}\mathrm{Hom}_{\mathcal{K}_i}(a_i, b_i)$. The shift functor and distinguished triangles are also defined componentwise. Each category $\mathcal{K}_i$ can be seen as a subcategory of $\coprod_{i\in I} \mathcal{K}_i$ and every object in the category $\coprod_{i\in I} \mathcal{K}_i$ is the finite direct sum of its components $\{a_i\}_{i\in I} = \bigoplus_{i\in I} a_i$. In this paper, we do not consider tensor structures on such coproducts.

**Lemma 3.16.** *Let $\mathcal{K}$ be a topologically noetherian tensor triangulated category. Let* $\dim : \mathrm{Spc}(\mathcal{K}) \longrightarrow \mathbb{Z}_{\geq 0} \cup \{+\infty\}$ *be a non-negative dimension function. Localization induces a well-defined functor:*

$$\begin{array}{cccc} q_0 & : & \mathcal{K}_{(0)} \longrightarrow & \displaystyle\coprod_{\mathcal{P} \in \mathrm{Spc}(\mathcal{K})_{(0)}} \mathrm{FL}(\mathcal{K}/\mathcal{P}) \\ & & a \longmapsto & \{ q_{\mathcal{P}}(a) \}_{\mathcal{P} \in \mathrm{Spc}(\mathcal{K})_{(0)}} \, . \end{array}$$

*Proof.* Let $a \in \mathcal{K}_{(0)}$. By Propositions 3.11 and 3.10, $q_{\mathcal{P}}(a) \in \mathrm{FL}(\mathcal{K}/\mathcal{P})$ for all $\mathcal{P} \in \mathrm{Spc}(\mathcal{K})_{(0)}$. Observe that $\mathrm{Spc}(\mathcal{K})_{(0)}$ consists of closed points since $\mathcal{Q} \subsetneq \mathcal{P} \in \mathrm{Spc}(\mathcal{K})_{(0)}$ would imply $\dim Q < 0$ by Definition 3.1. (For the dimension of Example 3.2, we have conversely that closed points belong to $\mathrm{Spc}(\mathcal{K})_{(0)}$ but this needs not be true for general dimension functions.) Therefore, the assumption $\dim(\mathrm{supp}(a)) = 0$ means that $\mathrm{supp}(a) = \cup_{i=1}^{n} \{\mathcal{P}_i\}$ for finitely many distinct closed points $\mathcal{P}_1, \ldots, \mathcal{P}_n \in \mathrm{Spc}(\mathcal{K})$ with $\dim(\mathcal{P}_i) = 0$; we use here that $\mathrm{Spc}(\mathcal{K})$ is noetherian. Hence $q_{\mathcal{P}}(a) = 0$ for all but finitely many $\mathcal{P} \in \mathrm{Spc}(\mathcal{K})$. So, the functor $q_0$ is well-defined. □

We now start with a special case of our second main result (Thm. 3.24).

**Theorem 3.17.** *Let $\mathcal{K}$ be a topologically noetherian, idempotent complete, strongly closed tensor triangulated category. Let* $\dim : \mathrm{Spc}(\mathcal{K}) \longrightarrow \mathbb{Z}_{\geq 0} \cup \{+\infty\}$ *be a non-negative dimension function. The localization functor as in Lemma 3.16*

$$q_0 \; : \; \mathcal{K}_{(0)} \longrightarrow \coprod_{\mathcal{P} \in \mathrm{Spc}(\mathcal{K})_{(0)}} \mathrm{FL}(\mathcal{K}/\mathcal{P})$$

*is an equivalence of categories. (For the notation* FL*, see Def. 3.7.)*

*Proof.* The functor $q_0$ has *trivial kernel*, that is, $q_0(a) = 0 \Rightarrow a = 0$. This is clear since for every non-zero $a \in \mathcal{K}_{(0)}$ we have $\mathrm{supp}(a) \neq \varnothing$, by Cororollary 2.5. Choose $\mathcal{P} \in \mathrm{supp}(a)$. Then $a \notin \mathcal{P}$ and so $q_{\mathcal{P}}(a) \neq 0$.

The functor $q_0$ is *essentially surjective.* Indeed, it suffices to see that for all $\mathcal{P} \in \mathrm{Spc}(\mathcal{K})_{(0)}$ and $b \in \mathcal{K}/\mathcal{P}$ with $\mathrm{supp}(b) = \{\mathcal{P}\}$, there exists $a \in \mathcal{K}$ with $\mathrm{supp}(a) = \{\mathcal{P}\}$ and $q_{\mathcal{P}}(a) \simeq b$. Let $c \in \mathcal{K}$ such that $b = q_{\mathcal{P}}(c)$. Let $\mathrm{supp}(c) = \overline{\{\mathcal{P}_1\}} \cup \cdots \cup \overline{\{\mathcal{P}_n\}}$ be a decomposition into irreducible closed subsets without redundancy, *i.e.* $\mathcal{P}_i \not\subset \mathcal{P}_j$ for $i \neq j$ (this exists by Definition 3.13 and Proposition 1.10). Since

$$\{\mathcal{P}\} = \mathrm{supp}(b) = \mathrm{supp}(q_{\mathcal{P}}(c)) = \mathrm{supp}(c) \cap \mathrm{Spc}(\mathcal{K}/\mathcal{P}) = \bigcup_{i=1}^{n} \overline{\{\mathcal{P}_i\}} \cap \mathrm{Spc}(\mathcal{K}/\mathcal{P}) \, ,$$

there is one index $j \in \{1, \ldots, n\}$ such that $\mathcal{P} \in \overline{\{\mathcal{P}_j\}}$ and so $\mathcal{P} \subset \mathcal{P}_j$ or equivalently $\mathcal{P}_j \in \mathrm{Spc}(\mathcal{K}/\mathcal{P})$. Therefore $\mathcal{P}_j = \mathcal{P}$ by the above equality read backwards. We can and will assume that $j = 1$. Since $\mathcal{P} \in \overline{\{\mathcal{P}_i\}}$ forces $\mathcal{P}_i = \mathcal{P} = \mathcal{P}_1$ and since we assumed the decomposition of $\mathrm{supp}(c)$ to be without redundancy, we obtain a decomposition of the support of $c$ into two disjoint closed subsets:

$$\mathrm{supp}(c) = \{\mathcal{P}\} \sqcup \Big( \bigcup_{i=2}^{n} \overline{\{\mathcal{P}_i\}} \Big) \, .$$

By Theorem 2.11, the object $c$ decomposes as $c \simeq a \oplus d$ with $\mathrm{supp}(a) = \{\mathcal{P}\}$ and $\mathrm{supp}(d) = \cup_{i=2}^{n} \overline{\{\mathcal{P}_i\}}$. In particular, $\mathcal{P} \notin \mathrm{supp}(d)$ and therefore $d \in \mathcal{P}$ or equivalently $q_{\mathcal{P}}(d) = 0$, which implies $q_{\mathcal{P}}(a) \simeq q_{\mathcal{P}}(c) = b$ as wanted.

The functor $q_0$ is *full.* Indeed, let $a, b \in \mathcal{K}_{(0)}$ and let $\mathcal{P}_1, \ldots, \mathcal{P}_n$ be the distinct points of dimension zero composing $\mathrm{supp}(a) \cup \mathrm{supp}(b)$. Then, by Theorem 2.11, we know that $a \simeq a_1 \oplus \cdots \oplus a_n$ and $b \simeq b_1 \oplus \cdots \oplus b_n$ with $\mathrm{supp}(a_i) \subset \{\mathcal{P}_i\}$ and $\mathrm{supp}(b_i) \subset \{\mathcal{P}_i\}$. We also know from Corollary 2.8 that $\mathrm{Hom}_{\mathcal{K}}(a_i, b_j) = 0$ for all $i \neq j$. Similarly, morphisms decompose "componentwise" in $\coprod \mathcal{K}/\mathcal{P}$, by definition of the coproduct, see Rem. 3.15. So, we can reduce the discussion to the case of $n = 1$ and $\mathrm{supp}(a) = \mathrm{supp}(b) = \{\mathcal{P}\}$. Consider a morphism between $q_{\mathcal{P}}(a)$ and $q_{\mathcal{P}}(b)$ in $\mathcal{K}/\mathcal{P}$, which can be represented by a fraction

$$a \xleftarrow{\;s\;} c \xrightarrow{\;f\;} b \tag{3.2}$$

of morphisms in $\mathcal{K}$ with $\mathrm{cone}(s) \in \mathcal{P}$. The latter means $\mathcal{P} \notin \mathrm{supp}(\mathrm{cone}(s))$. Consider a distinguished triangle in $\mathcal{K}$

$$\begin{tikzcd} c \arrow[r, "s"] & a \arrow[r, "s_1"] \arrow[l, bend left, dashed, "\exists t"] & \mathrm{cone}(s) \arrow[r, "s_2"] & T(c)\,. \end{tikzcd}$$

Since $\mathrm{supp}(a) \cap \mathrm{supp}(\mathrm{cone}(s)) = \varnothing$, we know by Corollary 2.8 that $s_1 = 0$. In other words, $s : c \to a$ is a split epimorphism whose cone belongs to $\mathcal{P}$. Choose a section $t : a \to c$ of the epimorphism $s$. Its cone is $T^{-1}(\mathrm{cone}(s)) \in \mathcal{P}$, so we can amplify the above fraction (3.2) by $t : a \to c$. We obtain the equivalent fraction

$$a =\!=\!= a \xrightarrow{\;f\,t\;} b\,,$$

which is the image in $\mathcal{K}/\mathcal{P}$ of a morphism between the objects $a$ and $b$ in $\mathcal{K}$.

The proof of the Theorem is finished, because of the following well-known general fact, whose proof is sketched below for the convenience of the reader. ☐

**Proposition 3.18.** *Let $F : \mathcal{K} \longrightarrow \mathcal{L}$ be a full, essentially surjective, exact functor with trivial kernel, between triangulated categories. Then $F$ is an equivalence.*

*Proof.* It suffices to show that $F$ is faithful. Let $f \in \mathrm{Hom}_{\mathcal{K}}(a, b)$ be such that $F(f) = 0$. Consider a distinguished triangle $a \xrightarrow{f} b \xrightarrow{f_1} c \xrightarrow{f_2} Ta$ in $\mathcal{K}$. Since $F$ is exact and $F(f) = 0$, the morphism $F(f_1)$ is a split monomorphism, say $g \circ F(f_1) = \mathrm{id}_{F(b)}$ for $g \in \mathrm{Hom}_{\mathcal{L}}(F(c), F(b))$. Since $F$ is full, there exists $h \in \mathrm{Hom}_{\mathcal{K}}(c, b)$ such that $g = F(h)$. Since $F(h\,f_1) = \mathrm{id}_{F(b)}$ is an isomorphism and since $\mathrm{Ker}(F) = 0$ the morphism $h\,f_1$ is an isomorphism. So $f_1$ is a split monomorphism and finally $f = 0$. ☐

*Remark* 3.19. Let $\mathcal{K}$ be a topologically noetherian, idempotent complete, strongly closed tensor triangulated category and let $\mathcal{P} \in \mathrm{Spc}(\mathcal{K})$ be a *closed* point. Then the subcategory $\mathrm{FL}(\mathcal{K}/\mathcal{P})$ of $\mathcal{K}/\mathcal{P}$ is idempotent complete. Indeed, let $\dim : \mathrm{Spc}(\mathcal{K}) \to \mathbb{Z}_{\geq 0} \cup \{+\infty\}$ be the Krull dimension of Example 3.2. The condition $\dim(\mathcal{P}) = 0$ is equivalent to $\mathcal{P}$ being a closed point of $\mathrm{Spc}(\mathcal{K})$. Since $\mathcal{K}_{(0)}$ is a thick subcategory of the idempotent complete category $\mathcal{K}$, it is idempotent complete. Hence the coproduct of Theorem 3.17 is idempotent complete. So, it suffices to prove that, if $\coprod_{i \in I} \mathcal{K}_i$ is idempotent complete, then each $\mathcal{K}_i$ is idempotent complete. The latter is an easy exercise. We do not know whether $\mathcal{K}/\mathcal{P}$ itself is idempotent complete.

**Proposition 3.20.** *Let* $\dim : \mathrm{Spc}(\mathcal{K}) \longrightarrow \mathbb{Z} \cup \{\pm\infty\}$ *be a dimension function. For every* $p \in \mathbb{Z} \cup \{\pm\infty\}$*, we have* $\mathcal{K}_{(p)} = \bigcap\limits_{\dim(\mathcal{P}) > p} \mathcal{P}$*, with the usual convention* $\bigcap\limits_{\varnothing} = \mathcal{K}$*.*

*Proof.* For an object $a \in \mathcal{K}$, we have equivalences:

$$\begin{aligned} a \in \mathcal{K}_{(p)} &\overset{(3.1)}{\iff} \Big(\forall\, \mathcal{P} \in \mathrm{supp}(a),\ \dim(\mathcal{P}) \le p\Big) \\ &\overset{\text{Def. 1.1}}{\iff} \Big(\forall\, \mathcal{P} \in \mathrm{Spc}(\mathcal{K}),\ a \notin \mathcal{P} \ \Rightarrow\ \dim(\mathcal{P}) \le p\Big) \\ &\iff \Big(\forall\, \mathcal{P} \in \mathrm{Spc}(\mathcal{K}),\ \dim(\mathcal{P}) > p \ \Rightarrow\ a \in \mathcal{P}\Big) \iff a \in \bigcap_{\dim(\mathcal{P})>p} \mathcal{P} \end{aligned}$$

which is the claim. □

**Proposition 3.21.** *Let* $\dim : \mathrm{Spc}(\mathcal{K}) \longrightarrow \mathbb{Z} \cup \{\pm\infty\}$ *be a dimension function and assume that* $\mathcal{K}$ *is topologically noetherian (Def. 3.13). Let* $\mathcal{P} \in \mathrm{Spc}(\mathcal{K})$ *and let* $p \in \mathbb{Z} \cup \{\pm\infty\}$. *Then we have* $\mathcal{P} \supset \mathcal{K}_{(p)}$ *if and only if* $\dim(\mathcal{P}) > p$.

*Proof.* From Proposition 3.20 it is clear that $\dim(\mathcal{P}) > p$ implies $\mathcal{P} \supset \mathcal{K}_{(p)}$.

Conversely, assume that $\mathcal{P} \supset \mathcal{K}_{(p)}$. Since $\mathrm{Spc}(\mathcal{K})$ is noetherian, the open $\mathrm{Spc}(\mathcal{K}) \smallsetminus \overline{\{\mathcal{P}\}}$ is quasi-compact. Then, by [1, Prop. 2.14 (b)], there exists an object $a \in \mathcal{K}$ such that $\mathrm{supp}(a) = \overline{\{\mathcal{P}\}}$. We clearly have $\dim(\mathrm{supp}(a)) = \dim(\mathcal{P})$ since $\overline{\{\mathcal{P}\}} = \{\mathcal{Q} \subset \mathcal{P}\}$, see Prop. 1.9 and Def. 3.1 (1). Now $\mathcal{P} \in \mathrm{supp}(a)$ means that $a \notin \mathcal{P}$ and in particular $a \notin \mathcal{K}_{(p)} \subset \mathcal{P}$. The property $a \notin \mathcal{K}_{(p)}$ means by definition that $\dim(\mathrm{supp}(a)) > p$. In short, $\dim(\mathcal{P}) = \dim(\mathrm{supp}(a)) > p$ as was to be shown. □

**Corollary 3.22.** *Let* $\dim : \mathrm{Spc}(\mathcal{K}) \longrightarrow \mathbb{Z} \cup \{\pm\infty\}$ *be a dimension function on a topologically noetherian tensor triangulated category* $\mathcal{K}$. *Let* $p \in \mathbb{Z} \cup \{\pm\infty\}$. *Consider* $\mathcal{M} := \mathcal{K}/\mathcal{K}_{(p)}$ *and the localization functor* $q : \mathcal{K} \longrightarrow \mathcal{M}$. *Consider also the idempotent completion* $\iota : \mathcal{M} \hookrightarrow \widetilde{\mathcal{M}}$. *Then via the identifications of Propositions 1.11 and 1.12,* $\mathrm{Spc}(\mathcal{M})$ *and* $\mathrm{Spc}(\widetilde{\mathcal{M}})$ *coincide with the following subspace*

$$\mathrm{Spc}(\mathcal{M}) = \mathrm{Spc}(\widetilde{\mathcal{M}}) = \{\mathcal{P} \in \mathrm{Spc}(\mathcal{K}) \mid \dim(\mathcal{P}) > p\}$$

*of the spectrum* $\mathrm{Spc}(\mathcal{K})$ *of the ambient category.* □

**Example 3.23.** Let $\mathcal{K}$ be a topologically noetherian tensor triangulated category and denote by $\dim_{\mathcal{K}} : \mathrm{Spc}(\mathcal{K}) \longrightarrow \mathbb{Z}_{\ge 0} \cup \{+\infty\}$ the Krull dimension (see Ex. 3.2). Let $p \in \mathbb{Z}_{\ge 0}$. Consider the two possible dimension functions on $\mathrm{Spc}(\mathcal{K}/\mathcal{K}_{(p)})$: the one restricted from the above $\dim_{\mathcal{K}}$ on $\mathrm{Spc}(\mathcal{K}/\mathcal{K}_{(p)}) \subset \mathrm{Spc}(\mathcal{K})$ (see Rem. 3.9) and secondly, the intrinsic Krull dimension $\dim_{\mathcal{K}/\mathcal{K}_{(p)}}$ for $\mathcal{K}/\mathcal{K}_{(p)}$. Then they differ exactly by $p$, namely for every prime $\mathcal{Q} \in \mathrm{Spc}(\mathcal{K})$ such that $\mathcal{Q} \supset \mathcal{K}_{(p)}$ we have

$$\dim_{\mathcal{K}}(\mathcal{Q}) = \dim_{\mathcal{K}/\mathcal{K}_{(p)}}(\mathcal{Q}) + p\,.$$

This follows immediately from the definition of the Krull dimension and from Proposition 3.21. This fact will not be used below, since we always deal with “abstract” dimension functions which are not necessarily the Krull one (nor co-one).

**Theorem 3.24.** *Let* $\dim : \mathrm{Spc}(\mathcal{K}) \longrightarrow \mathbb{Z} \cup \{\pm\infty\}$ *be a dimension function on a topologically noetherian, strongly closed tensor triangulated category* $\mathcal{K}$. *Let* $p \in \mathbb{Z}$. *Then we have a fully faithful functor:*

$$\mathcal{K}_{(p)} \,/\, \mathcal{K}_{(p-1)} \quad\longrightarrow \coprod_{\mathcal{P}\,\in\,\mathrm{Spc}(\mathcal{K})_{(p)}} \mathrm{FL}(\mathcal{K}/\mathcal{P})$$

*induced by localization. This functor is cofinal, hence induces an equivalence after idempotent completion on both sides. (For the notation* $\mathrm{FL}(\mathcal{K}/\mathcal{P})$*, see Def. 3.7.)*

*Proof.* Using the dimension function $\mathcal{P} \mapsto \dim(\mathcal{P}) - p$ instead of the given one, we can assume that $p = 0$.

Let $\mathcal{M} = \mathcal{K}/\mathcal{K}_{(-1)}$ and let $q : \mathcal{K} \longrightarrow \mathcal{M}$ be the localization functor. By Corollary 3.22, we can identify $\mathrm{Spc}(\mathcal{M})$ and $\mathrm{Spc}(\widetilde{\mathcal{M}})$ with the following subset of $\mathrm{Spc}(\mathcal{K})$ :

$$\mathrm{Spc}(\mathcal{M}) = \mathrm{Spc}(\widetilde{\mathcal{M}}) = \{\mathcal{P} \in \mathrm{Spc}(\mathcal{K}) \mid \dim(\mathcal{P}) \geq 0\}\,.$$

Observe that the quotient category considered in the statement is indeed

$$\mathcal{K}_{(0)}/\mathcal{K}_{(-1)} = \mathcal{M}_{(0)}\,. \tag{3.3}$$

To prove this, first observe that $\mathcal{K}_{(-1)} \subset \mathcal{K}_{(0)} \subset \mathcal{K}$, so we have an inclusion $\mathcal{K}_{(0)}/\mathcal{K}_{(-1)} \subset \mathcal{K}/\mathcal{K}_{(-1)} = \mathcal{M}$. For an object $a \in \mathcal{K}$, by the above description of $\mathrm{Spc}(\mathcal{M})$, the following subsets of $\mathrm{Spc}(\mathcal{K})$ coincide :

$$\begin{aligned}
&\{\mathcal{P} \in \mathrm{Spc}(\mathcal{K}) \mid \mathcal{P} \in \mathrm{supp}(a) \text{ and } \dim(\mathcal{P}) > 1\}\\
&\qquad = \{\mathcal{P} \in \mathrm{Spc}(\mathcal{M}) \mid \mathcal{P} \in \mathrm{supp}(a) \cap \mathrm{Spc}(\mathcal{M}) \text{ and } \dim(\mathcal{P}) > 1\}\\
&\qquad = \{\mathcal{P} \in \mathrm{Spc}(\mathcal{K}) \mid \mathcal{P} \in \mathrm{supp}(q(a)) \text{ and } \dim(\mathcal{P}) > 1\}\,,
\end{aligned}$$

the second equality coming from $\mathrm{supp}(q(a)) = \mathrm{supp}(a) \cap \mathrm{Spc}(\mathcal{M})$. Now the first of these subsets is empty exactly when $a \in \mathcal{K}_{(0)}$ whereas the third subset is empty exactly when $q(a) \in \mathcal{M}_{(0)}$. This establishes Equation (3.3). Let us now decompose $\mathcal{K}_{(0)}/\mathcal{K}_{(-1)} = \mathcal{M}_{(0)}$ as in the statement (for $p = 0$ of course).

Note that the dimension function restricted from $\mathrm{Spc}(\mathcal{K})$ to $\mathrm{Spc}(\mathcal{M})$ and to $\mathrm{Spc}(\widetilde{\mathcal{M}})$ is non-negative. Note also that $\mathrm{Spc}(\mathcal{M}) = \mathrm{Spc}(\widetilde{\mathcal{M}})$ is noetherian, since it is a subspace of the noetherian space $\mathrm{Spc}(\mathcal{K})$. So we can apply Lemma 3.16 both to $\mathcal{M}$ and to $\widetilde{\mathcal{M}}$ as well as Theorem 3.17 to the idempotent complete $\widetilde{\mathcal{M}}$. This gives the horizontal arrows in Diagram (3.4) below.

For every prime $\mathcal{P} \in \mathrm{Spc}(\mathcal{M})$, recall from Proposition 1.12 that the corresponding prime $\widetilde{\mathcal{P}} \in \mathrm{Spc}(\widetilde{\mathcal{M}})$ simply is the idempotent completion of $\mathcal{P}$. By Proposition 1.13, we have a natural functor

$$F_{\mathcal{P}} : \mathcal{M}/\mathcal{P} \longrightarrow \widetilde{\mathcal{M}}/\widetilde{\mathcal{P}}$$

which is fully faithful and cofinal. This functor restricts to a fully faithful and cofinal functor $F_{\mathcal{P}} : (\mathcal{M}/\mathcal{P})_{(0)} \longrightarrow (\widetilde{\mathcal{M}}/\widetilde{\mathcal{P}})_{(0)}$ by Remark 3.12, which also guarantees the cofinality of the inclusion $\iota_{\mathcal{M}} : \mathcal{M}_{(0)} \hookrightarrow (\widetilde{\mathcal{M}})_{(0)}$. We now have all the pieces of the following obviously commutative diagram :

$$\begin{array}{ccc}
\mathcal{M}_{(0)} & \xrightarrow{\;q_0\;} & \coprod\limits_{\mathcal{P} \in \mathrm{Spc}(\mathcal{M})_{(0)}} (\mathcal{M}/\mathcal{P})_{(0)} \\
\Big\downarrow{\scriptstyle \iota_{\mathcal{M}}} & & \Big\downarrow{\scriptstyle \coprod F_{\mathcal{P}}} \\
(\widetilde{\mathcal{M}})_{(0)} & \xrightarrow[\;q_0\;]{} & \coprod\limits_{\mathcal{P} \in \mathrm{Spc}(\mathcal{M})_{(0)}} (\widetilde{\mathcal{M}}/\widetilde{\mathcal{P}})_{(0)}\,.
\end{array} \tag{3.4}$$

We know by Theorem 3.17 that the lower $q_0$ functor in (3.4) is an equivalence and we have seen above that the two vertical functors are fully faithful and cofinal. This proves that the upper functor $q_0$ in (3.4) is fully faithful and cofinal.

For every prime $\mathcal{P} \in \mathrm{Spc}(\mathcal{M})$, we have $\mathcal{M}/\mathcal{P} \cong \mathcal{K}/q^{-1}(\mathcal{P})$, where $q^{-1}(\mathcal{P}) \in \mathrm{Spc}(\mathcal{K})$ is the prime corresponding to $\mathcal{P} \in \mathrm{Spc}(\mathcal{M})$, see Prop. 1.11. Using this and (3.3) to express $\mathcal{M}$ in terms of $\mathcal{K}$ in the upper row of (3.4), we have the result. $\square$

*Remark* 3.25. In the Introduction, the statement of Theorem 3.24 only involves Krull dimension. The link is made via Proposition 3.8.

Finally, we unfold the result for the dimension function of Example 3.3.

**Theorem 3.26.** *Let $\mathcal{K}$ be a topologically noetherian, strongly closed tensor triangulated category. Let* $\mathrm{codim} : \mathrm{Spc}(\mathcal{K}) \longrightarrow \mathbb{Z}_{\geq 0} \cup \{+\infty\}$ *be the Krull codimension. For each $q \in \mathbb{Z}_{\geq 0}$ consider* $\mathrm{Spc}(\mathcal{K})^{(q)} := \{\mathcal{P} \in \mathrm{Spc}(\mathcal{K}) \mid \mathrm{codim}(\mathcal{P}) = q\}$ *and*

$$\mathcal{K}^{(q)} := \{a \in \mathcal{K} \mid \mathrm{codim}(\mathrm{supp}(a)) \geq q\}\,.$$

*Then we have* $\cdots \subset \mathcal{K}^{(q+1)} \subset \mathcal{K}^{(q)} \subset \cdots \subset \mathcal{K}^{(0)} = \mathcal{K}$ *and fully faithful functors:*

$$\mathcal{K}^{(q)} \big/ \mathcal{K}^{(q+1)} \quad \longrightarrow \quad \coprod_{\mathcal{P} \in \mathrm{Spc}(\mathcal{K})^{(q)}} \mathrm{FL}(\mathcal{K}/\mathcal{P})$$

*induced by localization. These functors are cofinal, hence induce equivalences after idempotent completion on both sides. (For the notation* $\mathrm{FL}(\mathcal{K}/\mathcal{P})$*, see Def. 3.7.)*

*Proof.* We know that $\dim := -\mathrm{codim} : \mathrm{Spc}(\mathcal{K}) \to \mathbb{Z} \cup \{\pm\infty\}$ is a dimension function. Note that Definition 3.5 becomes here $\mathrm{codim}(Z) = \inf_{\mathcal{P}\in Z} \mathrm{codim}(\mathcal{P})$ as usual. We clearly have $\mathrm{Spc}(\mathcal{K})^{(q)} = \mathrm{Spc}(\mathcal{K})_{(-q)}$ and $\mathcal{K}^{(q)} = \mathcal{K}_{(-q)}$ for all $q \in \mathbb{Z}_{\geq 0}$. Hence the result follows from Theorem 3.24 applied to $p = -q$. $\square$

## 4. Applications to algebraic geometry and modular representation theory

4.1. **Schemes.** Let $X$ be a quasi-compact and quasi-separated scheme. Recall from [11, Cor. 6.1.13, p. 296] that being quasi-separated simply means that the intersection of two quasi-compact open subsets remains quasi-compact; this holds of course if the underlying space of $X$ is noetherian and *a fortiori* if $X$ is noetherian.

We denote by $\mathrm{D}^{\mathrm{perf}}(X)$ the derived category of perfect complexes over $X$. A concise introduction is given in Thomason [19, § 3.1] and a more detailed one in [20, § 2], where the reader will find further references to Grothendieck's original (SGA).

A complex of $\mathcal{O}_X$-modules is *perfect* if it is locally quasi-isomorphic to a bounded complex of vector bundles and we denote by $\mathrm{D}^{\mathrm{perf}}(X)$ the corresponding full subcategory of $\mathrm{D}(\mathcal{O}_X\!-\!\mathrm{Mod})$. If $\mathrm{D}^{\mathrm{b}}(\mathrm{VB}_X)$ stands for the derived category of bounded complexes of vector bundles over $X$, we have a fully faithful functor $\mathrm{D}^{\mathrm{b}}(\mathrm{VB}_X) \hookrightarrow \mathrm{D}^{\mathrm{perf}}(X)$ and this is an equivalence if $X$ has an ample family of line bundles, which is quite common in practice. If $X = \mathrm{Spec}(A)$ is affine, this boils down to $\mathrm{D}^{\mathrm{perf}}(X) \cong \mathrm{D}^{\mathrm{b}}(\mathrm{VB}_X) \cong \mathrm{K}^{\mathrm{b}}(A\!-\!\mathrm{proj})$ the homotopy category of bounded complexes of finitely generated projective $A$-modules (which are free if $A$ is local).

The category $\mathrm{D}^{\mathrm{perf}}(X)$ is contained in the subcategory $\mathrm{D}^{+}_{\mathrm{Qcoh}}(X)$ of cohomologically bounded below complexes with quasi-coherent cohomology and consists of the compact objects therein. However, this is wrong inside $\mathrm{D}(\mathcal{O}_X\!-\!\mathrm{Mod})$. (See more in Bondal–van den Bergh [4].) The category $\mathrm{D}^{\mathrm{perf}}(X)$ is also equivalent to the category of compact objects in $\mathrm{D}(\mathrm{Qcoh}_X)$, at least when $X$ is moreover separated, as explained in Neeman [16, § 2]. This is one possible approach to the following result in the spirit of Remark 2.2.

**Proposition 4.1.** *The category of perfect complexes* $\mathrm{D}^{\mathrm{perf}}(X)$ *is a strongly closed tensor triangulated category with the derived tensor product* $-\otimes^{L}_{\mathcal{O}_X}-$ *as tensor product and the derived Hom sheaf* $R\mathcal{H}om_{\mathcal{O}_X}(-,-)$ *as internal hom.*

*Proof.* The fact that $E\otimes^{L}_{\mathcal{O}_X}F$ and $R\mathcal{H}om_{\mathcal{O}_X}(E,F)$ are perfect for every perfect $E,F\in\mathrm{D}^{\mathrm{perf}}(X)$ can be found in the above references. The relevant adjunction (2.1), as well as strong dualizability (2.2), can be checked locally and are well-known for bounded complexes of finitely generated projective modules. We leave the details to the reader. □

We proved in [1, Cor. 5.6] that the map

$$\begin{aligned} X &\longrightarrow \mathrm{Spc}(\mathrm{D}^{\mathrm{perf}}(X)) \\ x &\longmapsto \mathcal{P}(x):=\mathrm{Ker}(\mathrm{D}^{\mathrm{perf}}(X)\to\mathrm{D}^{\mathrm{perf}}(\mathcal{O}_{X,x})) \end{aligned}$$

is a homeomorphism for $X$ topologically noetherian. Under this homeomorphism, the support coincides with the *homological support*, that is the support of the sum of the homology $\mathcal{O}_X$-modules. As already mentioned, the reader will find in [5] the extension of this result to $X$ quasi-compact and quasi-separated (*i.e.* whose underlying space is *spectral* in the sense of Hochster [13]). At this point, the first Corollary of the Introduction is under roof by means of Theorem 2.11. To apply our second main result (Thm. 3.24) though, we need to identify the local categories $\mathrm{D}^{\mathrm{perf}}(X)/\mathcal{P}(x)$ and this is where we need noetherianity of $X$. For every $x\in X$, from the definition of $\mathcal{P}(x)$, we obtain a canonical functor

$$\mathrm{D}^{\mathrm{perf}}(X)/\mathcal{P}(x)\longrightarrow\mathrm{D}^{\mathrm{perf}}(\mathcal{O}_{X,x})=\mathrm{K}^{\mathrm{b}}(\mathcal{O}_{X,x}-\mathrm{free}) \tag{4.1}$$

which is an equivalence. Indeed, for every open (affine) neighborhood $U\subset X$ of the point $x\in X$, we have a localization $\mathrm{D}^{\mathrm{perf}}(U)\longrightarrow\mathrm{D}^{\mathrm{perf}}(\mathcal{O}_{X,x})$ and $\mathrm{D}^{\mathrm{perf}}(\mathcal{O}_{X,x})$ is the "colimit" of those $\mathrm{D}^{\mathrm{perf}}(U)$ in the sense that every object and every morphism in $\mathrm{D}^{\mathrm{perf}}(\mathcal{O}_{X,x})$ extends to some neighborhood $U$ and that two such "germs" of objects or morphisms defined on some $U$ are equal in $\mathrm{D}^{\mathrm{perf}}(\mathcal{O}_{X,x})$ if and only if they become equal in some smaller open $V\ni x$. Thomason (see [20, § 5]) established that for every quasi-compact open $U\subset X$, if we denote by $Z=X\smallsetminus U$ its closed complement, we have a fully faithful functor $\mathrm{D}^{\mathrm{perf}}(X)/\,\mathrm{D}^{\mathrm{perf}}_{Z}(X)\longrightarrow\mathrm{D}^{\mathrm{perf}}(U)$ which is cofinal, meaning that a complex in $\mathrm{D}^{\mathrm{perf}}(U)$ can be lifted to $\mathrm{D}^{\mathrm{perf}}(X)$ if and only if its class in $K$-theory belongs to the image of $K_0(X)\to K_0(U)$. It is then easy to deduce from this discussion that the functor (4.1) is fully faithful and that its essential image consists of those complexes whose class belongs to the image of $K_0(X)\to K_0(\mathcal{O}_{X,x})$. But the latter homomorphism is always surjective since $K_0(\mathcal{O}_{X,x})\simeq\mathbb{Z}$. So, the functor (4.1) is an equivalence. Via this equivalence, the finite length objects of $\mathrm{D}^{\mathrm{perf}}(X)/\mathcal{P}(x)$, in the sense of Definition 3.7, correspond to those complexes in $\mathrm{D}^{\mathrm{perf}}(\mathcal{O}_{X,x})$ whose homology is supported on $\{x\}$, that is, those complexes whose homology is a finite length $\mathcal{O}_{X,x}$-module (we need here that the homology of a complex in $\mathrm{D}^{\mathrm{perf}}(\mathcal{O}_{X,x})$ is finitely generated and this holds if $\mathcal{O}_{X,x}$ is noetherian). Hence $\mathrm{FL}\left(\mathrm{D}^{\mathrm{perf}}(\mathcal{O}_{X,x})\right)=\mathrm{K}^{\mathrm{b}}_{\mathrm{fin.lg.}}(\mathcal{O}_{X,x}-\mathrm{free})$ as in the Introduction.

4.2. **Support varieties.** Let $k$ be a field of positive characteristic and let $G$ be a finite group whose order is divisible by the characteristic of $k$. (Otherwise $kG$ is semi-simple, *i.e.* $kG-\mathrm{mod}=kG-\mathrm{proj}$ and the stable category will be trivial.) Let $\mathrm{stab}(kG)=kG-\mathrm{mod}/kG-\mathrm{proj}$ be the *stable category*, whose objects are finitely generated (left) $kG$-modules and whose morphisms are the $kG$-module homomorphisms modulo those which factor via projective modules. This is a triangulated category. See details in Happel [12], Benson [3] or Carlson [7]. Indeed, $kG-\mathrm{mod}$ satisfies Krull-Schmidt, in the sense that every module has a unique decomposition as a direct sum of indecomposable $kG$-modules. Working in $\mathrm{stab}(kG)$ essentially means ignoring the projective summands. So, a non-projective indecomposable $kG$-module remains indecomposable in $\mathrm{stab}(kG)$. One can prove that $\mathrm{stab}(kG)$ consists of the compact objects in the bigger triangulated category $\mathrm{Stab}(kG) = kG-\mathrm{Mod}\,/kG-\mathrm{Proj}$ of not necessarily finitely generated $kG$-modules modulo projectives. This is one possible approach to the following result in the spirit of Remark 2.2.

**Proposition 4.2.** *The category* $\mathrm{stab}(kG)$ *is a strongly closed tensor triangulated category with* $-\otimes_k -$ *as tensor product and* $\mathrm{Hom}_k(-,-)$ *as internal hom.*

*Proof.* Recall that $M\otimes_k N$ is a $kG$-module with diagonal action $g\,(m\otimes n) = (g\,m)\otimes(g\,n)$ and similarly for $\mathrm{Hom}_k(M,N)$: for a $k$-linear homomorphism $f : M \to N$ we have $(g\,f)(m) = g\,(f(g^{-1}m))$. The adjunction (2.1) and strong dualizability (2.2) are easy exercises from the basic adjunction

$$\mathrm{Hom}_{kG}(M\otimes_k N, L) \simeq \mathrm{Hom}_{kG}(M, \mathrm{Hom}_k(N,L))$$

and from the fact that this adjunction preserves the property of being factorizable via a projective module. The latter is immediate from its naturality and from the fact that $M\otimes_k N$ and $\mathrm{Hom}_k(M,N)$ are projective as soon as $M$ or $N$ is. In $kG-\mathrm{mod}$, we already have $\mathrm{Hom}_k(M,N) \cong \mathrm{Hom}_k(M,k)\otimes_k N$. See [7, § 2 and Thm. 3.3] if necessary. □

Recall that $H^\bullet(G,k)$ is the graded cohomology ring of $G$ with coefficients in $k$ in characteristic two and only its even part in odd characteristic. This is done so that $H^\bullet(G,k)$ is commutative – and this is not important since $\mathrm{Proj}(-)$ does not see nilpotent homogeneous elements of positive degree anyway. One sets $\mathcal{V}_G(k) = \mathrm{Proj}(H^\bullet(G,k))$, the projective support variety of the group $G$ over the field $k$. This can be extended to finite group schemes as well, as explained in Friedlander-Pevtsova [10]. For a $kG$-module $M$, one can consider the closed subset $\mathcal{V}_G(M) \subset \mathcal{V}_G(k)$ defined by the homogeneous ideal $\mathrm{Ann}_{H^\bullet(G,k)}(\mathrm{Ext}^*_{kG}(M,M))$, where our ring $H^\bullet(G,k) = \mathrm{Ext}^\bullet_{kG}(k,k)$ acts on $\mathrm{Ext}^*_{kG}(M,M)$ by cup product, that is simply via the tensor product $-\otimes_k-$ in $\mathrm{stab}(kG)$ using that $M\otimes_k k = M$. Consider the map

$$\begin{aligned} \mathcal{V}_G(k) &\longrightarrow \mathrm{Spc}(\mathrm{stab}(kG)) \\ x &\longmapsto \mathcal{P}(x) := \{M \in \mathrm{stab}(kG) \mid x \notin \mathcal{V}_G(M)\}\,. \end{aligned}$$

In [1, Cor. 5.10], we proved that this map is a homeomorphism, under which $\mathcal{V}_G(M)$ corresponds to our support $\mathrm{supp}(M)$. This holds for any finite group scheme $G$.

Finally, for $G$ an ordinary finite group, recall that the *complexity* $c$ of a $kG$-module $M$ is defined to be the smallest integer such that there exists a projective resolution $\cdots \to P_2 \to P_1 \to P_0 \to M \to 0$ of $M$ for which the $k$-dimension of $P_i$ does not grow faster than $i^{c-1}$. The complexity of a projective module is zero. The complexity of $M$ is well-known to be equal to the Krull dimension $\dim_{\mathrm{Krull}}(V_G(M))$ of the homogeneous affine support variety of $M$ and is therefore equal to $\dim_{\mathrm{Krull}}(\mathcal{V}_G(M))+1$, since a projective variety $\mathcal{V}_G(M)$ has dimension one less than the corresponding homogeneous affine variety $V_G(M)$. See details in [3, Vol. 2, Chap. 5]. Indeed, Carlson, Donovan and Wheeler [8, Thm. 3.5] also give another description of the morphisms in $\mathcal{M}_{p+1}/\mathcal{M}_p$, at least when the field $k$ is algebraically closed. Further details will be left to the specialized reader.

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